Chapter 5

Derivations as homomorphisms and antihomomorphisms

5.1 Introduction

Let $S$ be a nonempty subset of a ring $R$ and $d$ be a derivation of $R$. If $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$) holds for all $x, y \in S$, then we say that $d$ acts as a homomorphism (resp. as an antihomomorphism) on $S$. In a paper [41], Bell and Kappe initiated the study of derivations which act as homomorphisms or antihomomorphisms on certain well behaved subsets of the ring. We establish parallel results in case of near rings.

In section 5.2, we study generalized derivation of a 3-prime near ring $N$ which acts as a homomorphism or an antihomomorphism on a semigroup ideal of $N$.

Finally, we study semiderivation of a 3-prime near ring $N$ acting as a homomorphism or antihomomorphism on a semigroup ideal of $N$ and prove that either semiderivation is zero or identity map or $N$ is a commutative ring.

5.2 Generalized derivations acting as homomorphisms or antihomomorphisms

In [41], Bell and Kappe initiated the study of derivations which act as homomorphisms or antihomomorphisms. In fact they proved that if $R$ is a semiprime
ring and \( d \) is a derivation on \( R \) which is either an endomorphism or an antiendomorphism on \( R \), then \( d = 0 \). Of course, derivations which are not endomorphisms or antiendomorphisms on \( R \) may behave as such on certain subsets of \( R \); for example; any derivation \( d \) behaves as the zero endomorphism on the subring \( C \) consisting of all constants (i.e., the elements \( x \) for which \( d(x) = 0 \)). In fact in a semiprime ring \( R \), \( d \) may behave as an endomorphism on a proper ideal of \( R \). However as noted in [41], the behaviour of \( d \) is somewhat restricted in the case of a prime ring. Recently Asma et.al. in [13] considered a \((\theta,\phi)\)-derivation \( d \) acting as a homomorphism or an antihomomorphism on a nonzero Lie ideal of a prime ring and concluded that \( d = 0 \). In this section we establish similar results in case of a semigroup ideal of a 3-prime near ring admitting a generalized derivation.

**Theorem 5.2.1** Let \( N \) be a 3-prime near ring and \( U \) be a nonzero semigroup ideal of \( N \). Let \( F \) be a nonzero generalized derivation on \( N \) with associated derivation \( d \). If \( F \) acts as a homomorphism on \( U \), then \( F \) is the identity map on \( N \) and \( d = 0 \).

**Proof.** By the hypothesis

\[
F(xy) = d(x)y + xF(y) = F(x)F(y) \quad \text{for all } x, y \in U.
\]

Replacing \( y \) by \( yz \) in the above relation, we get

\[
F(xyz) = d(x)yz + xF(yz) \quad \text{for all } x, y, z \in U,
\]

\[
F(xy)F(z) = d(x)yz + x(d(y)z + yF(z)) \quad \text{for all } x, y, z \in U.
\]

This implies that

\[
(d(x)y + xF(y))F(z) = d(x)yz + xd(y)z + xyF(z) \quad \text{for all } x, y, z \in U.
\]
Using Theorem 2.3.1 (ii), we get

\[ d(x)yF(z) + xF(y)F(z) = d(x)yz + xd(y)z + xyF(z) \quad \text{for all } x, y, z \in U, \]

\[ d(x)yF(z) + xF(yz) = d(x)yz + xd(y)z + xyF(z) \quad \text{for all } x, y, z \in U. \]

This implies that

\[ d(x)yF(z) + x(d(y)z + yF(z)) = d(x)yz + xd(y)z + xyF(z) \quad \text{for all } x, y, z \in U. \]

\[ d(x)yF(z) + xd(y)z + xyF(z) = d(x)yz + xd(y)z + xyF(z) \quad \text{for all } x, y, z \in U - \text{i.e.,} \]

\[ d(x)yF(z) = d(x)yz \quad \text{for all } x, y, z \in U. \]

Therefore

\[ d(x)y(F(z) - z) = 0 \quad \text{for all } x, y, z \in U, \]

which implies that

\[ d(x)U(F(z) - z) = \{0\} \quad \text{for all } x, z \in U. \]

It follows by Proposition 2.2.6 (i) that either \( d(x) = 0 \) or \( (F(z) - z) = 0 \), for all \( x, z \in U \) - i.e., \( d(U) = 0 \) or \( F(z) = z \) for all \( z \in U \).

