Chapter 2

Information-entropic analysis of chaotic time series

2.1 Introduction

Time-series data, representing the measurement of some variables of interest, have been extensively studied in the past two decades [1, 5, 23, 46, 65]. A large number of methods have been devised in order to extract as much information as possible from such data. Important issues in this analysis pertain to, for instance, whether the data is indicative of underlying chaotic dynamics, whether the variables arise from a low dimensional attractor, whether this attractor has a fractal dimension which can be inferred from the time-series data, etc. A host of such questions have been raised, and by now there are several robust methods to estimate fractal dimensions, Lyapunov exponents or the Lyapunov dimension from time-series data [33, 44, 100, 133].

The reconstruction of chaotic attractor or its crucial properties from time signals has been studied time and again [59, 100, 122]. Given two time-series of different variables, how can it be reliably determined that they originate from the same underlying dynamics, namely that they are governed by a common underlying attractor? Recent studies have addressed this issue, based on calculation of conditional entropy of two symbolic sequences, derived from the original time-series via a coarse-graining. In the coarse-graining technique, the phase space is partitioned into domains, each uniquely identified with a symbol. The time series is converted to a symbolic sequence by noting
the order in which the signal passes through successive domains. If the exact dynamics is known, a "generating partition" can be devised in order to get a faithful symbolic representation of the time signal. Clearly, most partitions will not be generating: finding the proper partition which gives a 1-1 correspondence between an orbit and a symbolic sequence can be difficult [22, 116, 120].

Even in the absence of knowledge of generating partition, though, some progress can be made. Recently Lehrman et al. [68] (LRW) have addressed a problem which applies to chaotic time signals whose underlying dynamics is not known; the only inputs are black box signals. Obtaining a generating partition thus is not possible in principle. However, they have shown that coarse-graining the dynamics by a non-generating partitions of the phase space in conjunction with the application of an information-entropic analysis can allow for some inference, in particular whether two different time signals have common underlying dynamics. This method has potential applications in the analysis of real data since frequently it is not known if there is indeed an underlying deterministic dynamical system. LRW make no attempt to find generating partitions for the numerical models studied: time series are converted into symbolic sequences via a simple coarse-graining procedure [68, 112, 113] which is based on the sequence in which the different dynamical variables pass through critical points in the phase space.

In this chapter, we show further utility of this technique. We find it is possible to extract time-delays which may be inherent in the dynamics. Delay phenomena, namely those where the response to a stimulus occurs after a time-lag, are intrinsic to many dynamical processes in optics [58], physiology [85], population biology [89, 97] etc., and are thus of considerable practical importance. We adapt the coarse-graining method to show that it is possible to obtain delay information present in the underlying dynamics by computing the conditional entropy of a sequence with respect to shifted copies of itself. The conditional entropy is minimized when the shift of the sequence matches the delay.

It should be noted that there are other methods which use information theoretic methods to determine delays [34, 123]. The main advantage of the present approach is its simplicity and robustness to additive noise. Applications are made to time series from the Mackey–Glass [85] and Ikeda equations [58], and we anticipate that such a method will be useful in the analysis of black box noisy signals.
2.2 The coarse-graining method

We briefly review the coarse-graining methodology as applied by Lehrman et al. [68] to chaotic time signals. A symbolic sequence \( \{S_1, S_2, S_3, \ldots \} \) is associated with a time-series \( \{X_1, X_2, X_3, \ldots \} \) via a coarse-graining [68, 112, 113] such that the information concerning the orbit is suitably encoded. This is accomplished by the following partitioning of the phase space.

Given a set of \( m \) symbols, \( \{S_0, S_1, \ldots, S_{m-1}\} \) and a set of \( m + 1 \) critical points, \( \{X_0, X_1, \ldots, X_m\} \) the time series \( \{X_1, X_2, X_3, \ldots\} \) is converted into a symbolic sequence by the rule

\[
S_j \equiv S_k \quad \text{if} \quad X_k < X_j < X_{k+1}. \tag{2.1}
\]

The association \( S_k \leftrightarrow k \) is convenient for defining the index of a subsequence of length \( L \), namely

\[
\ell_X(L, i) = \sum_{k=1}^{L} m^{-k} S_{k+i}, \tag{2.2}
\]

where it is clear that the subsequence starts at position \( i \) along the sequence \( \{S_1, S_2, S_3, \ldots\} \).

