Chapter 1
Preliminaries

1.1 Introduction

This chapter contains basic definitions and fundamental results in near rings theory which we shall need for the development of the subject in the subsequent chapters of the present thesis. The material for the present chapter has been collected mostly from the standard books like Clay[74], Meldrum[130] and Pilz[143].

1.2 Some near ring theoretic concepts

This section is aimed to collect some important terminology in near ring theory.

Definition 1.2.1 (Near ring) A left near ring $N$ is a triple $(N, +, \star)$ with two binary operation $+$ and $\star$ such that

(i) $(N, +)$ is a group (not necessarily abelian).

(ii) $(N, \star)$ is a semigroup.

(iii) $a \star (b + c) = a \star b + a \star c$, for all $a, b, c \in N$.

Analogously, if instead of (iii), we have the right distributive law

(iii)' $(a + b) \star c = a \star b + a \star c$, for all $a, b, c \in N$

holds, then $N$ is said to be a right near ring.

As in both the cases, the theory of near rings runs completely parallel, we may consider left near rings throughout and for simplicity call them as near rings.
Example 1.2.1  (i) The most natural example of a left near ring is the set of all identity preserving mappings acting from left of an additive group $G$ (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication. If these mapping act from right on $G$, then we get a right near ring.

(ii) $N = \{0, a\}$ with addition $+$ and multiplication $\ast$ table defined as follows:

$$
\begin{array}{c|cc}
+ & 0 & a \\
0 & 0 & a \\
a & a & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
\ast & 0 & a \\
0 & 0 & a \\
a & 0 & a \\
\end{array}
$$

It is easily checked that $(N, +, \ast)$ is a left near ring.

(iii) Let $(N, +)$ be any group. Define multiplication on $N$ by $a \cdot b = b$ for all $a, b \in N$. Then $(N, +, \cdot)$ is a left near ring and it is known as a constant near ring.

(iv) For more examples one may consult [73].

Definition 1.2.2 (Subnear ring) A nonvoid subset $S$ of a near ring $(N, +, \ast)$ is said to be a subnear ring of $N$ if $(S, +)$ is a subgroup of $(N, +)$ and $(S, \ast)$ is a subsemigroup of $(N, \ast)$.

Remark 1.2.1 A nonvoid set $S$ of a near ring $N$ is a subnear ring of $N$ if and only if for $s_1, s_2 \in S$, $s_1 - s_2 \in S$ and $s_1s_2 \in S$. 
Definition 1.2.3 (Characteristic of a near ring) The least positive integer \( n \) (if exists) such that \( nx = 0 \) for all \( x \in N \) is called the characteristic of the near ring \( N \) which is generally expressed as \( \text{char} N = n \). If no such positive integer exists, then \( N \) is said to have characteristic zero.

Definition 1.2.4 (Torsion free element) An element \( x \) in a near ring \( N \) is said to be \( n \)-torsion free if \( nx = 0 \), implies that \( x = 0 \). If \( nx = 0 \) implies that \( x = 0 \), for every \( x \in N \), then we say that \( N \) is \( n \)-torsion free.

Definition 1.2.5 (Nilpotent element) An element \( x \) of a near ring \( N \) is said to be nilpotent if there exists a positive integer \( n \) such that \( x^n = 0 \).

Definition 1.2.6 (Distributive element) An element \( x \) of a near ring \( N \) is called distributive if \( (y + z)x = yx + zx \), for all elements \( x, y \in N \).

Remark 1.2.2 In a near ring, \( 0 \) and the identity element \( 1 \) are distributive elements.

Definition 1.2.7 (Distributive near ring) A near ring \( N \) is said to be distributive if all of its elements are distributive.

Remark 1.2.3 In any near ring \( N \)
(i) \( x0 = 0 \), for all \( x \in N \), but not necessarily \( 0x = 0 \). However, if \( N \) is distributive, then \( 0x = 0 \).

(ii) \( x(-y) = -xy \), for all \( x, y \in N \), but not necessarily \( (-x)y = -xy \). However, if \( N \) is distributive, then \( (-x)y = -xy \).
Example 1.2.2 Let \((G,+)\) be a non-abelian group and \((R, +, \cdot)\) be a ring. Let \(N = G \oplus R\). Then \((N, +)\) is a non-abelian group. Define multiplication \(*\) in \(N\) as follows:

\[(g, r) \ast (g', r') = (0, rr').\]

Then \((N, +, \ast)\) is a distributive near ring.

Example 1.2.3 Let \(N = \{0, a, b, c, x, y\}\) with addition and multiplication defined as follows:

\[
\begin{array}{c|cccccc}
+ & 0 & a & b & c & x & y \\
0 & 0 & a & b & c & x & y \\
a & a & 0 & y & x & c & b \\
b & b & x & 0 & y & a & c \\
c & c & y & x & 0 & b & a \\
x & x & b & c & a & y & 0 \\
y & y & c & a & b & 0 & x \\
\end{array}
\quad
\begin{array}{c|cccccc}
\ast & 0 & a & b & c & x & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a & 0 & 0 \\
b & 0 & a & a & a & 0 & 0 \\
c & 0 & a & a & a & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then \((N, +, \ast)\) is a distributive near ring.