In fact, as we now show, both of these conditions hold.

Suppose that \( F(u) = u \) for all \( u \in U \). Then for all \( u \in U \) and \( x \in N \), \( F(xu) = xu = d(x)u + xF(u) = d(x)u + xu \); hence \( d(x)U = \{0\} \) for all \( x \in N \).

By Proposition 2.2.5 (i), we get \( d(x) = 0 \) for all \( x \in N \). Hence \( d = 0 \).

On the other hand, suppose that \( d(U) = \{0\} \), so that \( d = 0 \). Then for all \( x, y \in U \), \( F(xy) = F(x)y + xd(y) = F(x)F(y) \) - i.e., \( F(x)F(y) = F(x)y \), so that \( F(x)(y - F(y)) = 0 \). Replacing \( y \) by \( zy \), \( z \in N \) we get \( F(x)(zy - F(zy)) = F(x)(zy - F(z)y - zd(y)) = F(x)(zy - F(z)y) = 0 \) and noting that \( F(zy) = zF(y) \), we see that \( F(x)(zy - zF(y)) = 0 \) or \( F(x)z(y - F(y)) = 0 \) for all \( x, y \in U \) and \( z \in N \) - i.e., \( F(x)N(y - F(y)) = \{0\} \) for all \( x, y \in U \). Since \( N \) is 3-prime near ring,
we get either $F(x) = 0$ or $(y - F(y)) = 0$, for all $x, y \in U$. Therefore, $F(U) = \{0\}$ or $F$ is the identity on $U$. But $F(U) = \{0\}$ contradicts Theorem 2.3.4, so $F$ is the identity on $U$.

We now know that $F$ is the identity on $U$ and $d = 0$, we get $F(xy) = d(x)y + xf(y) = xf(y)$ for all $x, y \in N$. Consequently, $F(uv) = uv = d(u)x + uF(x)$ i.e., $ux = uF(x)$ for all $u \in U$ and $x \in N$ or $u(x - F(x)) = 0$ so that $U(x - F(x)) = \{0\}$ for all $x \in N$. By Proposition 2.2.5 $(i)$, we get $(x - F(x)) = 0$ i.e., $F(x) = x$ for all $x \in N$. It follows that $F$ is the identity on $N$.

**Theorem 5.2.2** Let $N$ be a 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. If $F$ is a nonzero generalized derivation on $N$ with associated derivation $d$. If $F$ acts as an antihomomorphism on $U$, then $d = 0$, $F$ is the identity map on $N$, and $N$ is a commutative ring.

**Proof.** We begin by showing that $d = 0$ if and only if $F$ is the identity map on $N$. Clearly if $F$ is the identity map on $N$, then $F(xy) = xy = F(x)y + xd(y) = xy + xd(y)$ i.e., $xd(y) = 0$ for all $x, y \in N$ or $Nd(x) = \{0\}$. By Proposition 2.2.5 $(i)$, we get $d(x) = 0$, for all $x \in N$ and hence $d = 0$.

Conversely, assume that $d = 0$, in which case $F(xy) = F(x)y + xd(y) = d(x)y + xf(y)$ i.e., $F(xy) = F(x)y = xF(y)$ for all $x, y \in N$. It follows that for any $x, y, z \in U$,

$$F(yxz) = F(z)F(yx) = F(z)yF(x) = F(zy)F(x) = zF(y)F(x) = zF(xy).$$

(5.2.1)

On the other hand,

$$F(yxz) = F(xz)F(y) = F(x)zF(y) = F(x)F(zy) = F(x)F(y)F(z) = F(yx)F(z)$$

$$= F(y)xF(z) = F(y)F(zz) = F(y)F(z)z = F(xy)z.$$  

(5.2.2)

77
Comparing (5.2.1) and (5.2.2), we obtain \( zF(xy) = F(xy)z \) or \([F(xy), z] = 0 \) for all \( x, y, z \in U \) - i.e., \([F(U^2), U] = \{0\} \) or \( F(U^2)U = UF(U^2) \) which implies that \( F(U^2) \) centralizes \( U \), so that \( F(U^2) \subseteq Z \) by Proposition 2.2.5 (iii).

