Chapter 4

Iterative Algorithms for a System of Nonlinear Variational Inequalities in Banach Spaces

In this chapter, we consider a general system of nonlinear variational inequalities (in short, GSNVI) in the setting of Banach spaces. We establish the equivalence between GSNVI and a system of fixed point problems. By utilizing this equivalence, we construct an implicit algorithm of Mann-type for solving GSNVI. We also propose an explicit algorithm of Mann-type for solving GSNVI. Finally, under very mild conditions, we prove the strong convergence of the sequences generated by the proposed algorithms.

4.1 Introduction and Formulation

Let $C \subseteq X$ be a nonempty closed convex set of a real Banach space $X$, $B_1, B_2 : C \to X$ be nonlinear mappings, and $\lambda, \mu$ be positive real numbers. The general system of nonlinear variational inequalities (in short, GSNVI), defined in the setting of Banach space, is to find $(x^*, y^*) \in C \times C$ such that

$$\left\{ \begin{array}{l}
\langle \lambda B_1 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\
\langle \mu B_2 x^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \quad \forall x \in C.
\end{array} \right. \quad (4.1.1)$$
It is considered and studied by Yao et al. [238]. They proposed and analyzed implicit and explicit iterative algorithms for solving GSNVI (4.1.1). They established equivalence between GSNVI (4.1.1) and the fixed point problem of some nonexpansive mappings defined on a Banach Space and then by using this equivalence, they constructed an implicit iterative algorithm and an explicit iterative algorithm for solving GSNVI (4.1.1). They proved the strong convergence of the sequences generated by the proposed algorithm. It is worth to mention that the system of variational inequalities plays an important role in game theory and economics. Namely, the Nash equilibrium problem can be formulated in the form of a system of variational inequalities, see for example [8, 7, 71, 134] and the references therein.

If $B_1 = B_2$ and $x^* = y^*$, the GSNVI (4.1.1) reduces to the following problem of finding $x^* \in C$ such that

$$\langle B_1 x^*, j(x-x^*) \rangle \geq 0, \quad \forall x \in C. \quad (4.1.2)$$

The following Lemma provides the equivalence between VIP (4.1.2) and its corresponding Minty-type variational inequality problem.

**Lemma 4.1.1.** [238] Let $C$ be a nonempty closed convex subset of a real Banach space $X$. Assume that the mapping $A : C \rightarrow X$ is accretive and weakly continuous along segments (that is, $A(x + ty) \rightharpoonup A(x)$ as $t \to 0$). Then, the variational inequality problem

$$x^* \in C, \quad \langle Ax^*, j(x-x^*) \rangle \geq 0, \quad \forall x \in C$$

is equivalent to the following Minty type variational inequality:

$$x^* \in C, \quad \langle Ax^*, j(x-x^*) \rangle \geq 0, \quad \forall x \in C.$$

Aoyama et al. [9] proposed an iterative scheme to find the approximate solution of (4.1.2) and they proved the weak convergence of the sequences generated by the proposed scheme. This problem is connected with the fixed point problem for a nonlinear mapping, the problem of finding a zero point of a nonlinear operator and so on.
It is an interesting problem that how to construct some algorithms with strong convergence for solving GSNVI (4.1.1).

Our purpose in this chapter is to continue the study of the iterative methods for finding the solutions of GSNVI (4.1.1). By utilizing the equivalence between GSNVI (4.1.1) and a fixed point problem, we construct an implicit algorithm of Mann-type for solving GSNVI (4.1.1). We also propose another explicit algorithm of Mann-type for solving GSNVI (4.1.1). Finally, under very mild conditions, we prove the strong convergence of the sequences generated by the proposed algorithms.

4.2 Some Basic Results

The following proposition will be used frequently throughout the chapter. For the sake of completeness, we include its proof.

Proposition 4.2.1. Let $X$ be a real smooth Banach space and $A : C \to X$ be a mapping.

(a) If $A$ is $\zeta$-strictly pseudo-contractive, then $A$ is Lipschitz continuous with constant $(1 + \frac{1}{\zeta})$.

(b) If $A$ is $\delta$-strongly accretive and $\zeta$-strictly pseudo-contractive with $\delta + \zeta > 1$, then $I - A$ is contractive with constant $\sqrt{\frac{1 - \delta}{\zeta}} \in (0, 1)$.

(c) If $A$ is $\delta$-strongly accretive and $\zeta$-strictly pseudo-contractive with $\delta + \zeta > 1$, then for any fixed number $\tau \in (0, 1)$, $I - \tau A$ is contractive with constant $1 - \tau \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right) \in (0, 1)$.

Proof. (a) Utilizing the definition of the $\zeta$-strict pseudo-contraction $A$, we derive for all $x, y \in C$,

$$\zeta \|(I - A)x - (I - A)y\|^2 \leq \langle (I - A)x - (I - A)y, j(x - y) \rangle$$

$$\leq \| (I - A)x - (I - A)y \| \| x - y \|,$$
which implies that
\[ \| (I - A)x - (I - A)y \| \leq \frac{1}{\zeta} \| x - y \|. \]

Thus,
\[
\| Ax - Ay \| \leq \| (I - A)x - (I - A)y \| + \| x - y \|
\leq \left( 1 + \frac{1}{\zeta} \right) \| x - y \|,
\]

and so \( A \) is Lipschitz continuous with constant \( (1 + \frac{1}{\zeta}) \).

(b) Since \( A \) is \( \delta \)-strongly accretive and \( \zeta \)-strictly pseudo-contractive, we have
\[
\zeta \| (I - A)x - (I - A)y \|^2 \leq \| x - y \|^2 - \langle Ax - Ay, j(x - y) \rangle
\leq (1 - \delta) \| x - y \|^2.
\]
Note that \( \delta + \zeta > 1 \iff \sqrt{\frac{1 - \delta}{\zeta}} \in (0, 1) \). Hence, we obtain
\[
\| (I - A)x - (I - A)y \| \leq \left( \sqrt{\frac{1 - \delta}{\zeta}} \right) \| x - y \|.
\]
This implies that \( I - A \) is contractive with constant \( \sqrt{\frac{1 - \delta}{\zeta}} \in (0, 1) \).

