CHAPTER 5

GLOBAL CHAOS SYNCHRONIZATION OF CHAOTIC SYSTEMS USING BACKSTEPPING CONTROL

5.1 Introduction

In this chapter, the global chaos synchronization is investigated for $n$-scroll Chua (Tang et al., 2001) and Lur’e (Sukens and Vandewalle, 1997) chaotic systems using backstepping control with recursive feedback.

Our theorems on synchronization for $n$-scroll Chua and Lur’e chaotic systems are established using Lyapunov stability theory.

The backstepping scheme is recursive procedure that links the choice of Lyapunov function with the design of a controller and guarantees global stability performance of strict-feedback chaotic systems.

Mainly this technique gives that the flexibility to construct a control law. Numerical simulations are also given to illustrate and validate the synchronization results derived in this chapter.

This chapter is organized as follows. In section 5.2 gives description of the chaotic systems discussed in this chapter. In Section 5.3, the chaos synchronization of two identical WINDMI chaotic systems using backstepping control design is discussed. Section 5.4, the chaos synchro-
nization of two identical Coullet systems using backstepping control is discussed. In Section 5.5, the chaos synchronization of WINDMI and Coullet chaotic systems using backstepping control is discussed. In Section 5.6, the chaos synchronization of two identical n-scroll Chua chaotic systems is discussed. In Section 5.7, the chaos synchronization of two identical Lur’e chaotic systems is discussed. In Section 5.8, the chaos synchronization of n-scroll Chua and Lur’e chaotic systems is discussed. Section 5.9 gives a summary of this chapter.

5.2 Systems Description

5.2.1 The WINDMI System

The WINDMI (Sprott, 2003) system is a complex driven-damped dynamical system. The WINDMI system describes the energy flow through the solar wind magnetosphere-ionosphere system.

The dynamics of the chaotic WINDMI system is described by

\[\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -ax_3 - x_2 + b - e^{x_1}
\end{align*}\]

where \(x_1, x_2, x_3\) are state variables and \(a, b\) are positive real constants.

The WINDMI system (5.1) is chaotic when

\[a = 0.7, \ b = 1.5\]

Figure 4.1. depicts the WINDMI chaotic attractor.

5.2.2 The Coullet System

The Coullet (Coullet et al., 1979) chaotic system was proposed by Coullet and Arneodo. The Coullet chaotic system is one of the paradigms of chaotic systems. It includes a simple cubic part and three positive parameters.

The dynamics of the chaotic Coullet system is described by

\[\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= ax_1 - bx_2 - cx_3 - x_1^3
\end{align*}\]
where $x_1, x_2, x_3$ are state variables and $a$, $b$, and $c$ are positive real constants.

The Coullet system (5.2) is chaotic when

$$a = 5.5, \ b = 3.5, \ \text{and} \ c = 1$$

Figure 4.2. depicts the Coullet chaotic attractor.

**5.2.3 The $n$-Scroll Chua system**

The $n$-scroll Chua system with sine function (Tang et al., 2001) is given by the dynamics

$$\begin{align*}
\dot{x}_1 &= \alpha (x_2 - f(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2,
\end{align*}$$

(5.3)

where $x_1, x_2, x_3$ are state variables, $\alpha, \beta$ are positive real constants, and $f(x_1)$ is given by

$$f(x_1) = \begin{cases} 
\frac{b\pi}{2a}(x_1 - 2ac), & \text{if } x_1 \geq 2ac, \\
-b\sin\left(\frac{\pi x_1}{2a} + d\right), & \text{if } -2ac \leq x_1 \leq 2ac, \\
\frac{b\pi}{2a}(x_1 + 2ac), & \text{if } x_1 \leq -2ac,
\end{cases}$$

(5.4)

where $a$, $b$, $c$, and $d$ are positive real constants.

The piecewise linear function is only nonlinearity in the system. A sine function is couched to obtain the nonlinearity needed for generating chaos in Chua system.

When

$$\alpha = 10.814, \ \beta = 14.0, \ a = 1.3, \ b = 0.11 \ \text{and} \ d = 0.$$ 

Furthermore, if we choose $c = 1, 2, 3$ and $5$, then we obtain 2-scroll, 3-scroll, 4-scroll and 6-scroll chaotic attractors respectively, as depicted in Figure 5.7, Figure 5.8, Figure 5.9. A maximum of six scrolls can be observed.

**5.2.4 Lur’e chaotic system**

The Lur’e (Sukens and Vandewalle, 1997) system with sign function is given by the dynamics

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= ax_1 + bx_2 + cx_3 + 12\phi(x_1)
\end{align*}$$

(5.5)
where $x_1, x_2, x_3$ are state variables, $a, b, c$ are real constants and $\phi(x_1)$ is given by

$$
\phi(x_1) = \begin{cases} 
  kx_1, & \text{if } |x_1| \geq \frac{1}{k}, \\
  \text{sign}(x_1), & \text{otherwise} 
\end{cases}
$$

(5.6)

where $k$ is a positive real constant.

When

$$a = -7.4, \ b = -4.1, \ c = -1, \ \text{and} \ k = 3.6$$

the chaotic attractors have been generated as shown in Figure 5.10, Figure 5.11, Figure 5.12, Figure 5.13.

### 5.3 Synchronization of identical WINDMI Chaotic Systems using Backstepping Control Design

In this section, the adaptive backstepping method is applied for the synchronization of two identical WINDMI chaotic systems (Sprott, 2003).

Thus, the master system is described by the chaotic WINDMI dynamics

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -ax_3 - x_2 + b - e^{x_1},
\end{align*}
$$

(5.7)

where $x_1, x_2, x_3$ are state variables and $a, b$ are positive parameters.

The slave system is also described by the chaotic WINDMI dynamics

$$
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= y_3, \\
\dot{y}_3 &= -ay_3 - y_2 + b - e^{y_1} + u,
\end{align*}
$$

(5.8)

where $y_1, y_2, y_3$ are state variables and $u$ is the backstepping controller to be designed.

The synchronization error is defined by

$$
e_1 = y_1 - x_1, \ e_2 = y_2 - x_2, \ e_3 = y_3 - x_3.
$$

(5.9)

Then the error dynamics is obtained as

$$
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3, \\
\dot{e}_3 &= -ae_3 - e_2 - e^{y_1} + e^{x_1} + u.
\end{align*}
$$

(5.10)
We introduce the backstepping procedure to design the controller $u$, where $u$ is control feedback, as long as these feedback stabilize system (5.10) converge to zero as the time $t \to \infty$.

First we consider the stability of the system

$$\dot{e}_1 = e_2, \quad (5.11)$$

where $e_2$ is regarded as virtual controller.