Now \( U^2 \) is a nonzero semigroup ideal by Proposition 2.2.5 (i), hence \( F(U^2) \neq 0 \) by Theorem 2.3.4. Choosing \( x, y \in U \) such that \( F(xy) \neq 0 \), we see that for any \( z \in U \), \( F(xy)z = F(xyz) = F(yz)F(x) = F(y)zF(x) = F(y)F(zx) = F(y)F(x)F(z) = F(xy)F(z) \) and hence \( F(xy)(z - F(z)) = 0 \). Since \( F(xy) \in Z \setminus \{0\} \), then by Proposition 2.2.3 (i) we conclude that \( (z - F(z)) = 0 \) for all \( z \in U \) - i.e., \( F(z) = z \) for all \( z \in U \) and it follows easily that \( F \) is the identity map on \( N \).

We note now that if the identity map on \( N \) acts as an antihomomorphism on \( U \), then \( U \) is commutative, so that by Propositions 2.2.5 (iii) and 2.2.7, \( N \) is a commutative ring.

To complete the proof of our theorem, we need only argue that \( d = 0 \). By our antihomomorphism hypothesis

\[
F(xy) = d(x)y + xF(y) = F(y)F(x) \quad \text{for all } x, y \in U.
\]
Replacing \( y \) by \( xy \) in the above relation, we get

\[
F(xy)F(x) = F(xxy) = d(x)xy + xF(xy) \quad \text{for all } x, y \in U.
\]
This implies that

\[
(d(x)y + xF(y))F(x) = d(x)xy + xF(y)F(x) \quad \text{for all } x, y \in U.
\]
Using Theorem 2.3.1 (ii), we get

\[
d(x)yF(x) + xF(y)F(x) = d(x)xy + xF(y)F(x) \quad \text{for all } x, y \in U.
\]
Thus

\[
d(x)yF(x) = d(x)xy \quad \text{for all } x, y \in U. \quad (5.2.3)
\]
Replacing \( y \) by \( yr \) in (5.2.3) and using (5.2.3), we get 
\[
d(x)yrF(x) = d(x)xyr \quad \text{or} \quad d(x)yrF(x) = d(x)yF(x)r
\]
-i.e.,
\[
d(x)yrF(x) - d(x)yF(x)r = 0 \quad \text{and so} \quad d(x)y[r, F(x)] = 0 \text{ for all } x, y \in U \text{ and } r \in N.
\]
-i.e.,
\[
d(x)U[r, F(x)] = \{0\} \text{ for all } x, y \in U \text{ and } r \in N.
\]
Application of Proposition 2.2.6 (i) yields that for each \( x \in U \) either \( d(x) = 0 \) or \( [r, F(x)] = 0 \) -i.e. \( d(x) = 0 \) or \( F(x) \in Z \).

Suppose there exists \( w \in U \) such that \( F(w) \in Z \setminus \{0\} \). Then for all \( v \in U \) such that \( d(v) = 0 \), \( F(wv) = F(w)v + wd(v) = F(v)F(w) = F(w)F(v) \) -i.e., \( F(w)v = F(w)F(v) \) and hence \( F(w)(v - F(v)) = 0 \). Since \( F(w) \in Z \setminus \{0\} \), then by Proposition 2.2.3 (i), we get \( (v - F(v)) = 0 \) or \( F(v) = v \) for all \( v \in U \). Now consider arbitrary \( x, y \in U \). If one of \( F(x), F(y) \) is in \( Z \), then \( F(xy) = F(x)F(y) \).

If \( d(x) = 0 = d(y) \), then \( d(xy) = d(x)y + xd(y) = 0 \), so \( F(xy) = xy = F(x)F(y) \).