Definition 1.2.8 (Distributively generated near ring) A near ring \(N\) is said to be a distributively generated near ring \((d - g)\) if it contains a multiplicative subsemigroup of distributive elements which generates the additive group \((N, +)\) of \(N\).

Example 1.2.4 The near ring generated additively by all endomorphisms of a group \((G, +)\) (not necessarily abelian), is a distributively generated near ring.
Remark 1.2.4 Each element of a distributively generated near ring can be expressed as a finite sum of distributive elements.

Remark 1.2.5 Every distributive near ring is a distributively generated near ring but not conversely.

Example 1.2.5 Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined as below:

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It is easy to check that $N$ is a near ring but $N$ is not a distributive near ring. Consider the set $S = \{0, a, c\}$. Then it is easy to check that $(S, \ast)$ is a subsemigroup of $(N, \ast)$ of distributive elements which generates $(N, +)$ with respect to $+$ only and $N$ is a distributively generated near ring.

Definition 1.2.9 (Division near ring or Near field) A division near ring or a near field is a near ring in which the nonzero elements form a group under multiplication.

Remark 1.2.6 Every division ring is a near field but converse need not be true in general.

Example 1.2.6 The near ring in Example 1.2.1 (ii) is a near field but not a
division ring.

**Definition 1.2.10** (Zero-symmetric near ring) A near ring $N$ is called zero-symmetric if $0x = 0$, for all $x \in N$ (Recall that left-distributivity yields $x0 = 0$).

**Example 1.2.7** The near ring in Example 1.2.1 (i) is a zero-symmetric near ring.

**Example 1.2.8** Let $N = \{0, 1, 2, 3\}$ with addition and multiplication tables defined as below:

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It can be verified that $(N, +, \times)$ is a zero-symmetric near ring.

**Remark 1.2.7** A $d-g$ near ring is always zero-symmetric.

**Definition 1.2.11** (Zero-commutative near ring) A near ring $N$ is called zero-commutative if $xy = 0$ implies $yx = 0$, for all $x, y \in N$.

**Example 1.2.9** Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined as below:
\[
\begin{array}{c|cccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]
\[
\begin{array}{c|cccc}
\times & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
b & b & 0 & 0 & 0 \\
c & c & 0 & c & 0 \\
\end{array}
\]

Then \((N, +, \times)\) is a zero-commutative near ring.

**Definition 1.2.12 (Additive commutator and Multiplicative commutator)** For any pair of elements \(x, y\) in a near ring \(N\), the additive commutator denoted by \((x, y) = x + y - x - y\) and multiplicative commutator denoted by \([x, y] = xy - yx\).

**Definition 1.2.13 (Multiplicative centre)** Multiplicative centre of a near ring \(N\) is the set of all those elements of \(N\) which commute with every elements of \(N\) under multiplication and is denoted by \(Z(N)\).

**Definition 1.2.14 (Additive centre)** The additive centre of a near ring \(N\) is the set of all those elements of \(N\) which commute with every element of \(N\) under addition is denoted by \(\xi(N)\).

**Definition 1.2.15 (Centralizer)** Let \(N\) be a near ring. For all \(x \in N\), \(C(x) = \{a \in N \mid ax = xa\}\) denotes the centralizer of \(x\) in \(N\).

**Definition 1.2.16 (Near ring homomorphism)** Let \((N_1, \oplus, \cdot)\) and \((N_2, +, \cdot)\) be two near rings. Then a mapping \(f : N_1 \rightarrow N_2\) is called a near ring homomorphism if

\((i)\ f(r_1 \oplus r_2) = f(r_1) + f(r_2)\)
(ii) \( f(r_1 \ast r_2) = f(r_1) \cdot f(r_2) \)
for all \( r_1, r_2 \in N_1 \).

**Remark 1.2.8** Image of a near ring under a near ring homomorphism is again a near ring.

**Definition 1.2.17 (Near ring antihomomorphism)** Let \((N_1, \oplus, \ast)\) and \((N_2, +, \cdot)\) be two near rings. Then a mapping \( f : N_1 \rightarrow N_2 \) is called a near ring antihomomorphism if

(i) \( f(s_1 \oplus s_2) = f(s_1) + f(s_2) \)

(ii) \( f(s_1 \ast s_2) = f(s_2) \cdot f(s_1) \)

for all \( s_1, s_2 \in N_1 \).

**Definition 1.2.18 (Ideal)** An ideal of a near ring \( N \) is defined to be a normal subgroup \( I \) of \((N, +)\) such that

(i) \( NI \subseteq I \).

(ii) \( (x + i)y = xy \in I \), for all \( x, y \in N \) and \( i \in I \).

Normal subgroups of \((N, +)\) satisfying (i) are called the left ideals and satisfying (ii) are called right ideals.

In case of a \( d - g \) near ring, the condition (ii) above may be replaced by

(ii)' \( IN \subseteq I \).