We consider the Lyapunov function defined by

$$V_1(e_1) = \frac{1}{2}e_1^2 \quad (5.12)$$

the derivative of $V_1$ is as following

$$\dot{V}_1 = e_1 e_2 \quad (5.13)$$

Assume the controller $e_2 = \alpha_1(e_1)$.

$$\alpha_1(e_1) = -e_1, \quad (5.14)$$

then

$$\dot{V}_1 = -e_1^2, \quad (5.15)$$

which is negative definite function.

Hence, the system (5.11) is globally asymptotically stable.

The function $\alpha_1(e_1)$ is an estimative function when $e_2$ is considered as a controller.

The error between $e_2$ and $\alpha_1(e_1)$ is

$$w_2 = e_2 - \alpha_1(e_1) = e_2 + e_1 \quad (5.16)$$

Consider the $(e_1, w_2)$ subsystem given by

$$\dot{e}_1 = e_2, \quad \dot{w}_2 = e_3 + e_2. \quad (5.17)$$

Let $e_3$ be a virtual controller in system (5.17).

Assume that when $e_3 = \alpha_2(e_1, w_2)$, the system (5.17) is made globally asymptotically stable.

Consider the Lyapunov function defined by

$$V_2(e_1, w_2) = V_1(e_1) + \frac{1}{2}w_2^2 \quad (5.18)$$

the derivative of $V_2(e_3, w_2)$ is obtained as

$$\dot{V}_2 = \dot{V}_1 + w_2(e_3 + e_2) \quad (5.19)$$
Substituting for $e_2$ from (5.16) into (5.19) and simplifying, we get

$$\dot{V}_2 = -e_1^2 + w_2^2 + w_2(\alpha_2(e_1, w_2)).$$ \hspace{1cm} (5.20)

We choose

$$\alpha_2(e_1, w_2) = -2w_2.$$ \hspace{1cm} (5.21)

Then it follows that

$$\dot{V}_2 = -e_1^2 - w_2^2.$$ \hspace{1cm} (5.22)

Thus, $\dot{V}_2$ is a negative definite function and hence the system (5.17) is globally exponentially stable.

The error between $e_3$ and $\alpha_2(e_1, w_2)$ is

$$w_3 = e_3 - \alpha_2(e_1, w_2) = e_3 + 2w_2.$$ \hspace{1cm} (5.23)

Consider the $(e_1, w_2, w_3)$ subsystem given by

$$\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{w}_2 &= e_2 + e_3, \\
\dot{w}_3 &= -ae_3 - e_2 - e^{y_1} + e^{x_1} + 2(e_2 + e_3) + u.
\end{align*}$$ \hspace{1cm} (5.24)

Consider the Lyapunov function defined by

$$V_3(e_1, w_2, w_3) = V_2(e_1, w_2) + \frac{1}{2}w_3^2.$$ \hspace{1cm} (5.25)

The derivative of $V_3(e_1, w_2, w_3)$ is obtained as

$$\dot{V}_3 = \dot{V}_2 + w_3\dot{w}_3.$$ \hspace{1cm} (5.26)

Substituting for $e_3$ from (5.23) into (5.26) and simplifying, we get

$$\dot{V}_3 = -e_1^2 - w_2^2 + w_3(-a\alpha_2 - \alpha_1 - e^{y_1} + e^{x_1} + 2e_2 + 2e_3 + u).$$ \hspace{1cm} (5.27)

We choose the backstepping control $u$ as follows

$$u = a\alpha_2 + \alpha_1 + e^{y_1} - e^{x_1} - 2e_2 - 2e_3.$$ \hspace{1cm} (5.28)

Substituting for $u$ from (5.28), into (5.27), we get

$$\dot{V}_3 = -e_1^2 - w_2^2 - aw_3^2.$$ \hspace{1cm} (5.29)

Thus, $\dot{V}_3$ is negative definite function.
Thus by Lyapunov stability theory (Hahn, 1967), the error dynamics (5.24) is globally exponentially stable for all initial conditions.

Hence, the states of the master and slave systems are globally and exponentially synchronized.

**Theorem 5.3.1.** The identical WINDMI systems (5.7) and (5.8) are globally and asymptotically synchronized with backstepping control

\[ u = a\alpha_2 + \alpha_1 + e^{y_1} - e^{x_1} - 2e_2 - 2e_3. \]  

(5.30)

5.3.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the system of differential equations (5.7) and (5.8) with the backstepping control \( u \) given by (5.30).

The parameters of the systems (5.7) and (5.8) are taken in the case of chaotic case as

\[ a = 0.7, \ b = 2.5 \]

The initial values of the master system (5.7) are chosen as

\[ x_1(0) = 0.984, \ x_2(0) = 0.345, \ x_3(0) = 0.789 \]

and slave system (5.8) are chosen as

\[ y_1(0) = 0.456, \ y_2(0) = 0.812, \ y_3(0) = 0.124. \]

Figure 5.1 depict the synchronization of identical WINDMI chaotic systems (5.7) and (5.8).

Figure 5.2 depicts the synchronization error between identical WINDMI chaotic systems (5.7) and (5.8).

5.4 Synchronization of Identical Coullet Chaotic Systems using Backstepping Control

In this section, the backstepping method is applied for the synchronization of two identical (Coullet et al., 1979) chaotic systems.

Thus, the master system is described by the chaotic Coullet dynamics

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= ax_1 - bx_2 - cx_3 - x_1^3.
\end{align*} \]

(5.31)
where $x_1, x_2, x_3$ are state variables and $a, b, c$ are positive constants.

The slave system is also described by the chaotic Coullet dynamics

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= y_3, \\
\dot{y}_3 &= ay_1 - by_2 - cy_3 - y_1^3 + u,
\end{align*}
\]

where $y_1, y_2, y_3$ are state variables and $u$ is the backstepping controller to be designed so as to synchronize the states of identical Coullet systems (5.31) and (5.32).

The synchronization error is defined by

\[ e_1 = y_1 - x_1, \ e_2 = y_2 - x_2, \ e_3 = y_3 - x_3. \]  

(5.33)

Then the error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3, \\
\dot{e}_3 &= ae_1 - be_2 - ce_3 - y_1^3 + x_1^3 + u.
\end{align*}
\]

(5.34)

The objective is to find the control law, so that the system (5.34) is globally stabilized at the origin.

We introduce the backstepping procedure to design the controller $u$, where $u$ is control feedback, as long as these feedback stabilize system (5.34) converge to zero as the time $t \to \infty$.

First we consider the stability of the system

\[ \dot{e}_1 = e_2, \]

(5.35)

where $e_2$ is regarded as virtual controller.