Therefore \( F(xy) = F(x)F(y) \) for all \( x, y \in U \) which implies that \( F \) acts as homomorphism on \( U \).

Now, we have \( F \) acts as homomorphism on \( U \), then
\[
F(xy) = d(x)y + xF(y) = F(x)F(y) \text{ for all } x, y \in U.
\]
Replacing \( y \) by \( yz \) in the above relation, we get
\[
F(xyz) = d(x)yz + xF(yz) \quad \text{for all } x, y, z \in U,
\]
\[
F(xy)F(z) = d(x)yz + x(d(y)z + yF(z)) \quad \text{for all } x, y, z \in U.
\]
This implies that
\[
(d(x)y + xF(y))F(z) = d(x)yz + xd(y)z + xyF(z) \text{ for all } x, y, z \in U.
\]

79
Using Theorem 2.3.1 (ii), we get

\[ d(x) y f(z) + x F(y) F(z) = d(x) y z + x d(y) z + x y F(z) \text{ for all } x, y, z \in U, \]

\[ d(x) y F(z) + x F(y z) = d(x) y z + x d(y) z + x y F(z) \text{ for all } x, y, z \in U. \]

This implies that

\[ d(x) y f(z) + x (d(y) z + y F(z)) = d(x) y z + x d(y) z + x y F(z) \text{ for all } x, y, z \in U. \]

\[ d(x) y F(z) + x d(y) z + x y F(z) = d(x) y z + x d(y) z + x y F(z) \text{ for all } x, y, z \in U - i.e., \]

\[ d(x) y F(z) = d(x) y z \text{ for all } x, y, z \in U. \]

Therefore

\[ d(x) y (F(z) - z) = 0 \text{ for all } x, y, z \in U, \]

which implies that

\[ d(x) U(F(z) - z) = \{0\} \text{ for all } x, z \in U. \]

It follows by Proposition 2.2.3 (i) that either \( d(x) = 0 \) or \( (F(z) - z) = 0 \), for all \( x, z \in U \) - i.e., \( d(U) = 0 \) or \( F(z) = z \) for all \( z \in U \).

In fact, as we now show, both of these conditions hold.

Suppose that \( F(u) = u \) for all \( u \in U \). Then for all \( u \in U \) and \( x \in N \),

\[ F(x u) = x u = d(x) u + x F(u) = d(x) u + x u; \text{ hence } d(x) U = \{0\} \text{ for all } x \in N. \]

By Proposition 2.2.5 (i), we get \( d(x) = 0 \) for all \( x \in N \). Hence \( d = 0 \).

On the other hand, suppose that \( d(U) = \{0\} \), so that \( d = 0 \). Then for all \( x, y \in U \), \( F(x y) = F(x) y + x d(y) = F(x) F(y) \) - i.e., \( F(x) F(y) = F(x) y \), so that \( F(x)(y - F(y)) = 0 \). Replacing \( y \) by \( z y \), \( z \in N \) we get \( F(x)(z y - F(z y)) = F(x)(z y - F(z) y - z d(y)) = F(x)(z y - F(z) y) = 0 \) and noting that \( F(z y) = z F(y) \), we see that \( F(x)(z y - z F(y)) = 0 \) or \( F(x) z (y - F(y)) = 0 \) for all \( x, y \in U \) and \( z \in N - i.e., \( F(x) N (y - F(y)) = \{0\} \) for all \( x, y \in U \). Since \( N \) is 3-prime near ring,
we get either $F(x = 0 \text{ or } (y - F(y)) = 0$, for all $x, y \in U$. Therefore, $F(U) = \{0\}$ or $F$ is the identity on $U$. But $F(U) = \{0\}$ contradicts Theorem 2.3.4, so $F$ is the identity on $U$.

We now know that $F$ is the identity on $U$ and $d = 0$, we get $F(xy) = d(x)y + xF(y) = xF(y)$ for all $x, y \in N$. Consequently, $F(ux) = ux = d(u)x + uF(x)$-i.e., $ux = uF(x)$ for all $u \in U$ and $x \in N$ or $u(x - F(x)) = 0$ so that $U(x - F(x)) = \{0\}$ for all $x \in N$. By Proposition 2.2.5 (i), we get $(x - F(x)) = 0$ -i.e., $F(x) = x$ for all $x \in N$. It follows that $F$ is the identity on $N$.