**Remark 1.2.9** Ideals may also be defined as the kernels of a near ring homomorphism.
Example 1.2.10 Consider $N = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$, the near ring of $2 \times 2$ upper triangular matrices over a near field $F$. Then $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in F \right\}$, $B = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in F \right\}$ are left ideals of $N$ and $C = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in F \right\}$ is a right ideal of $N$.

Example 1.2.11 Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined as below:

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It is easy to verify that $A = \{0, a\}$ and $B = \{0, b\}$ are ideals of $N$.

Definition 1.2.19 (Semigroup Ideal) A nonempty subset $U$ of a near ring $N$ is said to be right (resp. left) semigroup ideal of $N$ if $UN \subseteq U$ (resp. $NU \subseteq U$) and $U$ is said to be a semigroup ideal if it is both a right semigroup ideal and a left semigroup ideal of $N$.

Example 1.2.12 Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined as below:
If we take $A = \{0, a\}$, $B = \{0, a, b\}$ and $C = \{0, a, c\}$, then $B, C$ are semigroup right ideals of $N$ and $A$ is a semigroup ideal of $N$.

**Definition 1.2.20 (Nilpotent Ideal)** A right (left, two sided) ideal $I$ of a near ring $N$ is said to be nilpotent if there exists a positive integer $n > 1$ such that $I^n = \{0\}$.

**Definition 1.2.21 (Prime Ideal)** An ideal $P$ of a near ring $N$ is said to be prime if for any ideals $A, B$ in $N$, whenever $AB \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.

**Example 1.2.13** In the near ring $N$ of Example 1.2.1(iii) each normal subgroup of $(N, +)$ is a prime ideal of $N$.

**Definition 1.2.22 (Completely Prime Ideal)** An ideal $P$ in a near ring $N$ is said to be completely prime if for any $a, b \in N$, $ab \in P$, implies that $a \in P$ or $b \in P$.

**Definition 1.2.23 (Semiprime Ideal)** An ideal $P$ in a near ring $N$ is said to be semiprime if for any ideal $A$ in $N$, $A^2 \subseteq P$ implies that $A \subseteq P$.

**Remark 1.2.10** (i) A prime ideal is necessarily semiprime but converse need
not be true in general.

(ii) Intersection of prime (semiprime) ideals is semiprime.

**Definition 1.2.24 (3-Prime near ring)** A near ring $N$ is said to be 3-prime if zero ideal $\{0\}$ is a prime ideal in $N$.

**Remark 1.2.11** Equivalently a near ring $N$ is 3-prime if and only if for any $a, b \in N$, $aNb = \{0\}$, implies that either $a = 0$ or $b = 0$.

**Example 1.2.14** Let $C = \{a + ib \mid a, b \in R\}$ be the set of all complex numbers. Addition is the usual addition of complex numbers. Then $(C, +)$ is a group. Define multiplication $\ast$ on $C$ by $a \ast b = |a|b$. Then $(C, +, \ast)$ is a 3-prime near ring.

**Example 1.2.15** The near ring in Example 1.2.1(iii) is a 3-prime near ring.

**Definition 1.2.25 (3-Semiprime near ring)** A near ring $N$ is said to be 3-semiprime if zero ideal $\{0\}$ is a semiprime ideal in $N$.

**Remark 1.2.12** Equivalently a near ring $N$ is 3-semiprime if and only if for $a \in N$, $aNa = \{0\}$ implies that $a = 0$.

**Remark 1.2.13** A near ring $N$ is a 3-semiprime if and only if it has no nonzero nilpotent ideals.

**Example 1.2.16** Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a near ring under addition modulo 8 and multiplication defined as below:
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Then \((N, \Theta, \ast)\) is a 3-semiprime near ring.

### 1.3 Some key results

In this section we state some well-known results which may be frequently referred in the subsequent text. For their proofs, the references are mentioned against respective results for those who develop interest in them.

**Theorem 1.3.1 (Neumann [136])** The additive group of a division near ring is abelian.

**Theorem 1.3.2 (Frohlic [82])** A \(d-g\) near ring \(N\) is distributive if and only if \(N^2\) is additively commutative.

**Theorem 1.3.3 (Frohlic [82])** A \(d-g\) near ring \(N\) with unity 1 is a ring if \((N, \ast)\) is abelian or if \(N\) is distributive.

**Theorem 1.3.4 (Bell [32])** If a near ring \(N\) is zero-commutative, then the following hold:
(i) \(ab = 0\), implies \(axb = 0\) for all \(x \in N\).

(ii) The annihilator of any nonempty subset of \(N\) is an ideal.

(iii) The set \(N\) of all nilpotent elements is an ideal if and only if it is a subgroup of \((N, +)\).

**Theorem 1.3.5 (Bell [31])** Let \(N\) be a zero-commutative near ring having no nonzero nilpotent elements. Then \(N\) contains a family of completely prime ideals with trivial intersection.

**Theorem 1.3.6 (Bell [31])** Let \(N\) be a zero-symmetric near ring having no nonzero nilpotent elements. Then

(i) every distributive idempotent is central.

(ii) for every idempotent \(e\) and every element \(x \in N\), \(ex^2 = (ex)^2\).

(iii) if \(N\) has a multiplicative identity element, then all idempotents are central.