(c) Since \( I - A \) is contractive with constant \( \sqrt{\frac{1 - \delta}{\zeta}} \), for each fixed number \( \tau \in (0, 1) \), we have
\[
\| (x - y) - \tau (Ax - Ay) \| = \| (1 - \tau)(x - y) + \tau[(I - A)x - (I - A)y] \|
\leq (1 - \tau) \| x - y \| + \tau \| (I - A)x - (I - A)y \|
\leq (1 - \tau) \| x - y \| + \tau \left( \sqrt{\frac{1 - \delta}{\zeta}} \right) \| x - y \|
= \left( 1 - \tau \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \right) \| x - y \|.
\]
This shows that \( I - \tau A \) is contractive with constant \( 1 - \tau \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \in (0, 1) \). \( \square \)
The following Lemma provides the equivalence between GSNVI (4.1.1) and the fixed point problem.

Lemma 4.2.1. [238] Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$ and $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mappings $B_1, B_2 : C \to X$ be $\alpha$-inverse-strongly accretive and $\beta$-inverse-strongly accretive, respectively. $(x^*, y^*) \in C \times C$ is a solution of the GSNVI (4.1.1) if and only if

$$x^* = \Pi_C(y^* - \lambda B_1 y^*),$$

where $y^* = \Pi_C(x^* - \mu B_2 x^*)$.

Remark 4.2.1. From Lemma 4.2.1 and equation (2.3.4), we have

$$x^* = \Pi_C[\Pi_C(x^* - \mu B_2 x^*) - \lambda B_1 \Pi_C(x^* - \mu B_2 x^*)] = G(x^*),$$

which implies that $x^*$ is a fixed point of the mapping $G$.

Throughout the chapter, we denote $\Omega$ the set of fixed points of the mapping $G$.

In order to solve GSNVI (4.1.1), we first introduce an implicit algorithm of Mann-type. Let $C$ be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space $X$ and $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_1, B_2 : C \to X$ be $\alpha$-inverse-strongly accretive and $\beta$-inverse-strongly accretive, respectively, and $A : C \to X$ be $\delta$-strongly accretive and $\zeta$-strictly pseudo-contractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\delta})$ and $\mu \in (0, \frac{\beta}{\delta})$ where $\kappa$ is the 2-uniformly smooth constant of $X$ (see Lemma 2.3.2). For each $t \in (0, 1)$, choose a number $\theta_t \in (0, 1)$ arbitrarily. For any $x \in C$, we consider the following mapping

$$W_t x := \{t \Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)$$

$$+ (1 - t)\Pi_C(I - \theta_t A)\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)\} x.$$  

We note that $\Pi_C(I - \lambda B_1)$ and $\Pi_C(I - \mu B_2)$ are nonexpansive (by Lemma 2.3.4), $G = \Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)$ is also nonexpansive (by Lemma 2.3.5), and $I - \theta_t A$ is contractive.
with efficient \(1 - \theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right) \in (0, 1)\) (by Proposition 4.2.1 (c)). Hence, for all \(x, y \in C\),

\[
\|W_t x - W_t y\| = \|\{t \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_2) + (1-t) \Pi_C(I - \theta_t A) \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_2)\} x \\
- \{t \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_2) + (1-t) \Pi_C(I - \theta_t A) \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_2)\} y\|
\]

\[
= \|t(G(x) - G(y)) + (1-t)[\Pi_C(I - \theta_t A)G(x) - \Pi_C(I - \theta_t A)G(y)]\|\]

\[
\leq t\|G(x) - G(y)\| + (1-t)\|\Pi_C(I - \theta_t A)G(x) - \Pi_C(I - \theta_t A)G(y)\|
\]

\[
\leq t\|x - y\| + (1-t)\|(I - \lambda B_1)G(x) - (I - \theta_t A)G(y)\|
\]

\[
\leq t\|x - y\| + (1-t)\left(1 - \theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right)\right)\|G(x) - G(y)\|
\]

\[
\leq t\|x - y\| + (1-t)\left(1 - \theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right)\right)\|x - y\|
\]

\[
= \left[1 - (1-t)\theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right)\right]\|x - y\|.
\]

Since \(\theta_t \in (0, 1), \forall t \in (0, 1)\) and \(\delta + \zeta > 1\) with \(\delta, \zeta \in (0, 1)\), we obtain

\[
0 < \theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right) < 1
\]

and so,

\[
0 < 1 - (1-t)\theta_t \left(1 - \sqrt{\frac{1-\delta}{\zeta}}\right) < 1.
\]

This means that the mapping \(W_t\) is a contraction.

### 4.3 Iterative Methods and Convergence Results

In this section, we study of the iterative methods for computing the approximate solutions of GSNVI (4.1.2). We introduce the implicit and explicit algorithms of Mann-type for
solving GSNVI (4.1.2). We show the strong converge theorems for the sequences generated by the proposed algorithms.

The following implicit algorithm of Mann-type for solving GSNVI (4.1.1) is well defined.

**Algorithm 4.3.1.** For each \( t \in (0, 1) \), choose a number \( \theta_t \in (0, 1) \) arbitrarily. The net \( \{x_t\} \) is generated by the following implicit method:

\[
x_t = (t\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)
+ (1 - t)\Pi_C(I - \theta_t A)\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2))x_t, \quad \forall t \in (0, 1),
\]

where \( x_t \) is a unique fixed point of the contraction

\[
W_t = t\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2) + (1 - t)\Pi_C(I - \theta_t A)\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2).
\]

We prove that the sequence generated by Algorithm 4.3.1 converges strongly to a solution of a variational inequality defined on \( \Omega \), the fixed point set of mapping \( G \) defined by (2.3.4).