The remaining possibility is that for each $x \in U$, either $d(x) = 0$ or $F(x) = 0$. Let $u \in U \setminus \{0\}$, and let $U_1 = uN$. Then $U_1$ is a nonzero semigroup right ideal contained in $U$, and $U_1$ is an additive subgroup of $N$. The sets $\{x \in U_1 \mid d(x) = 0\}$ and $\{x \in U_1 \mid F(x) = 0\}$ are additive subgroups of $U_1$ with union equal to $U_1$, so $d(U_1) = \{0\}$ or $F(U_1) = \{0\}$. If $d(U_1) = \{0\}$, then $d = 0$ by Proposition 2.2.5 (ii). Suppose, then that $F(U_1) = \{0\}$. Then for arbitrary $x, y \in N$,

$F(uxy) = F(ux)y + uxd(y) = 0 = uxd(y)$, so $uNd(y) = \{0\}$, and again $d = 0$. This completes the proof.

5.3 Semiderivations acting as homomorphisms or antihomomorphisms

In this section we take semiderivation acting as homomorphism or antihomomorphism on a semigroup ideal of a 3-prime near ring. Using this we prove either semiderivation is zero or identity map or near ring a commutative ring.

**Theorem 5.3.1** Let $N$ be a zero-symmetric 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. Suppose $f$ is a semiderivation of $N$ associated with a map $g$ such that $g(uv) = g(u)g(v)$, for all $u, v \in U$. If $f$ acts as a hom-
morphism on $U$, then either $f$ is an identity map on $U$ or $f = 0$.

**Proof.** By the hypothesis

$$f(uv) = f(u)f(v), \quad \text{for all } u, v \in U.$$  

$$f(u)g(v) + uf(v) = f(u)f(v), \quad \text{for all } u, v \in U. \quad (5.3.1)$$

Replacing $u$ by $wu$ in (5.3.1), we get

$$f(wu)v + wuf(v) = f(wu)f(v), \quad \text{for all } u, v, w \in U - \text{i.e.,}$$

$$f(w)f(u)g(v) + wuf(v) = f(w)f(u)f(v), \quad \text{for all } u, v, w \in U.$$  

From (5.3.1), we obtain

$$f(w)f(u)g(v) + wuf(v) = f(w)(f(u)g(v) + uf(v)), \quad \text{for all } u, v, w \in U. \quad (5.3.2)$$

Using Theorem 4.2.1, the relation (5.3.2) yields that

$$f(w)f(u)g(v) + wuf(v) = f(w)f(u)g(v) + f(w)uf(v) - \text{i.e.,}$$

$$wuf(v) = f(w)uf(v), \quad \text{for all } u, v, w \in U,$$

$$(f(w) - w)uf(v) = 0, \quad \text{for all } u, v, w \in U - \text{i.e.,}$$

$$(f(w) - w)uf(v) = \{0\}, \quad \text{for all } v, w \in U.$$

It follows by Proposition 2.2.6(i) that either $f(U) = \{0\}$ or $f(w) = w$ for all $w \in U$. In the first case, $f = 0$ by Theorem 4.2.2 and in the latter case, $f$ is an identity map on $U$, so our proof is complete.

**Theorem 5.3.2** Let $N$ be a zero-symmetric 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. Suppose $f$ is a semiderivation of $N$ associated
with a map \( g \) such that \( g(uv) = g(u)g(v) \), for all \( u, v \in U \). If \( f \) acts as an anti-homomorphism on \( U \), then either \( f = 0 \), or \( N \) is a commutative ring and \( f \) is the identity map on \( U \).