We consider the Lyapunov function defined by

\[ V_1(e_1) = \frac{1}{2} e_1^2. \]

(5.36)

The derivative of $V_1$ is obtained as follows

\[ \dot{V}_1 = e_1 e_2. \]

(5.37)

Assume the controller $e_2 = \alpha_1(e_1)$.

\[ \alpha_1(e_1) = -e_1, \]

(5.38)

then

\[ \dot{V}_1 = -e_1^2, \]

(5.39)
which is a negative definite function.

Hence, the system (5.35) globally asymptotically stable.

The function $\alpha_1(e_1)$ is an estimative function when $e_2$ is considered as a controller.

The error between $e_2$ and $\alpha_1(e_1)$ is

$$w_2 = e_2 - \alpha_1(e_1) = e_2 + e_1.$$  \hfill (5.40)

Consider the $(e_1, w_2)$ subsystem given by

$$\dot{e}_1 = w_2 - e_1,$$
$$\dot{w}_2 = w_2 - e_1 + e_3.$$ \hfill (5.41)

Let $e_3$ be a virtual controller in system (5.41).

Assume that when $e_3 = \alpha_2(e_1, w_2)$, the system (5.41) is made globally asymptotically stable.

Consider the Lyapunov function defined by

$$V_2(e_1, w_2) = V_1(e_1) + \frac{1}{2}w_2^2.$$ \hfill (5.42)

The derivative of $V_2(e_3, w_2)$ is obtained as

$$\dot{V}_2 = \dot{V}_1 + w_2\dot{w}_2.$$ \hfill (5.43)

Substituting $e_2$ from (5.40) into (5.43) and simplifying, we get

$$\dot{V}_2 = -k_1e_1^2 + w_2(\alpha_2(e_1, w_2)).$$ \hfill (5.44)

Assume the virtual controller $e_3 = \alpha_2(e_1, w_2)$.

We choose

$$\alpha_2(e_1, w_2) = -2w_2.$$ \hfill (5.45)

Then it follows that

$$\dot{V}_2 = -e_1^2 - w_2^2.$$ \hfill (5.46)

Thus, $\dot{V}_2$ is a negative definite function and hence the system (5.41) is globally exponentially stable.

The error between $e_3$ and $\alpha_2(e_1, w_2)$ is

$$w_3 = e_3 - \beta_2(e_1, w_2) = e_3 + 2w_2.$$ \hfill (5.47)

Consider the $(e_1, w_2, w_3)$ subsystem given by

$$\dot{e}_1 = w_2 - e_1,$$
$$\dot{w}_2 = w_3 - w_2 - e_1,$$
$$\dot{w}_3 = (a - 2)e_1 - be_2 - ce_3 - 2w_2 + 2w_3 - y_1^2 + x_1^3 + u.$$ \hfill (5.48)
Consider the Lyapunov function defined by

$$V_3(e_1, w_2, w_3) = V_2(e_3, w_2) + \frac{1}{2} w_3^2.$$  \hspace{1cm} (5.49)

The derivative of $V_3(e_1, w_2, w_3)$ is obtained as

$$\dot{V}_3 = \dot{V}_2 + w_3 \dot{w}_3.$$  \hspace{1cm} (5.50)

Substituting for $e_3$ from (5.47) into (5.50) and simplifying, we get

$$\dot{V}_3 = -e_1^2 - w_2^2 + w_3[(a + b - 2)e_1 + (2c - b - 1)w_2 + (2 - c)w_3 - y_1^3 + x_1^3] + u.$$  \hspace{1cm} (5.51)

We choose the control $u$ as follows.

$$u = -(a + b - 2)e_1 - (2c - b - 1)w_2 - (2 - c)w_3 + y_1^3 - x_1^3.$$  \hspace{1cm} (5.52)

Substituting for $u$ from (5.52) into (5.51), we get

$$\dot{V}_3 = -e_1^2 - w_2^2 - aw_3^2.$$  \hspace{1cm} (5.53)

Thus, $\dot{V}_3$ is a negative definite function.

Thus by a Lyapunov stability theory (Hahn, 1967), the error dynamics (5.48) is globally asymptotically stable for all initial conditions.

Hence, the states of the master and slave systems are globally and exponentially synchronized.

**Theorem 5.4.1.** The Identical Coullet chaotic systems (5.31) and (5.32) are globally and asymptotically synchronized with backstepping control

$$u = -(a + b - 2)e_1 - (2c - b - 1)w_2 - (2 - c)w_3 + y_1^3 - x_1^3.$$  \hspace{1cm} (5.54)

### 5.4.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the differential equations (5.31) and (5.32) with the backstepping control $u$ given by (5.54).

The initial values of the master system (5.31) are chosen as

$$x_1(0) = 0.125, \ x_2(0) = 0.625, \ x_3(0) = 0.825,$$

and slave system (5.32) are chosen as

$$y_1(0) = 0.924, \ y_2(0) = 0.498, \ y_3(0) = 0.032.$$
Figure 5.3 depict the synchronization of identical Coullet chaotic systems (5.31) and (5.32). Figure 5.4 depict the synchronization error of identical Coullet chaotic systems (5.31) and (5.32).

5.5 Synchronization of WINDMI and Coullet Chaotic Systems using Backstepping Control Design

In this section, the adaptive backstepping control design is applied for the synchronization of two different chaotic systems described by WINDMI (Sprott, 2003) system as the master system and Coullet (Coullet et al., 1979) system as the slave system.

The dynamics of the Coullet system, taken as master system is described by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= ax_1 - bx_2 - cx_3 - ex_1^3,
\end{align*}
\]

where \( x_1, x_2, x_3 \) are state variables, \( a, b, c \) are positive parameters.

The dynamics of the WINDMI system, taken as the slave system, is described by

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= y_3, \\
\dot{y}_3 &= \alpha y_3 - y_2 - \beta - e^{y_1} + u,
\end{align*}
\]

where \( u \) is the backstepping controller to be designed so as to synchronize the states of the different chaotic systems (5.55) and (5.56).

The synchronization error is defined by

\[
e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3.
\]

The error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= e_3, \\
\dot{e}_3 &= -\alpha y_3 + cx_3 - y_2 + bx_2 - ax_1 + x_1^3 + \beta - e^{y_1} + u.
\end{align*}
\]

The objective is to find the control law, so the that the system (5.58) is globally asymptotically stabilized at the origin.

We introduce the backstepping procedure to design the controller \( u \), where \( u \) is control feedback, as long as these feedback stabilize system (5.58) converge to zero as the time \( t \to \infty \).
First we consider the stability of the system

\[ \dot{e}_1 = e_2, \]

(5.59)

where \( e_2 \) is regarded as virtual controller.

We consider the Lyapunov function defined by

\[ V_1(e_1) = \frac{1}{2} e_1^2. \]

(5.60)

The derivative of \( V_1 \) is obtained as following

\[ \dot{V}_1 = e_1 e_2. \]

(5.61)

Assume the controller \( e_2 = \alpha_1(e_1) \).