**Theorem 4.3.1.** The net \( \{x_t\} \) generated by Algorithm 4.3.1 converges in norm, as \( t \to 0^+ \), to a unique solution \( \bar{x} \) of the following variational inequality:

\[
\bar{x} \in \Omega, \quad \langle A\bar{x}, j(\bar{x} - z) \rangle \leq 0, \quad \forall z \in \Omega,
\]

provided \( \lim_{t \to 0^+} \theta_t = 0 \).

**Proof.** Set \( x_t = \Pi_C(I - \mu B_2)x_t \) and \( y_t = \Pi_C(I - \lambda B_1)x_t \) for all \( t \in (0, 1) \). Then, we have

\[
x_t = ty_t + (1 - t)\Pi_C(I - \theta_t A)y_t.
\]

Let \( x^* \in \Omega \), then from Lemma 4.2.1, we have

\[
x^* = \Pi_C[\Pi_C(x^* - \mu B_2x^*) - \lambda B_1\Pi_C(x^* - \mu B_2x^*)].
\]
Set \( y^* = \Pi_C(x^* - \mu B_2x^*) \). Then, \( x^* = \Pi_C(y^* - \lambda B_1y^*) \). From Lemma 2.3.4, we know that \( \Pi_C(I - \lambda B_1) \) and \( \Pi_C(I - \mu B_2) \) are nonexpansive. Hence, we have

\[
\|y_t - x^*\| = \|\Pi_C(I - \lambda B_1)x_t - \Pi_C(I - \lambda B_1)y^*\| \\
\leq \|x_t - y^*\| = \|\Pi_C(I - \mu B_2)x_t - \Pi_C(I - \mu B_2)x^*\| \\
\leq \|x_t - x^*\|.
\]

So, by Proposition 4.2.1 (c), we get

\[
\|x_t - x^*\| \\
= \|ty_t + (1 - t)\Pi_C(I - \theta_t A)y_t - (tx^* + (1 - t)\Pi_C(x^*))\| \\
= \|t(y_t - x^*) + (1 - t)\Pi_C(I - \theta_t A)y_t - \Pi_C(x^*)\| \\
\leq t\|y_t - x^*\| + (1 - t)\|\Pi_C(I - \theta_t A)y_t - \Pi_C(x^*)\| \\
\leq t\|x_t - x^*\| + (1 - t)\|\Pi_C(I - \theta_t A)y_t - \Pi_C(x^*)\| \\
= t\|x_t - x^*\| + (1 - t)\|\Pi_C(I - \theta_t A)y_t - (I - \theta_t A)x^* - \theta_t Ax^*\| \\
\leq t\|x_t - x^*\| + (1 - t)\|\Pi_C(I - \theta_t A)y_t - (I - \theta_t A)x^* - \theta_t Ax^*\| \\
\leq t\|x_t - x^*\| + (1 - t)\left(\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)\right)\|y_t - x^*\| + \theta_t\|Ax^*\|\right) \\
\leq t\|x_t - x^*\| + (1 - t)\left(\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)\right)\|x_t - x^*\| + \theta_t\|Ax^*\|\right) \\
= \left[1 - (1 - t)\theta_t\left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)\right]\|x_t - x^*\| + (1 - t)\theta_t\|Ax^*\|. \\
\]

It follows that

\[
\|x_t - x^*\| \leq \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)^{-1}\|A(x^*)\|.
\]

Therefore, \( \{x_t\} \) is bounded. Hence \( \{y_t\}, \{x_t\}, \{B_1y_t\}, \{B_2x_t\} \) and \( \{Ay_t\} \) are also bounded. We observe that
\[ \| x_t - y_t \| = \| ty_t + (1 - t)\Pi_C(I - \theta_t A)y_t - (ty_t + (1 - t)\Pi_C y_t) \|
\leq \| (I - \theta_t A)y_t - \Pi_C y_t \|
= \theta_t \| A(y_t) \| \to 0 \text{ as } t \to 0^+. \quad (4.3.3) \]

From Lemma 2.3.5, it is known that \( G : C \to C \) is nonexpansive. Thus, we have
\[
\| y_t - \hat{G}(y_t) \| = \| \Pi_C[\Pi_C(x_t - \mu B_2 x_t) - \lambda B_1 \Pi_C(x_t - \mu B_2 x_t)] - G(y_t) \|
\leq \| x_t - y_t \| \to 0 \text{ as } t \to 0^+. \]

Therefore,
\[
\lim_{t \to 0^+} \| x_t - G(x_t) \| = 0. \quad (4.3.4) \]

Next, we show that \( \{ x_t \} \) is relatively norm-compact as \( t \to 0^+ \). Assume that \( \{ t_n \} \subset (0, 1) \) is such that \( t_n \to 0^+ \) as \( n \to \infty \). Put \( x_n := x_{t_n} \), \( y_n := y_{t_n} \) and \( \theta_n := \theta_{t_n} \). It follows from (4.3.4) that
\[
\| x_n - G(x_n) \| \to 0, \quad \text{as} \ n \to \infty. \quad (4.3.5) \]

We can rewrite (4.3.1) as
\[
x_t = ty_t + (1 - t) [\Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t + (I - \theta_t A)y_t].
\]

For any \( x^* \in \Omega \subset C \), by Lemma 2.3.3, we have
\[
\langle x_t - (I - \theta_t A)y_t, j(x_t - x^*) \rangle
= t \langle y_t - (I - \theta_t A)y_t, j(x_t - x^*) \rangle
+ (1 - t) \langle \Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(x_t - x^*) \rangle
\]
\[
\begin{align*}
&= t\theta_t\langle Ay_t, j(x_t - x^*) \rangle \\
&+ (1 - t)\langle \Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(\Pi_C(I - \theta_t A)y_t - x^*) \rangle \\
&+ \langle \Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*) \rangle \\
&\leq t\theta_t\langle Ay_t, j(x_t - x^*) \rangle \\
&+ (1 - t)\langle \Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*) \rangle \\
&\leq t\theta_t\langle Ay_t, j(x_t - x^*) \rangle \\
&+ (1 - t)\|\Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t\|\|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
&\leq t\theta_t\langle Ay_t, j(x_t - x^*) \rangle \\
&+ (1 - t)\|\Pi_C(I - \theta_t A)y_t - (I - \theta_t A)y_t\| + \theta_t\|Ay_t\|\|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
&\leq t\theta_t\|Ay_t\|\|x_t - x^*\| + 2\theta_t\|Ay_t\|\|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
\end{align*}
\]