**Proof.** By the hypothesis

\[
f(uv) = f(v)f(u), \quad \text{for all} \quad u, v \in U - \text{i.e.,}
\]

\[
f(u)g(v) + uf(v) = f(v)f(u), \quad \text{for all} \quad u, v \in U.
\] (5.3.3)

Replacing \( u \) by \( uv \) in (5.3.3), we get \( f(uv)g(v) + uvf(v) = f(v)f(uv) \), for all \( u, v \in U \) or \( f(u)f(u)g(v) + uvf(v) = f(v)(f(u)g(v) + uf(v)) \). Using Theorem 4.2.1, we get \( f(v)f(u)g(v) + uvf(v) = f(v)f(u)g(v) + f(v)uf(v) \) -i.e.,

\[
uvf(v) = f(v)uf(v), \quad \text{for all} \quad u, v \in U.
\] (5.3.4)

Replacing \( u \) by \( ru \) in (5.3.4), we get

\[
ruf(v) = f(v)ruf(v), \quad \text{for all} \quad u, v \in U \text{ and } r \in N.
\] (5.3.5)

Using (5.3.4), the relation (5.3.5) gives that \( rf(v)uf(v) = f(v)ruf(v) \) or \((f(u)r - rf(v))uf(v) = 0 \) or \([f(v), r]uf(v) = 0 \), for all \( u, v \in U \) and \( r \in N \). This implies that \([f(v), r]uf(v) = 0 \), for all \( v \in U \) and \( r \in N \). Application of Proposition 2.2.6 (i) yields that either \( f(v) = 0 \) or \([f(v), r] = 0 \). In the first case \( f(v) = 0 \) for all \( v \in U \). Since \( 0 \in Z \), then \( f(v) \in Z \). In the latter case \([f(v), r] = 0 \), for all \( v \in U \) and \( r \in N \), then \( f(v) \in Z \). Thus in the both cases, we get \( f(v) \in Z \). Then in the given hypothesis, we get \( f(uv) = f(v)f(u) = f(u)f(v) \). Therefore, \( f \) acts as a homomorphism on \( U \).

Now we have \( f \) acts as a homomorphism on \( U \) and \( g(uv) = g(u)g(v) \), for all \( u, v \in U \), we show that either \( f = 0 \) or \( f \) is the identity map on \( U \).
We have \( f \) acts as a homomorphism on \( U \), then
\[
f(uv) = f(u)f(v), \quad \text{for all } u, v \in U.
\]
\[
f(ug(v) + uf(v) = f(u)f(v), \quad \text{for all } u, v \in U. \tag{5.3.6}
\]
Replacing \( u \) by \( uv \) in (5.3.6), we get
\[
f(uw)g(v) + wuf(v) = f(w)g(v), \quad \text{for all } u, v, w \in U - \text{i.e.,}
\]
\[
f(u)g(v) + wuf(v) = f(w)f(u)f(v), \quad \text{for all } u, v, w \in U.
\]
From (5.3.6), we obtain
\[
f(u)g(v) + wuf(v) = f(w)(g(v) + uf(v)), \quad \text{for all } u, v, w \in U. \tag{5.3.7}
\]
Using Theorem 4.2.1, the relation (5.3.7) yields that
\[
f(w)f(u)g(v) + wuf(v) = f(w)f(u)g(v) + f(w)uf(v) - \text{i.e.,}
\]
\[
wuf(v) = f(w)uf(v), \quad \text{for all } u, v, w \in U,
\]
\[
(f(w) - w)uf(v) = 0, \quad \text{for all } u, v, w \in U - \text{i.e.,}
\]
\[
(f(w) - w)Uf(v) = \{0\}, \quad \text{for all } v, w \in U.
\]
It follows by Proposition 2.2.6 (i) that either \( f(U) = \{0\} \) or \( f(w) = w \) for all \( w \in U \). In the first case, \( f = 0 \) by Theorem 4.2.2 and in the latter case, \( f \) is an identity map on \( U \).