A particular subsequence is uniquely characterized by \( \ell_X(L, i) \). If \( P_{tx} \) is the probability of finding a particular value of \( \ell_X \), then the information entropy is defined as

\[
E = -\frac{1}{L} \sum_{tx} P_{tx} \ln P_{tx} \tag{2.3}
\]

(we omit the arguments of \( \ell_X \) for simplicity). To get the critical points \( \{X\} \) for a particular signal, one needs to maximize this entropy with respect to all possible partitions. Increasing the number of critical points will increase the information, but the optimum language for the symbolic analysis is the one having a sufficient number of critical points (and further addition of critical points does not increase the information entropy).

For two signals \( \{X\} \) and \( \{Z\} \), the conditional information entropy is defined as

\[
E(Z/X) = -\frac{1}{N_{\ell}} \sum_{tx} \frac{1}{L} \sum_{tz} P(\ell_z/\ell_x) \ln P(\ell_z/\ell_x) \tag{2.4}
\]

where \( \ell_z \) is defined in a manner similar to \( \ell_x \), and \( P(\ell_z/\ell_x) \) is the probability for the variable \( Z \) to occupy state \( \ell_z \) when the variable \( X \) occupies state \( \ell_x \), and \( N_\ell \) is the total number of different \( \ell_x \) values that are observed.
2.3 Delay Determination

If two signals, \{X\} and \{Z\} follow the same dynamics, for these time-series, \(\ell_X(L, i)\) and \(\ell_Z(L, i)\) are necessarily coupled. However, shifting the two sequences with respect to one another will in general destroy the mutual coupling. If the indices \(\ell_X(L, i)\) and \(\ell_Z(L, i + n_0)\) are compared as a function of the shift parameter \(n_0\), a distinctive minimum in the conditional entropy \(E(Z/X)\) for zero shift indicates that the two sequences have a common dynamical origin [68].

Lehrman et al. applied the above method to the time-signals obtained from the Lorenz model [83]. The Lorenz equations (1.2)–(1.4) are solved by fourth order Runge-Kutta numerical method with time step \(\Delta t = .005\) and the initial conditions \(X_0 = 6, Y_0 = 6, Z_0 = 13.5\). The \(X\) and \(Z\) time-signals are shown in Fig. 2.1 for time \(t = 25\). The two signals look quite different and it is difficult to decide if they arise from the same underlying dynamics.

The coarse-graining method as detailed in Section 2.2 above is applied. The time series \(X(t)\) is discretized as

\[
X_n = X(n\tau'), \quad Z_n = Z(n\tau'), \quad n = 0, 1, 2, \ldots, \quad (2.5)
\]

the value of \(\tau'\) being chosen to be larger than the decorrelation time. This
Figure 2.2: The Conditional entropy $E(Z/X)$ as a function of shift parameter $n_0$ corresponding to $X$ and $Z$ signals of Lorenz model.

The coarse-graining methodology can be used to extract time-delays that may be inherent in the underlying dynamics. Consider a signal $\{X\}$ and its $j$-shifted copy, namely $\{Y_1, Y_2, \ldots, Y_k = X_{k+j}, \ldots\}$. Clearly the conditional entropy $E(Y/X)$ will be a minimum for $j = 0$.

If there is a time-delay which is present in the underlying dynamics, however, the conditional entropy also shows a minimum when the shift is equal to the delay.

We apply this technique to a time-series obtained from the solution of
the Mackey–Glass equation [34, 85],

\[ \dot{X}(t) = -X(t) + f(X(t - \tau)) \]
\[ f(X) = \lambda X/(1 + X^c) \]

which has been numerically integrated with time step \( \delta t = 0.005 \), and other parameters for which the system yields a chaotic attractor, are taken to be \([34, 123]\) \( \tau = 1, \epsilon = 0.05, \lambda = 2 \) and \( c = 10 \). The time-series obtained, \( X(t) \), is shown in Fig. 2.3, and is the only available input information for subsequent analysis.

To determine the delay, we first apply the coarse-graining method. The time series \( X(t) \) is discretized as above with \( \tau = 0.05 \).

Adopting again the binary symbolic language, \( m = 2 \) and \( L = 5 \), the critical points are determined as \( \{X_0, X_1, X_2\} = \{X_{\text{min}}, 0.92, X_{\text{max}}\} \). The conditional entropy \( E(Y/X) \) where \( \{Y\} \) is the \( j \)-shifted series \( \{X\} \) is calculated for different values of time lag \( \tau = j\tau' \), and is shown in Fig. 2.4.