With this fact, we deduce that

\[
\|x_t - x^*\|^2 \\
= \langle x_t - x^*, j(x_t - x^*) \rangle \\
= \langle x_t - (I - \theta_t A)y_t, j(x_t - x^*) \rangle + \langle (I - \theta_t A)y_t - x^*, j(x_t - x^*) \rangle \\
\leq t\theta_t\|Ay_t\|\|x_t - x^*\| + 2\theta_t\|Ay_t\|\|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
+ \langle (I - \theta_t A)y_t - (I - \theta_t A)x^*, j(x_t - x^*) \rangle - \theta_t\langle Ax^*, j(x_t - x^*) \rangle \\
\leq t\theta_t\|Ay_t\|\|x_t - x^*\| + 2\theta_t\|Ay_t\|\|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
+ \|((I - \theta_t A)y_t - (I - \theta_t A)x^*)\|\|x_t - x^*\| - \theta_t\langle Ax^*, j(x_t - x^*) \rangle \\
\]
\begin{align*}
&\leq t\theta_t \|Ay_t\| \|x_t - x^*\| + 2\theta_t \|Ay_t\| \|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
&\quad + (1 - \theta_t (1 - \sqrt{1 - \frac{1}{\zeta}})) \|y_t - x^*\| \|x_t - x^*\| - \theta_t (Ax^*, j(x_t - x^*)) \\
&\leq t\theta_t \|Ay_t\| \|x_t - x^*\| + 2\theta_t \|Ay_t\| \|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\| \\
&\quad + (1 - \theta_t (1 - \sqrt{1 - \frac{1}{\zeta}})) \|x_t - x^*\|^2 - \theta_t (Ax^*, j(x_t - x^*)). \quad (4.3.6)
\end{align*}

It turns out that

\begin{align*}
\|x_t - x^*\|^2 \\
&\leq \left(1 - \sqrt{1 - \frac{1}{\zeta}}\right)^{-1} \left[(Ax^*, j(x^* - x_t)) + t\|Ay_t\| \|x_t - x^*\| \\
&\quad + 2\|Ay_t\| \|j(x_t - x^*) - j(\Pi_C(I - \theta_t A)y_t - x^*)\|\right], \quad \forall x^* \in \Omega. \quad (4.3.7)
\end{align*}

In particular,

\begin{align*}
\|x_n - x^*\|^2 \\
&\leq \left(1 - \sqrt{1 - \frac{1}{\zeta}}\right)^{-1} \left[(Ax^*, j(x^* - x_n)) + t_n\|Ay_n\| \|x_n - x^*\| \\
&\quad + 2\|Ay_n\| \|j(x_n - x^*) - j(\Pi_C(I - \theta_n A)y_n - x^*)\|\right], \quad \forall x^* \in \Omega. \quad (4.3.8)
\end{align*}

Since \(\{x_n\}\) is bounded, without loss of generality we may assume that \(\{x_n\}\) converges weakly to a point \(\bar{x} \in C\). Noticing (4.3.5) we can use Lemma 2.3.1 to get \(\bar{x} \in \Omega\). Therefore, we can substitute \(\bar{x}\) for \(x^*\) in (4.3.8) to get

\begin{align*}
\|x_n - \bar{x}\|^2 \\
&\leq \left(1 - \sqrt{1 - \frac{1}{\zeta}}\right)^{-1} \left[(Ax, j(\bar{x} - x_n)) + t_n\|Ay_n\| \|x_n - \bar{x}\| \\
&\quad + 2\|Ay_n\| \|j(x_n - \bar{x}) - j(\Pi_C(I - \theta_n A)y_n - \bar{x})\|\right]. \quad (4.3.9)
\end{align*}

Note that

\[\|(x_n - \bar{x}) - (\Pi_C(I - \theta_n A)y_n - \bar{x})\| = \|x_n - \Pi_C(I - \theta_n A)y_n\|\]
\[ t_n \| \Pi_C y_n - \Pi_C (I - \theta_n A) y_n \| \leq t_n \| y_n - (I - \theta_n A) y_n \| = t_n \theta_n \| A y_n \| \to 0 \quad \text{as } n \to \infty. \]

Since \( X \) is uniformly smooth, we get that
\[
\| j(x_n - \bar{x}) - j(\Pi_C (I - \theta_n A) y_n - \bar{x}) \| \to 0 \quad \text{as } n \to \infty.
\]

Consequently, the weak convergence of \( \{x_n\} \) to \( \bar{x} \) together with (4.3.9), actually implies that \( x_n \to \bar{x} \) strongly. This has proved the relative norm compactness of the net \( \{x_t\} \) as \( t \to 0^+ \).