We choose

\[ \alpha_1(e_1) = -e_1. \]

(5.62)

Then we get

\[ \dot{V}_1 = -e_1^2, \]

(5.63)

which is a negative definite function.

Hence, the system (5.59) globally asymptotically stable.

The function \( \alpha_1(e_1) \) is an estimative function when \( e_2 \) is considered as a controller.

The error between \( e_2 \) and \( \alpha_1(e_1) \) is

\[ w_2 = e_2 - \alpha_1(e_1) = e_2 + e_1. \]

(5.64)

Consider \((e_1, w_2)\) subsystem given by

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{w}_2 &= e_2 + e_3.
\end{align*}
\]

(5.65)

Let \( e_3 \) be a virtual controller in system (5.65).

Assume that when \( e_3 = \alpha_2(e_1, w_2) \), the system (5.65) is made globally asymptotically stable.

Consider the Lyapunov function defined by

\[ V_2(e_1, w_2) = V_1(e_1) + \frac{1}{2} w_2^2. \]

(5.66)

Assume the virtual controller \( e_3 = \alpha_2(e_1, w_2) \).

The derivative of \( V_2(e_3, w_2) \) is obtained as

\[ \dot{V}_2 = \dot{V}_1 + w_2 \dot{w}_2. \]

(5.67)
Substituting $e_2$ from (5.64) into (5.67) and simplifying, we get

$$\dot{V}_2 = -e_1^2 + w_2^2 + w_2(\alpha_2(e_1, w_2)).$$

(5.68)

We choose

$$\alpha_2(e_1) = -2w_2.$$  

(5.69)

Then we get

$$\dot{V}_2 = -e_1^2 - w_2^2.$$  

(5.70)

Then $\dot{V}_2$ is a negative definite function and hence the system (5.65) is globally asymptotically stable.

Define the error variable $w_3$ as

$$w_3 = e_3 - \alpha_2(e_1, w_2) = e_3 + 2w_2.$$  

(5.71)

Consider $(e_1, w_2, w_3)$ subsystem given by

$$\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{w}_2 &= e_2 + e_3, \\
\dot{w}_3 &= -\alpha y_3 + cx_3 - y_2 + bx_2 - ax_1 + x_1^3 + \beta - e_3 + 2e_2 + 2e_3 + u.
\end{align*}$$

(5.72)

Consider the Lyapunov function defined by

$$V_3(e_1, w_2, w_3) = V_2(e_3, w_2) + \frac{1}{2}w_3^2.$$  

(5.73)

The derivative of $V_3(e_1, w_2, w_3)$ is

$$\begin{align*}
\dot{V}_3 &= \dot{V}_2 + w_3\dot{w}_3, \\
&= \dot{V}_2 + w_3(-\alpha y_3 + cx_3 - y_2 + bx_2 - ax_1 + x_1^3 + \beta - e_3 + 2e_2 + 2e_3 + u).
\end{align*}$$

(5.74)

Substituting $e_3$ from (5.71) into (5.74) and simplifying, we get

$$\dot{V}_3 = -e_1^2 - w_2^2 + w_3(-\alpha y_3 + cx_3 - y_2 + bx_2 - ax_1 + x_1^3 + \beta - e_3 - 2e_1 + 2w_3 - w_2 + u)$$

(5.75)

We choose

$$u = \alpha y_3 - cx_3 + y_2 - bx_2 + ax_1 - x_1^3 - \beta + e_3 + w_2 + 2e_1 - 3w_3.$$  

(5.76)

Then we get

$$\dot{V}_3 = -e_1^2 - w_2^2 - w_3^2.$$  

(5.77)

Thus, $\dot{V}_3$ is a negative definite function.
Thus by a Lyapunov stability theory (Hahn, 1967), the error dynamics (5.72) is globally asymptotically stable and satisfied for all initial conditions.

Hence, the states of the master and slave systems are globally and exponentially synchronized.

**Theorem 5.5.1.** The WINDMI chaotic system (5.55) and Coullet chaotic system (5.56) are globally and asymptotically synchronized with backstepping control

\[ u = \alpha y_3 - cx_3 + y_2 - bx_2 + ax_1 - x_1^3 - \beta + e^{y_1} + w_2 + 2e_1 - 3w_3. \]  

(5.78)

5.5.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the differential equations (5.55) and (5.56) with the backstepping control \( u \).

The parameters of the master system (5.55) are taken in the case of chaotic as

\[ a = 0.7, \quad b = 2.5, \]

and the slave system (5.56) are taken in the case of chaotic as

\[ \alpha = 5.5, \quad \beta = 3.5, \quad \gamma = 1. \]

The initial values of the master system (5.55) are chosen as

\[ x_1(0) = 0.341, \quad x_2(0) = 0.598, \quad x_3(0) = 0.928, \]

and slave system (5.56) are chosen as

\[ y_1(0) = 792, \quad y_2(0) = 0.734, \quad y_3(0) = 0.253. \]

Figure 5.5 depicts the synchronization of WINDMI system (5.55) and Coullet chaotic system (5.56).

Figure 5.6 depicts the synchronization error of WINDMI system (5.55) and Coullet chaotic system (5.56).

5.6 Synchronization of identical \( n \)-scroll Chua systems using Backstepping Control Design with Recursive Feedback

In this section, the backstepping control design with recursive feedback is applied for the synchronization of identical \( n \)-scroll Chua systems.
The $n$-scroll Chua system (Tang et al., 2001) is taken as the master system, which is described by
\begin{align}
\dot{x}_1 &= \alpha(x_2 - f(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2,
\end{align}
(5.79)
where $x_1, x_2, x_3$ are state variables, $\alpha, \beta$ are positive real constants and $f(x_1)$ is given by
\begin{align}
f(x_1) &= \begin{cases}
\frac{bn}{2a}(x_1 - 2ac) & \text{if } x_1 \geq 2ac, \\
-b \sin\left(\frac{\pi x_1}{2a} + d\right) & \text{if } -2ac \leq x_1 \leq 2ac, \\
\frac{bn}{2a}(x_1 + 2ac) & \text{if } x_1 \leq -2ac,
\end{cases}
\end{align}
(5.80)
a, b, c and $d$ are positive constants.