Finally, we show that \( N \) is a commutative ring. Since, we have shown that \( f(u) \in Z \), for all \( u \in U \), then we get \( f(U) \subseteq Z \). We begin by showing that \((N, +)\) is abelian, which by Proposition 2.2.3 (ii) is accomplished by producing \( z \in Z \setminus \{0\} \) such that \( z + z \in Z \). Let \( a \) be an element of \( U \) such that \( f(a) \neq 0 \).
Then for all $x \in N$, $xa \in U$, $f(xa) \in f(U)$, $f(U) \subseteq Z$, then $f(xa) \in Z$ and $xa + xa = (x + x)a \in U$, so that $f(xa) \in Z$ and $f(xa) + f(xa) \in Z$; hence we need only show that there exists $x \in N$ such that $f(xa) \neq 0$. Suppose this is not the case, so that $f((xa)a) = 0 = f(xa)g(a) + xaf(a)$ or $xf(a) = 0$ for all $x \in N$. Since $f(a)$ is not a zero divisor by Proposition 2.2.3 (i), we get $xa = 0$ for all $x \in N$ or $Na = \{0\}$, so that $a = 0$; a contradiction. Thus our assumption $f(xa) = 0$ is wrong. Thus $f(xa) \neq 0$ so that $f(xa) \in Z$ and $f(xa) + f(xa) \in Z$. Therefore $(N, +)$ is abelian by Proposition 2.2.3 (ii).

Since $f(U) \subseteq Z$, then we are given that $[f(u), x] = 0$ for all $u \in U$ and $x \in N$. Replacing $u$ by $uv$, we get $[f(uv), x] = 0$, which yields $[f(u)v + g(u)f(v), x] = 0$ for all $u, v \in U$ and $x \in N$. This implies that $(f(u)v + g(u)f(v))x = x(f(u)v + g(u)f(v)) = 0$, now applying Theorem 4.2.1, we get $f(u)vx + g(u)f(v)x - xf(u)v - xg(u)f(v) = 0$ for all $u, v \in U$ and $x \in N$. Since $f(U) \subseteq Z$, this implies that $f(u)x = xf(u)$, for all $u \in U$ and $x \in N$. Which yields that $f(u)vx + g(u)xf(v) - f(u)xv - xg(u)f(v) = 0$ for all $u, v \in U$ and $x \in N$. Since $(N, +)$ is abelian, we obtain $f(u)xv - f(u)xv + g(u)xf(v) - xg(u)f(v) = 0$ or $f(u)(vx - xv) + (g(u)x - xg(u))f(v) = 0$, this gives

$$f(u)[v, x] + [g(u), x]f(v) = 0 \text{ for all } u, v \in U \text{ and } x \in N. \quad (5.3.8)$$

Replacing $x$ by $g(u)$, we obtain $f(u)[v, g(u)] + [g(u), g(u)]f(v) = 0$ for all $u, v \in U$ or $f(u)[v, g(u)] = 0$ for all $u, v \in U$; and choosing $u \in U$ such that $f(u) \neq 0$ and applying Proposition 2.2.3 (i), we get $[v, g(u)] = 0$ for all $u, v \in U$. Replacing $v$ by $uv$, for $r \in N$, we get $v[r, g(u)] + [v, g(u)]r = 0$ or $v[r, g(u)] = 0$, for all $u, v \in U$ and $x \in N$. This implies that $U[r, g(u)] = \{0\}$ for all $u \in U$ and $x \in N$. Then by Proposition 2.2.5 (i), we get $[r, g(u)] = 0$, for all $u \in U$ and $r \in N$. Which gives $g(u) \in Z$. It then follows from (5.3.8) that is
\[ f(u)[v, x] = 0 \text{ for all } u, v \in U \text{ and } x \in N \text{ or } f(U)[v, x] = \{0\}, \text{ now using Theorem 4.2.3, we get } [v, x] = 0, \text{ for all } v \in U \text{ and } x \in N; \text{ therefore } U \subseteq Z \text{ and Proposition 2.2.7 gives } N \text{ is a commutative ring. Thus proof of theorem is complete.} \]

**Remark 5.3.1** The following example shows that the above theorems do not hold for an arbitrary near ring.

**Example 5.3.1** In Example 4.2.1, consider \( N_2 \) a commutative ring and \( U \) to be the set of all elements \((x, 0)\) of \( N \). Then the nontrivial semiderivation \( f \) acts as a homomorphism and an antihomomorphism on \( U \). However \( N \) is not a 3-prime near ring.