We next show that \( \bar{x} \) solves the variational inequality (4.3.2). From (4.3.1), we have
\[
x_t = t y_t + (1 - t) \| \Pi_C (I - \theta_t A) y_t - (I - \theta_t A) y_t \|
\Rightarrow
x_t = t y_t + (1 - t) \| \Pi_C (I - \theta_t A) y_t - (I - \theta_t A) y_t \|
- \langle (I - \theta_t A)x_t - (I - \theta_t A) y_t, x_t - \theta_t A x_t \rangle
\Rightarrow
A x_t = -\frac{t (x_t - y_t)}{(1 - t) \theta_t} + \frac{1}{\theta_t} \| \Pi_C (I - \theta_t A) y_t - (I - \theta_t A) y_t \|
- \langle (I - \theta_t A)x_t - (I - \theta_t A) y_t, x_t - \theta_t A x_t \rangle.
\]

For any \( z \in \Omega \), we have
\[
\langle A x_t, j(x_t - z) \rangle
= -\frac{t}{(1 - t) \theta_t} \langle x_t - y_t, j(x_t - z) \rangle + \frac{1}{\theta_t} \langle \Pi_C (I - \theta_t A) y_t - (I - \theta_t A) y_t, j(x_t - z) \rangle
\]
\[
- \frac{1}{\theta_t} \langle (I - \theta_t A)x_t - (I - \theta_t A) y_t, j(x_t - z) \rangle
\]
\[
= -\frac{t}{(1 - t) \theta_t} \langle x_t - y_t, j(x_t - z) \rangle + \frac{1}{\theta_t} \langle \Pi_C (I - \theta_t A) y_t \rangle
- \langle (I - \theta_t A)y_t, j(\Pi_C (I - \theta_t A)y_t - z) \rangle
+ \frac{1}{\theta_t} \langle \Pi_C (I - \theta_t A)y_t - (I - \theta_t A)y_t, j(x_t - z) - j(\Pi_C (I - \theta_t A)y_t - z) \rangle
- \frac{1}{\theta_t} \langle (I - \theta_t A)x_t - (I - \theta_t A)y_t, j(x_t - z) \rangle.
\]
\[ \leq - \frac{t}{(1-t)\theta_t} \langle x_t - y_t, j(x_t - z) \rangle + \frac{1}{\theta_t} \langle \Pi_{\mathcal{C}}(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(\Pi_{\mathcal{C}}(I - \theta_t A)y_t - z) \rangle \]
\[ + 2\|Ay_t\|\|j(x_t - z) - j(\Pi_{\mathcal{C}}(I - \theta_t A)y_t - z)\| - \frac{1}{\theta_t} \langle (x_t - y_t), j(x_t - z) \rangle + \langle Ax_t - Ay_t, j(x_t - z) \rangle. \] (4.3.10)

We now prove that \( \langle (x_t - y_t), j(x_t - z) \rangle \geq 0 \). Indeed, we can write \( y_t = G(x_t) \). At the same time, we note that \( z = G(z) \). So,
\[ \langle x_t - y_t, j(x_t - z) \rangle = \langle x_t - G(x_t) - (z - G(z)), j(x_t - z) \rangle. \]

Since \( I - G \) is accretive (as \( G \) is nonexpansive (Lemma 2.3.5)), we can deduce immediately that
\[ \langle x_t - y_t, j(x_t - z) \rangle = \langle x_t - G(x_t) - (z - G(z)), j(x_t - z) \rangle \geq 0. \]

Furthermore, utilizing Lemma 2.3.3 and Proposition 4.2.1 (a), we have
\[ \langle \Pi_{\mathcal{C}}(I - \theta_t A)y_t - (I - \theta_t A)y_t, j(\Pi_{\mathcal{C}}(I - \theta_t A)y_t - z) \rangle \leq 0 \]

and
\[ \|Ax_t - Ay_t\| \leq \left(1 + \frac{1}{\zeta}\right)\|x_t - y_t\|. \]

It follows from (4.3.10) that
\[ \langle Ax_t, j(x_t - z) \rangle \leq 2\|Ay_t\|\|j(x_t - z) - j(\Pi_{\mathcal{C}}(I - \theta_t A)y_t - z)\| + \left(1 + \frac{1}{\zeta}\right)\|x_t - y_t\|\|x_t - z\|. \] (4.3.11)

Since \( A \) is \( \delta \)-strongly accretive, we have
\[ 0 \leq \delta\|x_t - z\|^2 \leq \langle Ax_t - Az, j(x_t - z) \rangle. \]

Therefore,
\[ \langle Az, j(x_t - z) \rangle \leq \langle Ax_t, j(x_t - z) \rangle. \] (4.3.12)
Combining (4.3.11) and (4.3.12), we get
\[
\langle Az, j(x_t - z) \rangle \leq 2\|Ay_t\|\|j(x_t - z) - j(\Pi_C(I - \theta_t A)y_t - z)\| + (1 + \frac{1}{\xi})\|x_t - y_t\|\|x_t - z\|.
\] (4.3.13)

Replace \( t \) in (4.3.13) by \( t_n \), and notice that as \( n \to \infty \), \( x_{t_n} - y_{t_n} \to 0 \) and \( j(x_{t_n} - z) - j(\Pi_C(I - \theta_{t_n} A)y_{t_n} - z) \to 0 \), yields
\[
\langle Az, j(\bar{x} - z) \rangle \leq 0, \quad \forall z \in \Omega,
\]
which is equivalent to the Minty type variational inequality (see Lemma 4.1.1)
\[
\langle A\bar{x}, j(\bar{x} - z) \rangle \leq 0, \quad \forall z \in \Omega. \tag{4.3.14}
\]

That is, \( \bar{x} \in \Omega \) is a solution of (4.3.2).

Now we show that the solution set of (4.3.2) is a singleton. As a matter of fact, we assume that \( \bar{x} \in \Omega \) is also a solution of (4.3.2). Then, we have
\[
\langle A(\bar{x}), j(\bar{x} - \bar{x}) \rangle \leq 0.
\]

From (4.3.14), we have
\[
\langle A\bar{x}, j(\bar{x} - \bar{x}) \rangle \leq 0.
\]

So, by \( \delta \)-strong accretiveness of \( A \), we have
\[
\langle A\bar{x}, j(\bar{x} - \bar{x}) \rangle + \langle A(\bar{x}), j(\bar{x} - \bar{x}) \rangle \leq 0
\]
\[
\Rightarrow \langle A\bar{x} - A\bar{x}, j(\bar{x} - \bar{x}) \rangle \leq 0
\]
\[
\Rightarrow \delta\|\bar{x} - \bar{x}\|^2 \leq 0.
\]

Therefore, \( \bar{x} = \bar{x} \). In summary, we have shown that each cluster point of \( \{x_t\} \) (as \( t \to 0 \)) equals to \( \bar{x} \). Therefore, \( x_t \to \bar{x} \) as \( t \to 0 \). \( \square \)

We now introduce an explicit method which is the discretization of the implicit method (4.3.1).
Algorithm 4.3.2. Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_1, B_2, A : C \to X$ be three nonlinear mappings. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

\[
x_{n+1} = \beta_n x_n + \gamma_n \Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)x_n \\
+ (1 - \beta_n - \gamma_n) \Pi_C(I - \alpha_n A)\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)x_n,
\]

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ such that $\beta_n + \gamma_n \leq 1$, $\forall n \geq 0$, and $\lambda$, $\mu$ are two real numbers.