The $n$-scroll Chua system is also taken as the slave system, which is described by
\begin{align}
\dot{y}_1 &= \alpha(y_2 - f(y_1)) + u_1, \\
\dot{y}_2 &= y_1 - y_2 + y_3 + u_2, \\
\dot{y}_3 &= -\beta y_2 + u_3,
\end{align}
(5.81)
where $f(y_1)$ is given by
\begin{align}
f(y_1) &= \begin{cases}
\frac{bn}{2a}(y_1 - 2ac) & \text{if } y_1 \geq 2ac, \\
-b \sin\left(\frac{\pi y_1}{2a} + d\right) & \text{if } -2ac \leq y_1 \leq 2ac, \\
\frac{bn}{2a}(y_1 + 2ac) & \text{if } y_1 \leq -2ac,
\end{cases}
\end{align}
(5.82)
and $u = [u_1, u_2, u_3]^T$ is the backstepping controller with recursive feedback to be designed so as to synchronize the states of identical $n$-scroll Chua systems (5.79) and (5.81).

The synchronization error is defined by
\begin{align}
e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3.
\end{align}
(5.83)

The error dynamics is obtained as
\begin{align}
\dot{e}_1 &= \alpha e_2 - \alpha [f(y_1) - f(x_1)] + u_1 \\
\dot{e}_2 &= e_1 - e_2 + e_3 + u_2 \\
\dot{e}_3 &= -\beta e_2 + u_3
\end{align}
(5.84)

Now the objective is to find the control laws $u_i, i = 1, 2, 3$ for stabilizing the error system (5.83) at the origin.

First we consider the stability of the system
\begin{align}
\dot{e}_3 &= -\beta e_2 + u_3,
\end{align}
(5.85)
where $e_2$ is regarded as virtual controller.

We consider the Lyapunov function defined by

$$V_1(e_3) = \frac{1}{2}e_3^2.$$  \hspace{1cm} (5.86)

The derivative of $V_1$ is obtained as

$$\dot{V}_1 = e_3\dot{e}_3 = -\beta e_3 e_2 + e_3 u_3.$$  \hspace{1cm} (5.87)

Assume the controller $e_2 = \alpha_1(e_3)$.

If we choose

$$\alpha_1(e_3) = e_3 \text{ and } u_3 = 0,$$  \hspace{1cm} (5.88)

then

$$\dot{V}_1 = -\beta e_3^2,$$  \hspace{1cm} (5.89)

which is negative definite function. Hence, the system (5.85) asymptotically stable.

The function $\alpha_1(e_3)$ is an estimative function when $e_2$ is considered as a controller.

The error between $e_2$ and $\alpha_1(e_3)$ is

$$w_2 = e_2 - \alpha_1(e_3).$$  \hspace{1cm} (5.90)

Consider the $(e_3, w_2)$ subsystem given by

$$\begin{align*}
\dot{e}_3 &= -\beta w_2 - \beta e_3, \\
\dot{w}_2 &= e_1 - (1 - \beta)e_2 + e_3 + u_2.
\end{align*}$$  \hspace{1cm} (5.91)

Let $e_1$ be a virtual controller in system (5.91).

Assume that when $e_1 = \alpha_2(e_3, w_2)$, the system (5.91) is made globally asymptotically stable.

Consider the Lyapunov function defined by

$$V_2(e_3, w_2) = V_1(e_3) + \frac{1}{2}w_2^2.$$  \hspace{1cm} (5.92)

The derivative of $V_2(e_3, w_2)$ is

$$\dot{V}_2 = \dot{V}_1 + w_2\dot{w}_2 = -\beta e_3 e_2 + w_2[e_1 - (1 - \beta)e_2 + e_3 + u_2].$$  \hspace{1cm} (5.93)

Substituting for $e_2$ from (5.90) into (5.94) and simplifying, we get

$$\dot{V}_2 = -\beta e_3^2 + w_2[e_1 - (1 - \beta)w_2 + u_2].$$  \hspace{1cm} (5.94)
Assume the virtual controller \( e_1 = \alpha_2(e_3, w_2) \).

We choose
\[
\alpha_2(e_1) = 0, \text{ and } u_2 = 0.
\] (5.95)

Then it follow that
\[
\dot{V}_2 = -\beta e_3^2 - w_2^2.
\] (5.96)

Thus, \( \dot{V}_2 \) is a negative definite function and hence the system (5.91) globally asymptotically stable.

The function \( \alpha_2(e_3, w_2) \) is an estimative function when \( e_1 \) is considered as a controller.

The error between \( e_1 \) and \( \alpha_2(e_3, w_2) \) is
\[
w_3 = e_1 - \alpha_2(e_3, w_2)
\] (5.97)

Consider \((e_3, w_2, w_3)\) system given by
\[
\begin{align*}
\dot{e}_3 &= -\beta w_2 - \beta e_3, \\
\dot{w}_2 &= e_1 - (1 - \beta)e_2 + \beta e_3, \\
\dot{w}_3 &= \alpha w_2 + \alpha e_3 - \alpha(f(y_1) - f(x_1)) + \beta w_2 - \beta w_2 + \beta^2 e_3 + u_1,
\end{align*}
\] (5.98)

which can be expressed as
\[
\begin{align*}
\dot{e}_3 &= -\beta w_2 - \beta e_3, \\
\dot{w}_2 &= w_3 - w_2 + \beta e_3, \\
\dot{w}_3 &= \alpha w_2 + \alpha e_3 - \alpha(f(y_1) - f(x_1)) + \beta w_2 - \beta w_2 + \beta^2 e_3 + u_1,
\end{align*}
\] (5.99)

Consider the Lyapunov function defined by
\[
V_3(e_3, w_2, w_3) = V_2(e_3, w_2) + \frac{1}{2}w_3^2.
\] (5.100)

The derivative of \( V_3(e_3, w_2, w_3) \) is
\[
\dot{V}_3(e_3, w_2, w_3) = \dot{V}_2(e_3, w_2) + \dot{w}_3w_3
\] (5.101)

i.e.
\[
\dot{V}_3 = -\beta e_3^2 - w_2[e_1 - (1 - \beta)w_2] + w_3[\alpha w_2 + \alpha e_3 - \alpha(f(y_1) - f(x_1)) + \beta w_3 - \beta w_2 + \beta^2 e_3 + u_1].
\] (5.102)

Substituting for \( e_1 \) from (5.97) into (5.102) and simplifying, we get
\[
\dot{V}_3 = -\beta e_3^2 - w_2^2 + w_3[(1 + \alpha - \beta)w_2 + (\alpha + \beta^2)e_3 - \alpha(f(y_1) - f(x_1)) + \beta w_3 + u_1].
\] (5.103)
We choose

\[ u_1 = -2\beta w_3 - (1 + \alpha - \beta)w_2 - (\alpha + \beta^2)e_3 + \alpha(f(y_1) - f(x_1)). \]  \tag{5.104}

Then we obtain

\[ \dot{V}_3 = -\beta e_3^2 - w_2^2 - \beta w_3^2. \]  \tag{5.105}

Thus, \( \dot{V}_3 \) is a negative definite function on \( \mathbb{R}^3 \). Thus, by Lyapunov stability theory (Hahn, 1967), the error dynamics (5.99) is globally asymptotically stable for all initial conditions \( e(0) \in \mathbb{R}^3 \).