A sharp minimum is seen for time lag \( \tau = 20\tau' \equiv 1 \) namely, when the lag matches the delay. Note that there are other minima as well: apart from the minimum at \( \tau = 0 \), there are also minima at multiples of \( \tau \), but these decrease in amplitude. It is clear that this adaptation of the methodology of symbolic conversion is able to determine the inherent time–delay in the dynamics.

To rule out the possibility of this minimum at \( \tau = 1 \) due to recurrence in the dynamics, we have looked at the power spectrum of the data (see Fig. 2.5). We observed no prominent maxima in the neighbourhood of frequency corresponding to the delay \( (\equiv 1) \) of the Mackey–Glass time series.
Figure 2.4: Conditional entropy as a function of the time lag, $\tau_1$. The time series shown in Fig. 2.3 has been coarse-grained as described in Section 2.2. At $\tau_1 = 0$, the minimum $E(Y/X) = 0$ is not shown.

Figure 2.5: Power spectrum $P(f)$ versus frequency $f (= 1/T)$, for the discretized time series, of Mackey–Glass equation (2.7)-(2.8).
The method works even in the presence of additive noise. Random white noise with zero mean and variance $\sigma_n^2$ [123] is added to the signal $\{X\}$. The measurement noise intensity is characterized by the signal to noise ratio $SNR = \sqrt{\sigma_s^2/\sigma_n^2}$, where $\sigma_s$ is the standard deviation of the noise free chaotic series and $\sigma_n$ that of noise. In our analysis, we take $SNR \sim 2(\sigma_s^2 \sim 0.1, \sigma_n^2 \sim 0.025)$. The above procedure is applied to noisy time-series thus generated (see Fig. 2.6) and appears equally successful in obtaining the delay (see Fig. 2.7) even for relatively high levels of noise.

A similar result is obtained by applying the method to time series from the Ikeda laser equation [58]

$$\dot{X}(t) = -X(t) + \pi \nu \sin[X(t - \tau) - X_0],$$

(2.9)

which is numerically integrated as above to yield chaotic time-series; the parameter values taken are $\nu = 2.1$ and $\tau = 20$ [58]. The coarse-graining is done as described above with critical points $\{X_0 = X_{\text{min}}, X_1 = -0.16, X_2 = X_{\text{max}}\}$. A sharp fall in $E(Y/X)$ can be seen for time lag $\tau = 20$ (see Fig. 2.8).

For the signal with added noise with $SNR = 2$, again the dip in the conditional entropy at the same lag, shown in Fig. 2.9 confirms the previous result.

Figure 2.6: Noisy time series of the Mackey–Glass system. White noise has been added to the signal shown in Fig. 2.3.
Figure 2.7: As in Fig. 2.4, for the time series data of Fig. 2.6.

Figure 2.8: As in Fig. 2.4, for the time series data of Ikeda equation, Eq. (2.9).
Discussion

In this Chapter, we have shown how an information-theoretic method can be usefully applied to the coarse-grained dynamics in order to determine time-delays which may be intrinsic to the dynamics. We also show the utility of the technique in determining the commonality of dynamics of different chaotic time signals. Both these objectives are of considerable current interest.

The methodology is based on obtaining a symbolic representation of the dynamics via a coarse-graining procedure, and subsequently computing the conditional probability. The method appears robust to additive noise, and is thus relatively stable to errors in the dynamics or errors in the coding.

The extraction of time-delays from signals is an important goal in the analysis of time-series data. Time-delays are routinely used in modeling a number of natural phenomena—the dynamics of diseases [86], population biology [49], ecology [87], medical disciplines [94] or economics [82]—to cite a few instances. We have analyzed time signals coming from two different delay models—Mackey-Glass equation and the Ikeda equation. Our results show that a method based on coarse-graining method and Shannon information theory has been found to gather delay information from chaotic time signals.

Given the wealth of observational dynamical data that is now becoming available in various areas (finance, climate, medicine, etc.), determining the common origin of two given chaotic time-series is an issue of considerable
practical significance. One potential application of the information–theoretic method can be to experimentally obtained climatic time series data. In modeling complex systems—and the climate is a good example—the present technique is likely to be of help in this regard. Further, in climatology, delay is an important factor in determining the dynamical coupling of different variables and extraction of delay information from climatic time series is another wishful objective. A detailed study of dynamical coupling of climatic time signals is given in the next chapter.