In particular, if $B_1 \equiv B_2$, then (4.3.15) reduces to the following iterative scheme:

\[
x_{n+1} = \beta_n x_n + \gamma_n \Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_2)x_n \\
+ (1 - \beta_n - \gamma_n) \Pi_C(I - \alpha_n A)\Pi_C(I - \lambda B_1)\Pi_C(I - \mu B_1)x_n.
\]

We prove that the sequences generated by the Algorithm 4.3.2 converge strongly to a solution of a variational inequality defined on $\Omega$ the fixed point set of mapping $G$ defined by (2.3.4).

Theorem 4.3.2. Let $C$ be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space $X$ and let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mappings $B_1, B_2 : C \to X$ be $\alpha$-inverse-strongly accretive and $\beta$-inverse-strongly accretive, respectively. Let $A : C \to X$ be $\delta$-strongly accretive and $\zeta$-strictly pseudo-contractive with $\delta + \zeta > 1$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (4.3.15). Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) = 0$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} (\beta_n + \gamma_n) < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$ which solves the variational inequality (4.3.2).

Proof. Set $z_n = \Pi_C(I - \mu B_2)x_n$ and $y_n = \Pi_C(I - \lambda B_1)z_n$ for all $n \geq 0$. Then

$$x_{n+1} = \beta_n x_n + \gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n A)y_n, \quad \forall n \geq 0.$$  

We take a point $x^* \in \Omega$ arbitrarily. From Lemma 2.3.4, we know that $\Pi_C(I - \lambda B_1)$ and $\Pi_C(I - \mu B_2)$ are nonexpansive. Hence, we have

$$\|y_n - x^*\| = \|\Pi_C(I - \lambda B_1)z_n - \Pi_C(I - \lambda B_1)y^*\|$$
$$\leq \|z_n - y^*\| = \|\Pi_C(I - \mu B_2)x_n - \Pi_C(I - \mu B_2)x^*\|$$
$$\leq \|x_n - x^*\|.$$  

So, by Proposition 4.2.1 (c), we get

$$\|x_{n+1} - x^*\|$$
$$= \|\beta_n x_n + \gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_n A)y_n - x^*\|$$
$$\leq \beta_n \|x_n - x^*\| + \gamma_n \|y_n - x^*\| + (1 - \beta_n - \gamma_n)\|\Pi_C(I - \alpha_n A)y_n - \Pi_Cx^*\|$$
$$\leq \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| + (1 - \beta_n - \gamma_n)\|I - \alpha_n A\|y_n - (I - \alpha_n A)x^*\|$$
$$\leq (\beta_n + \gamma_n)\|x_n - x^*\| + (1 - \beta_n - \gamma_n)\|I - \alpha_n A\|y_n - (I - \alpha_n A)x^*\|$$
$$+ \alpha_n (1 - \beta_n - \gamma_n)\|Ax^*\|$$
$$\leq (\beta_n + \gamma_n)\|x_n - x^*\|$$
$$+ (1 - \beta_n - \gamma_n)\left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)\right)\|y_n - x^*\|$$
$$+ \alpha_n (1 - \beta_n - \gamma_n)\|Ax^*\|.$$
\[
\leq \left[ 1 - \alpha_n (1 - \beta_n - \gamma_n) \left( 1 - \sqrt{1 - \frac{1 - \delta}{\zeta}} \right) \right] \| x_n - x^* \|
+ \alpha_n (1 - \beta_n - \gamma_n) \left( 1 - \sqrt{1 - \frac{1 - \delta}{\zeta}} \right) \frac{\| Ax^* \|}{(1 - \sqrt{1 - \frac{1 - \delta}{\zeta}})}.
\]

By induction, we conclude that
\[
\| x_{n+1} - x^* \| \leq \max \left\{ \| x_0 - x^* \|, \left( 1 - \sqrt{1 - \frac{1 - \delta}{\zeta}} \right)^{-1} \frac{\| Ax^* \|}{(1 - \sqrt{1 - \frac{1 - \delta}{\zeta}})} \right\}.
\]

Therefore, \( \{ x_n \} \) is bounded. Hence \( \{ y_n \} \), \( \{ z_n \} \), \( \{ B_1 y_n \} \) and \( \{ B_2 x_n \} \) are also bounded. We observe that
\[
\| y_{n+1} - y_n \| = \| \Pi_C (I - \lambda B_1) z_{n+1} - \Pi_C (I - \lambda B_1) z_n \|
\leq \| z_{n+1} - z_n \|
= \| \Pi_C (I - \mu B_2) x_{n+1} - \Pi_C (I - \mu B_2) x_n \|
\leq \| x_{n+1} - x_n \|.
\]

Set \( x_{n+1} = \beta_n x_n + (1 - \beta_n) v_n \) for all \( n \geq 0 \).

Then, \( u_n = \gamma_n y_n + (1 - \beta_n - \gamma_n) \Pi_C (I - \alpha_n A) y_n \frac{1 - \beta_n}{1 - \beta_n} \).