Thus, the states of the master and slave systems are globally and asymptotically synchronized.

**Theorem 5.6.1.** The identical \( n \)-scroll Chua’s systems (5.79) and (5.81) are globally and exponentially synchronized for any initial conditions with the backstepping controls with recursive feedback \( u \) defined by

\[
\begin{align*}
  u_1 &= -2\beta w_3 - (1 + \alpha - \beta)w_2 - (\alpha + \beta^2)e_3 + \alpha(f(y_1) - f(x_1)), \\
  u_2 &= u_3 = 0.
\end{align*}
\]  \tag{5.106}

### 5.6.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the system of differential equations (5.79) and (5.81) with the backstepping controls \( u_1, u_2 \) and \( u_3 \) given by (5.106).

The parameters of the systems (5.79) and (5.81) are taken in the case of chaotic case as

\[
\alpha = 10.814, \quad \beta = 14.0, \quad a = 1.3, \quad b = 0.11, \quad c = 3, \quad d = 0.
\]

The initial values of the master system (5.79) are chosen as

\[
x_1(0) = 0.125, \quad x_2(0) = 0.625, \quad x_3(0) = 0.941.
\]

The initial value of the slave system (5.81) are chosen as

\[
y_1(0) = 0.321, \quad y_2(0) = 0.487, \quad y_3(0) = 0.965.
\]

Figure 5.14 describes the chaos synchronization of identical \( n \)-scroll Chua’s systems (5.79) and (5.81).

Figure 5.15 describes the chaos synchronization error between identical \( n \)-scroll Chua’s systems (5.79) and (5.81).
5.7 Synchronization of identical Lur’e Systems using Backstepping Control Design with Recursive Feedback

In this section, the backstepping control design with recursive feedback method is applied for the synchronization of identical Lur’e (Sukens and Vandewalle, 1997) chaotic systems.

The Lur’e system is taken as master system, which is described by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= ax_1 + bx_2 + cx_3 + 12\phi(x_1),
\end{align*}
\] (5.107)

where \(x_1, x_2, x_3\) are state variables, \(a, b, c\) are constants and \(\phi(x_1)\) is given by

\[
\phi(x_1) = \begin{cases} 
  kx_1 & \text{if } |x_1| \geq \frac{1}{k}, \\
  \operatorname{sign}(x_1) & \text{otherwise}.
\end{cases}
\] (5.108)

The Lur’e system is also taken as the slave system, which is described by

\[
\begin{align*}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= y_3 + u_2, \\
\dot{y}_3 &= ey_1 + fy_2 + gy_3 + 12\phi(y_1) + u_3,
\end{align*}
\] (5.109)

where \(\phi(y_1)\) is given by

\[
\phi(y_1) = \begin{cases} 
  ky_1 & \text{if } |y_1| \geq \frac{1}{k}, \\
  \operatorname{sign}(y_1) & \text{otherwise},
\end{cases}
\] (5.110)

and \(u = [u_1, u_2, u_3]^T\) is the backstepping controller with recursive feedback to be designed so as to synchronize the states of identical Lur’e systems (5.107) and (5.109).

The synchronization error is defined by

\[
e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3.
\] (5.111)

The error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= e_2 + u_1, \\
\dot{e}_2 &= e_3 + u_2, \\
\dot{e}_3 &= ae_1 + be_2 + ce_3 + 12[\phi(y_1) - \phi(x_1)] + u_3.
\end{align*}
\] (5.112)

Now the objective is to find the control laws \(u_i, i = 1, 2, 3\) for stabilizing the error system (5.112) at the origin.
First we consider the stability of the system

\[ \dot{e}_3 = -\beta e_2 + u_1, \quad (5.113) \]

where \( e_2 \) is regarded as virtual controller.

We consider the Lyapunov function defined by

\[ V_1(e_3) = \frac{1}{2} e_2^2. \quad (5.114) \]

The derivative of \( V_1 \) is obtained as

\[ \dot{V}_1 = e_1 (e_2 + u_1). \quad (5.115) \]

Assume the controller \( e_2 = \alpha_1(e_1) \).

If we choose

\[ \alpha_1(e_1) = -e_1, \text{ and } u_1 = 0, \quad (5.116) \]

then

\[ \dot{V}_1 = -\beta e_1^2, \quad (5.117) \]

which is a negative definite function.

Hence the system (5.113) is globally asymptotically stable.

The function \( \alpha_1(e_1) \) is an estimative function when \( e_2 \) is considered as a controller.

The error between \( e_2 \) and \( \alpha_1(e_1) \) is

\[ w_2 = e_2 - \alpha_1(e_1) = e_2 + e_1 \quad (5.118) \]

Consider the \((e_1, w_2)\) subsystem given by

\[ \begin{aligned}
\dot{e}_1 &= w_2 - e_1, \\
\dot{w}_2 &= e_2 + e_3 + u_2.
\end{aligned} \quad (5.119) \]

Let \( e_3 \) be a virtual controller in system (5.119).

Assume that when \( e_3 = \alpha_2(e_1, w_2) \), the system (5.119) is made globally asymptotically stable.

Consider the Lyapunov function defined by

\[ V_2(e_1, w_2) = V_1(e_1) + \frac{1}{2} w_2^2 \quad (5.120) \]

The derivative of \( V_2(e_1, w_2) \) is obtained as

\[ \dot{V}_2 = \dot{V}_1 + w_2 \dot{w}_2 = e_1 e_2 + w_2 (e_2 + e_3 + u_2). \quad (5.121) \]
Substituting for $e_2$ from (5.118) into (5.121) and simplifying, we get

\[ \dot{V}_2 = -e_1^2 + w_2(e_2 + e_3 + u_2) \]  

(5.122)

Assume the virtual controller $e_3 = \alpha_2(e_1, w_2)$.

We choose

\[ \alpha_2(e_1) = -2w_2, \text{ and } u_2 = 0. \]  

(5.123)

Then it follows that

\[ \dot{V}_2 = -e_1^2 - w_2^2. \]  

(5.124)

Thus, $\dot{V}_2$ is a negative definite function and hence the system (5.119) is globally asymptotically stable.