Note that
\[
\| \Pi_C (I - \alpha_{n+1} A) y_{n+1} - \Pi_C (I - \alpha_n A) y_n \| \leq \| (I - \alpha_{n+1} A) y_{n+1} - (I - \alpha_n A) y_n \|
= \| y_{n+1} - y_n - \alpha_{n+1} A (y_{n+1}) + \alpha_n A (y_n) \|
\leq \| y_{n+1} - y_n \| + \alpha_{n+1} \| A (y_{n+1}) \| + \alpha_n \| A (y_n) \|
\leq \| x_{n+1} - x_n \| + \alpha_{n+1} \| A (y_{n+1}) \| + \alpha_n \| A (y_n) \|.
\]

Hence,
\[
\| u_{n+1} - u_n \|
\]
\[
\begin{align*}
&= \left\| \frac{\gamma_{n+1}y_{n+1} + (1 - \beta_{n+1} - \gamma_{n+1})\Pi_C(I - \alpha_{n+1}A)y_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n y_n + (1 - \beta_n - \gamma_n)\Pi_C(I - \alpha_nA)y_n}{1 - \beta_n} \right\| \\
&\leq \left\| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} y_{n+1} - \frac{\gamma_n}{1 - \beta_n} y_n \right\| \\
&\quad + \left\| \frac{(1 - \beta_{n+1} - \gamma_{n+1})}{1 - \beta_{n+1}} \Pi_C(I - \alpha_{n+1}A)y_{n+1} - \frac{(1 - \beta_n - \gamma_n)}{1 - \beta_n} \Pi_C(I - \alpha_n A)y_n \right\| \\
&\leq \left\| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right\| \| y_{n+1} \| + \gamma_n \left\| y_{n+1} - y_n \right\| \\
&\quad + \left\| \frac{(1 - \beta_{n+1} - \gamma_{n+1})}{1 - \beta_{n+1}} \right\| \| \Pi_C(I - \alpha_{n+1}A)y_{n+1} \| \\
&\quad + \left\| \frac{(1 - \beta_n - \gamma_n)}{1 - \beta_n} \right\| \| \Pi_C(I - \alpha_n A)y_n \| \\
&\leq \left\| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right\| \left( \| y_{n+1} \| + \| \Pi_C(I - \alpha_{n+1}A)y_{n+1} \| \right) \\
&\quad + \gamma_n \left\| y_{n+1} - x_n \right\| \\
&\quad + \left\| \frac{(1 - \beta_{n+1} - \gamma_{n+1})}{1 - \beta_{n+1}} \right\| \left( \| y_{n+1} \| + \| \Pi_C(I - \alpha_{n+1}A)y_{n+1} \| \right) \\
&\quad + \| x_{n+1} - x_n \| + \alpha_{n+1}\| A(y_{n+1}) \| + \alpha_n\| A(y_n) \| \\
&\leq \left\| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right\| \left( \| y_{n+1} \| + \| \Pi_C(I - \alpha_{n+1}A)y_{n+1} \| \right) \\
&\quad + \| x_{n+1} - x_n \| + \alpha_{n+1}\| A(y_{n+1}) \| + \alpha_n\| A(y_n) \|,
\end{align*}
\]

Since, \{y_n\} and \{A(y_n)\} are bounded, we have that \{\|y_n\| + \|\Pi_C(I - \alpha_n A)y_n\|\} is bounded.

It follows from conditions (i) and (ii) that

\[
\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Hence, by Lemma 2.1.7, we get \(\|y_n - x_n\| \to 0\). Consequently,

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|y_n - x_n\| = 0.
\]
We also note that
\[
\|u_n - y_n\| \leq \frac{\gamma_n y_n + (1 - \beta_n - \gamma_n) \Pi_C(I - \alpha_n A) y_n - y_n}{1 - \beta_n}
\]
\[
= \frac{\|\gamma_n y_n + (1 - \beta_n - \gamma_n) \Pi_C(I - \alpha_n A) y_n - (1 - \beta_n) y_n\|}{1 - \beta_n}
\]
\[
\leq \frac{1 - \beta_n - \gamma_n}{1 - \beta_n} \|\Pi_C(I - \alpha_n A) y_n - y_n\| + \|\Pi_C(I - \alpha_n A) y_n - \Pi_C y_n\|
\]
\[
\leq \alpha_n \|A y_n\| \to 0 \quad \text{as } n \to \infty.
\]

It follows that
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0.
\]

From Lemma 2.3.5, we know that $G : C \to C$ is nonexpansive. Thus, we have
\[
\|y_n - G(y_n)\| = \|\Pi_C[\Pi_C(x_n - \mu B_2 x_n) - \lambda B_1 \Pi_C(x_n - \mu B_2 x_n)] - G(y_n)\|
\]
\[
= \|\Pi_C(x_n) - G(y_n)\|
\]
\[
\leq \|x_n - y_n\| \to 0 \quad \text{as } n \to \infty.
\]

Therefore, $\lim_{n \to \infty} \|x_n - G(x_n)\| = 0$.

Set $u_n = \Pi_C(I - \alpha_n A) y_n$ for all $n \geq 0$. We note that
\[
\|u_n - G(u_n)\| \leq \|u_n - x_n\| + \|x_n - G(x_n)\| + \|G(x_n) - G(u_n)\|
\]
\[
\leq 2\|u_n - x_n\| + \|x_n - G(x_n)\|
\]
\[
= 2\|\Pi_C(I - \alpha_n A) y_n - \Pi_C x_n\| + \|x_n - G(x_n)\| \quad (4.3.16)
\]
\[
\leq 2(\|y_n - x_n\| + \alpha_n \|A y_n\|) + \|x_n - G(x_n)\|
\]
\[
\to 0 \quad \text{as } n \to \infty.
\]

Next, we show that
\[
\limsup_{n \to \infty} (A \bar{x}, j(\bar{x} - u_n)) \leq 0,
\]
where $\bar{x} \in \Omega$ is the unique solution of the variational inequality (4.3.2).
Indeed, we first take a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that

\[
\limsup_{n \to \infty} \langle A \bar{x}, j(\bar{x} - u_n) \rangle = \lim_{k \to \infty} \langle A \bar{x}, j(\bar{x} - u_{n_k}) \rangle.
\]

We may also assume that \( u_{n_k} \rightharpoonup z \). Note that \( z \in \Omega \) by virtue of Lemma 2.3.1 and (4.3.16).