The error between $e_3$ and $\alpha_2(e_1, w_2)$ is

\[ w_3 = e_3 - \alpha_2(e_1, w_2) = e_3 + 2w_2. \]  

(5.125)

Consider the $(e_1, w_2, w_3)$ system given by

\[
\begin{align*}
\dot{e}_1 &= w_2 - e_1, \\
\dot{w}_2 &= w_3 - w_2 - e_1, \\
\dot{w}_3 &= (a - b - 2)e_1 + (b - 2 - 2c)w_2 + (2 + c)w_3 \\
&\quad + 12[\phi(y_1) - \phi(x_1)] + u_3.
\end{align*}
\]  

(5.126)

Consider the Lyapunov function defined by

\[ V_3(e_1, w_2, w_3) = V_2(e_1, w_2) + \frac{1}{2}w_3^2. \]  

(5.127)

The derivative of $V_3(e_1, w_2, w_3)$ is obtained as

\[ \dot{V}_3(e_1, w_2, w_3) = \dot{V}_2(e_1, w_2) + w_3\dot{w}_3 \]  

(5.128)

i.e.

\[ \dot{V}_3 = -e_1^2 - w_2^2 + w_3[(a - b - 2)e_1 + (b - 2c - 1)w_2 + (2 + c)w_3 + 12(\phi(y_1) - \phi(x_1)) + u_3]. \]  

(5.129)

We choose the control $u_3$ as follows.

\[ u_3 = -(a - b - 2)e_1 - (b - 2 - 2c + 1)w_2 - 2(3 + c)w_3 - 12(\phi(y_1) - \phi(x_1)). \]  

(5.130)

Substituting for $u_3$ from (5.130) into (5.129), we get

\[ \dot{V}_3 = -\beta e_3^2 - w_2^2 - w_3^2. \]  

(5.131)
Thus, $\dot{V}_3$ is a negative definite function on $\mathbb{R}^3$.

Thus, by Lyapunov stability theory (Hahn, 1967), the error dynamics (5.126) is globally asymptotically stable for all initial conditions $e(0) \in \mathbb{R}^3$.

Hence, the states of the master and slave systems are globally and asymptotically synchronized.

**Theorem 5.7.1.** The identical Lur’e systems (5.107) and (5.109) are globally and asymptotically synchronized for any initial conditions with the backstepping controls with recursive feedback $u$ defined by

\[
\begin{align*}
    u_1 &= u_2 = 0, \\
    u_3 &= -(a - b - 2)e_1 - (b - 2 - 2c + 1)w_2 - 2(2 + c)w_3 - 12(\phi(y_1) - \phi(x_1)).
\end{align*}
\]  

(5.132)

5.7.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the differential equations (5.107) and (5.109) with the backstepping controls $u_1$, $u_2$ and $u_3$ given by (5.132).

The parameters of the systems (5.107) and (5.109) are taken in the chaotic case as

\[ a = -7.4, \quad b = -4.1, \quad c = -1, \quad k = 3.6. \]

The initial values of the master system (5.107) are chosen as

\[ x_1(0) = 1.125, \quad x_2(0) = 1.025, \quad x_3(0) = 1.941. \]

The initial values of the slave system (5.109) are chosen as

\[ y_1(0) = 1.253, \quad y_2(0) = 1.558, \quad y_3(0) = 1.756. \]

Figure 5.16 depicts the synchronization of identical Lur’e systems (5.107) and (5.109).

Figure 5.17 depicts the synchronization error between identical Lur’e systems (5.107) and (5.109).

5.8 Synchronization of n-scroll Chua System and Lur’e system using Backstepping Control Design with Recursive Feedback

In this section, the backstepping control design with recursive feedback function is applied for the synchronization of two different chaotic systems described by $n$-scroll Chua system (Tang
et al., 2001) as master system and the Lur’è (Sukens and Vandewalle, 1997) system as the slave system.

The dynamics of the \( n \)-scroll Chua system, taken as master system, is described by

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - f(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2,
\end{align*}
\tag{5.133}
\]

where \( x_1, x_2, x_3 \) are state variables, \( \alpha, \beta \) are positive constants and \( f(x_1) \) is given by

\[
f(x_1) = \begin{cases} 
\frac{bn}{2a}(x_1 - 2ac) & \text{if } x_1 \geq 2ac, \\
-b\sin\left(\frac{\pi x_1}{2a} + d\right) & \text{if } -2ac \leq x_1 \leq 2ac, \\
\frac{bn}{2a}(x_1 + 2ac) & \text{if } x_1 \leq -2ac,
\end{cases}
\tag{5.134}
\]

where \( a, b, c \) and \( d \) are positive constants.

The dynamics of the Lur’è system taken as the slave system, is described by

\[
\begin{align*}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= y_3 + u_2, \\
\dot{y}_3 &= \delta y_1 + \eta y_2 + \theta y_3 + 12\phi(y_1) + u_3,
\end{align*}
\tag{5.135}
\]

where \( y_1, y_2, y_3 \) are state variables, \( \delta, \eta, \theta \) are real constants and \( \phi(y_1) \) is given by

\[
\phi(y_1) = \begin{cases} k y_1 & \text{if } |y_1| \geq \frac{1}{k}, \\
\text{sign}(y_1) & \text{if otherwise},
\end{cases}
\tag{5.136}
\]

where \( k \) is a positive constant and \( u = [u_1, u_2, u_3]^T \) is the backstepping controller with recursive feedback to be designed so as to synchronize the states of the system (5.133) and (5.135).

The synchronization error is defined by

\[
e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3.
\tag{5.137}
\]

The error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= e_2 + (1 - \alpha)x_2 + \alpha f(x_1) + u_1, \\
\dot{e}_2 &= e_3 - x_1 + x_2 + u_2, \\
\dot{e}_3 &= \delta y_1 + \eta y_2 + \theta y_3 + 12\phi(y_1) + \beta x_2 + u_3.
\end{align*}
\tag{5.138}
\]

Now the objective is to find control laws \( u_i, \ i = 1, 2, 3 \), for stabilizing the error variables of the system (5.138) at the origin.
First we consider the stability of the system

\[ \dot{e}_1 = e_2 + (1 - \alpha)x_2 + \alpha f(x_1) + u_1, \]  

(5.139)

where \( e_2 \) is regarded as virtual controller.

We consider the Lyapunov function defined by

\[ V_1(e_1) = \frac{1}{2}e_1^2 \]  

(5.140)

The derivative of \( V_1 \) is obtained as

\[ \dot{V}_1 = e_1(e_2 + (1 - \alpha)x_2 + \alpha f(x_1) + u_1) \]  

(5.141)

Assume the controller \( e_2 = \alpha_1(e_1) \).

We choose

\[ \alpha_1(e_1) = 0, \text{ and } u_1 = -2e_2 - (1 - \alpha)x_2 - \alpha f(x_1) - e_1. \]  

(5.142)

Then, we get

\[ \dot{V}_1 = -e_1^2, \]  

(5.143)

which is a negative definite function. Hence, the system (5.139) is globally asymptotically stable.

The function \( \alpha_1(e_1) \) is an estimative function when \( e_2 \) is considered as a controller.

The error between \( e_2 \) and \( \alpha_1(e_1) \) is

\[ w_2 = e_2 - \alpha_1(e_1) = e_2 \]  

(5.144)

Consider \((e_1, w_2)\) system given by

\[ \begin{align*}
\dot{e}_1 &= -e_1 - w_2, \\
\dot{w}_2 &= e_3 - x_1 + x_2 + u_2.
\end{align*} \]  

(5.145)

Let \( e_3 \) be a virtual controller in system (5.145).

Assume that when \( e_3 = \alpha_2(e_3, w_2) \), the system (5.145) is made globally asymptotically stable.

Consider the Lyapunov function defined by

\[ V_2(e_1, w_2) = V_1(e_1) + \frac{1}{2}w_2^2 \]  

(5.146)

The derivative of \( V_2(e_1, w_2) \) is obtained as

\[ \dot{V}_2 = \dot{v}_1 + w_2\dot{w}_2 = e_1(-e_2 - e_1) + w_2(e_3 - x_1 + x_2 + u_2). \]  

(5.147)
Substituting for $e_2$ from (5.144 into (5.147) and simplifying, we get

$$\dot{V}_2 = -e_1^2 + w_2(-e_1 + e_3 - x_1 + x_2 + u_2). \quad (5.148)$$

We choose

$$\alpha_2(e_1, w_2) = 0, \text{ and } u_2 = e_1 + x_1 - x_2 - w_2. \quad (5.149)$$

Then we get

$$\dot{V}_2 = -e_1^2 - w_2^2. \quad (5.150)$$

Then $\dot{V}_2$ is a negative definite function and hence the system (5.145) is globally asymptotically stable.

Define the error variable $w_3$ as

$$w_3 = e_3 - \alpha_2(e_1, w_2) \quad (5.151)$$

Consider $(e_1, w_2, w_3)$ system given by

$$\begin{align*}
\dot{e}_3 &= -e_1 - w_2, \\
\dot{w}_2 &= e_3 + e_1 - w_2, \\
\dot{w}_3 &= \delta y_1 + \eta y_2 + \theta y_3 + 12\phi(y_1) + \beta x_2 + u_3.
\end{align*} \quad (5.152)$$

Consider the Lyapunov function defined by

$$V_3(e_3, w_2, w_3) = V_2(e_1, w_2) + \frac{1}{2}w_3^2, \quad (5.153)$$

The derivative of $V_3(e_3, w_2, w_3)$ is obtained as

$$\dot{V}_3 = \dot{V}_3(e_3, w_2, w_3) + w_3\dot{w}_3, \quad (5.154)$$

i.e.

$$\dot{V}_3 = -e_1^2 + w_2(e_3 - w_2) + w_3[w_2 + \delta y_1 + \eta y_2 + \theta y_3 + 12\phi(y_1) + \beta x_2 + u_3]. \quad (5.155)$$

We choose

$$u_3 = -w_2 - \delta y_1 - \eta y_2 - \theta y_3 - 12\phi(y_1) - \beta x_2 - w_3. \quad (5.156)$$

Then we get

$$\dot{V}_3 = -e_1^2 - w_2^2 - w_3^2. \quad (5.157)$$

Thus, $\dot{V}_3$ is a negative definite function on $\mathbb{R}^3$.

Thus, by Lyapunov stability theory (Hahn, 1967), the error dynamics (5.152) is globally asymptotically stable for all initial conditions $e(0) \in \mathbb{R}^3$.
Theorem 5.8.1. The $n$-scroll Chua’s system (5.133) and Lur’e chaotic system (5.135) are globally and asymptotically synchronized with the backstepping controls design with recursive feedback inputs

\begin{align}
    u_1 &= -2e_2 - (1 - \alpha)x_2 - \alpha f(x_1) - e_1, \\
    u_2 &= e_1 + x_1 - x_2 - w_2, \\
    u_3 &= -w_2 - \delta y_1 - \eta y_2 - \theta y_3 - 12\phi(y_1) - \beta x_2 - w_3.
\end{align}

(5.158)

5.8.1 Numerical Simulation

For the numerical simulations, the fourth order Runge-Kutta method is used to solve the differential equations (5.133) and (5.135) with the backstepping controls $u_1$, $u_2$ and $u_3$ given by (5.158).

The parameters of the systems (5.133) and (5.135) are taken in the chaotic case as

$\alpha = 10.814$, $\beta = 14.0$, $a = 1.3$, $b = 0.11$, $c = 3$, $d = 0$

and

$\delta = -7.4$, $\eta = -4.1$, $\theta = -1$, $k = 3.6$.

The initial values of the master system (5.133) are chosen as

$x_1(0) = .125$, $x_2(0) = .025$, $x_3(0) = .941$.

The initial values of slave system (5.135) are chosen as

$y_1(0) = 0.253$, $y_2(0) = 0.558$, $y_3(0) = 0.756$.

Figure 5.18 depicts the synchronization of $n$-scroll Chua system (5.133) and Lur’e system (5.135).

5.9 Summary

In this chapter, backstepping control method has been applied to achieve global chaos synchronization for $n$-scroll Chua and Lur’e chaotic systems. Since the Lyapunov exponents are not required for these calculations, the backstepping method is very effective and convenient to achieve global chaos synchronization for $n$-scroll Chua and Lur’e systems. Numerical simulations have been given to validate and illustrate the effectiveness of the backstepping control based synchronization schemes of the $n$-scroll Chua and Lur’e chaotic systems.
Figure 5.1: Synchronization of Identical WINDMI Chaotic Systems

Figure 5.2: Synchronization Error between Identical WINDMI Chaotic Systems
Figure 5.3: Synchronization of Identical Coullet Chaotic Systems

Figure 5.4: Synchronization Error between Identical Coullet Chaotic Systems
Figure 5.5: Synchronization of WINDMI and Couillet Systems

Figure 5.6: Synchronization error between WINDMI and Couillet Systems
Figure 5.7: Chaotic Portrait of 2-Scroll Chua System

Figure 5.8: Chaotic Portrait of 3-Scroll Chua System
Figure 5.9: Chaotic Portrait of the 4-scroll Chua System

Figure 5.10: Chaotic Portrait of Lur’e System
Figure 5.11: Chaotic Portrait of Lur’e System

Figure 5.12: Chaotic Portrait of Lur’e System
Figure 5.13: Chaotic Portrait of Lur’e System

Figure 5.14: Synchronization of Identical $n$-scroll Chua’s Systems
Figure 5.15: Synchronization Error between Identical \( n \)-scroll Chua’s Systems

Figure 5.16: Synchronization of Identical Lur’e Systems
Figure 5.17: Synchronization Error between Identical Lur’e Systems

Figure 5.18: Synchronization of \(n\)-scroll Chua and Lur’e Systems