It follows from the variational inequality (4.3.2) that

\[
\limsup_{n \to \infty} \langle A \bar{x}, j(\bar{x} - u_n) \rangle = \lim_{k \to \infty} \langle A \bar{x}, j(\bar{x} - u_{n_k}) \rangle = \langle A \bar{x}, j(\bar{x} - z) \rangle \leq 0.
\]

Since \( u_n = \Pi_C(I - \alpha_n A) y_n \), according to Lemma 2.3.3, we have

\[
\langle (I - \alpha_n A) y_n - \Pi_C(I - \alpha_n A) y_n, j(\bar{x} - u_n) \rangle \leq 0. \tag{4.3.17}
\]

From (4.3.17), we have

\[
\begin{align*}
\|u_n - \bar{x}\|^2 &= \langle \Pi_C(I - \alpha_n A) y_n - \bar{x}, j(u_n - \bar{x}) \rangle \\
&= \langle \Pi_C(I - \alpha_n A) y_n - (I - \alpha_n A) y_n, j(u_n - \bar{x}) \rangle + \langle (I - \alpha_n A) y_n - \bar{x}, j(u_n - \bar{x}) \rangle \\
&\leq \langle (I - \alpha_n A) y_n - \bar{x}, j(u_n - \bar{x}) \rangle \\
&= \langle (I - \alpha_n A) y_n - (I - \alpha_n A) \bar{x}, j(u_n - \bar{x}) \rangle + \alpha_n \langle A \bar{x}, j(\bar{x} - u_n) \rangle \\
&\leq \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \right) \|y_n - \bar{x}\|\|u_n - \bar{x}\| + \alpha_n \langle A \bar{x}, j(\bar{x} - u_n) \rangle \\
&\leq \frac{1}{2} \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \right)^2 \|y_n - \bar{x}\|^2 + \frac{1}{2} \|u_n - \bar{x}\|^2 + \alpha_n \langle A \bar{x}, j(\bar{x} - u_n) \rangle.
\end{align*}
\]

It follows that

\[
\begin{align*}
\|u_n - \bar{x}\|^2 &\leq \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \right) \|y_n - \bar{x}\|^2 + 2\alpha_n \langle A \bar{x}, j(\bar{x} - u_n) \rangle \\
&\leq \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \right) \right) \|x_n - \bar{x}\|^2 + 2\alpha_n \langle A \bar{x}, j(\bar{x} - u_n) \rangle. \tag{4.3.18}
\end{align*}
\]
Finally, we prove \( x_n \to \bar{x} \). As a matter of fact, from (4.3.1) and (4.3.18), we have

\[
\begin{align*}
\|x_{n+1} - \bar{x}\|^2 &
\leq \beta_n \|x_n - \bar{x}\|^2 + \gamma_n \|y_n - \bar{x}\|^2 + (1 - \beta_n - \gamma_n) \|u_n - \bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + \gamma_n \|x_n - \bar{x}\|^2 \\
&\quad + (1 - \beta_n - \gamma_n) \left[ \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right)\right) \|x_n - \bar{x}\|^2 + 2 \alpha_n \langle A\bar{x}, j(\bar{x} - u_n) \rangle \right] \\
&= \left[ 1 - \alpha_n (1 - \beta_n - \gamma_n) \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right) \right] \|x_n - \bar{x}\|^2 \\
&\quad + \alpha_n (1 - \beta_n - \gamma_n) \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right) \left(\frac{2}{1 - \sqrt{\frac{1 - \delta}{\zeta}}} \right) \langle A\bar{x}, j(\bar{x} - u_n) \rangle.
\end{align*}
\]  

(4.3.19)

Since \( \sum_{n=0}^{\infty} \alpha_n = \infty \), \( \limsup_{n \to \infty} (\beta_n + \gamma_n) < 1 \) and \( 1 - \sqrt{\frac{1 - \delta}{\zeta}} \in (0, 1) \), we get

\[
\sum_{n=0}^{\infty} \alpha_n (1 - \beta_n - \gamma_n) \left(1 - \sqrt{\frac{1 - \delta}{\zeta}}\right) = \infty.
\]

Taking into account \( \limsup_{n \to \infty} \langle A\bar{x}, j(\bar{x} - u_n) \rangle \leq 0 \), we can apply Lemma 2.1.1 to the relation (4.3.19) and conclude that \( x_n \to \bar{x} \). \( \square \)

We use \( \Phi \) to denote the solution set of the variational inequality (4.1.2). We can derive easily the following corollaries.

**Corollary 4.3.1.** Let \( \theta_t \in (0, 1), \forall t \in (0, 1) \) such that \( \lim_{t \to 0^+} \theta_t = 0 \). The net \( \{x_t\} \) generated by the implicit method

\[
x_t = \{t \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_1) + (1 - t) \Pi_C(I - \theta_t A) \Pi_C(I - \lambda B_1) \Pi_C(I - \mu B_1)\}x_t,
\]

(4.3.20)

for all \( t \in (0, 1) \), converges in norm, as \( t \to 0^+ \), to \( \bar{x} \in \Phi \) which is the unique solution of the following variational inequality:

\( \bar{x} \in \Phi : \langle A\bar{x}, j(\bar{x} - z) \rangle \leq 0, \quad \forall z \in \Phi. \)
Corollary 4.3.2. Let $C$ be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space $X$ and let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_1 : C \to X$ be $\alpha$-inverse-strongly accretive. Let $A : C \to X$ be $\delta$-strongly accretive and $\zeta$-strictly pseudo-contractive with $\delta + \zeta > 1$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (4.3.16). Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) = 0$;

(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} (\beta_n + \gamma_n) < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Phi$, which solves the following variational inequality:

$$\bar{x} \in \Phi : \langle A\bar{x}, j(\bar{x} - z) \rangle \leq 0, \quad \forall z \in \Phi.$$