CHAPTER - V

SEMIMODULES OVER SEMIRINGS

In this chapter we study semimodules over hemirings and also over additive inverse semirings. Moreover, we study exact sequences of semimodules over hemirings and AC - congruence free semidules.

1. Semimodules over hemirings

Definition 1.1. Let S be a hemiring. A left S-semimodule M is a commutative additive semigroup (M, +) which has a zero element o, together with an operation $S \times M \longrightarrow M$, denoted by $(a, x) \longrightarrow ax$ such that for all $a, b \in S$ and $x, y \in M$,

(i) $a(x + y) = ax + ay$ (ii) $(a + b)x = ax + bx$

(iii) $(ab)x = a(bx)$ (iv) $0x = o = ao$.

If $1 \in S$ and also $1x = x$ for all $x \in M$, M is said to be a unitary left S - semimodule.

A right S - semimodule is defined in an analogous manner.

J. S Golan [16] studied S-semimodules on the assumption that S contains 1. But in our study S may not contain 1.

Every left (right) ideal of a hemiring S is a left(right) semimodule over S.

In this section M denotes a left S-semimodule. Henceforth we confine to the study of left S-semimodules M. The corresponding

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results for right S-semimodules will be assumed without explicit mention.

Definition 1.2. A subsemimodule of M is a non-empty subset A of M such that

(i) $x + y \in A$ for all $x, y \in A$.

and (ii) $ax \in A$ for all $a \in S$ and for all $x \in A$.

Definition 1.3. : A subsemimodule A of M is said to be a k-subsemimodule of M iff $a, a + b \in A$ and $b \in M$ implies $b \in A$.

Proposition 1.4. For each subsemimodule A of M, the k-closure $\bar{A}$ of A defined by

$$\bar{A} = \{x \in M : x + y \in A \text{ for some } y \in A\}$$ is a k-subsemimodule of M.

Proof. Let $a, b \in \bar{A}$. Then there exist $x, y \in A$ such that $a + x \in A$, $b + y \in A$. Now $a + x + b + y = (a + b) + (x + y) \in A$ where $x + y \in A$. Hence $a + b \in \bar{A}$.

Again for $r \in S$, $r(a + x) \in A$. Then $ra + rx \in A$ where $ra \in M$ and $rx \in A$. Hence $ra \in \bar{A}$.

Consequently, $\bar{A}$ is a subsemimodule of M.

Next let, $a, a+b \in \bar{A}$. Then $a + x, a + b + y \in A$ for some $x, y \in A$. Now $b + (a + x) + y = (a + b + y) + x \in A$. This implies $b \in \bar{A}$. Consequently, $\bar{A}$ is a k-subsemimodule of M.

The k-closure of a subsemimodule satisfies the following properties.

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Theorem 1.5. Let M be a left S-semimodule. Then

(i) $A \subseteq \bar{A}$ for any subsemimodule A of M;

(ii) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$ for subsemimodules A and B of M ;

(iii) $A = \bar{A}$ iff A is a k-subsemimodule of M;

(iv) $\bar{A} = \bar{\bar{A}}$ for any subsemimodule A of M;

(v) $\bar{A}$ is the smallest k-subsemimodule containing A.

Proof. Similar to the proof of theorem 1.4 of Chapter I.

Proposition 1.6. Let A and B be subsemimodules of M.

Then $\bar{A} + \bar{B} \subseteq \overline{A + B}$.

Proof: Let $x \in \bar{A} + \bar{B}$. Then $x = y + t$ for $y \in \bar{A}$ and $t \in \bar{B}$. Hence there exist $y_1 \in A$ and $t_1 \in B$ such that $y + y_1 = y_2$ and $t + t_1 = t_2$. Consequently $x + y_1 + t_1 = y + y_1 + t + t_1 = \overline{y_2 + t_2}$ yields $x \in \overline{A + B}$. Hence $\bar{A} + \bar{B} \subseteq \overline{A + B}$.

Note that $\{o\}$ and M are subsemimodules of M. They are called trivial subsemimodules. These are also k-subsemimodules of M called trivial k-subsemimodules of M.

Definition 1.7. A left S-semimodule M is called k-simple iff

(i) $SM \neq \{o\}$

and (ii) M has only trivial k-subsemimodules.

Theorem 1.8. A non-zero left S-semimodule M is k-simple iff $\overline{Sa} = M$ for all $o \neq a \in M$.

Proof. Suppose $\overline{Sa} = M$ for all $o \neq a \in M$. Then clearly, $SM \neq \{o\}$. Let A be a left k-subsemimodule of M. Suppose $A \neq \{o\}$ and $o \neq a \in A$.

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Then $Sa \subseteq A$. Hence $Sa \subseteq \bar{A} = A$. But $\overline{Sa} = M$. Consequently, $M \subseteq A$.

This implies that $A = M$. As a result, M is k-simple.

Conversely, assume that M is k-simple. Define

$T = \{x \in M : Sx = o\}$ Then $T \subseteq M$ and T is a k-subsemimodule

of M. We claim that $T = \{o\}$.

If $T \neq \{o\}$ then $T = M$. Hence $SM = \{o\}$, a contradiction.

Consequently, $T = \{o\}$. This shows that $Sa \neq \{o\}$ for any $o \neq a \in M$.

Hence $\overline{Sa} \neq \{o\}$. But $\overline{Sa}$ is a left k-subsemimodule of M. Consequently,

$\overline{Sa} = M$ for all $o \neq a \in M$.

Proposition 1.9. If M is a k-simple left S-semimodule, then

$I_a = \{x \in S : xa = o\}$ forms a left k-ideal of S for

each $o \neq a \in M$.

Proof: Let $o \neq a \in M$. Hence by theorem 1.8, $M = \overline{Sa}$.

Since $o \in I_a$, $I_a \neq \phi$. Let $x, y \in I_a$. Then $xa = o$ and $ya = o$

and hence $(x + y)a = o$ yields $x + y \in I_a$. Also for $r \in S$, $rx \in I_a$

for all $r \in S$. Hence I_a is a left ideal of S. Next let, $x, x + y \in I_a$

Then $(x + y)a = o$ and $xa = o$. Hence $y \in I_a$. Consequently I_a is a left

k-ideal of S.

Definition 1.10. Let M and N be left S-semimodules.

A homomorphism (more precisely, semimodule homomorphism) from M to

N is a map

$f : M \longrightarrow N$ such that

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(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$

(ii) $f(am) = a f(m)$

for all $m, m_1, m_2 \in M$ and all $a \in S$.

Kernel of f denoted by kerf is defined by

$\ker f = \{m \in M : f(m) = o\}$.

Since $f(om) = o f(m) = o$, $o \in$ kerf.

Hence kerf $\neq \phi$. ker f is a k-subsemimodule of M.

$f(M) = \{b \in N : b = f(a)$ for some $a \in M\}$ is a subsemimodule of N.

A homomorphism f is called a monomorphism or an epimorphism according as f is injective or surjective and an isomorphism iff f is both a monomorphism and an epimorphism.

Definition 1.11. Let M be a left S-semimodule. An equivalence relation ρ on M is said to be a congruence relation iff $(a, b) \in \rho$ implies $(a+c, b+c) \in \rho$ for all $c \in M$ and $(ra, rb) \in \rho$ for all $r \in S$.

Let M be a left S-semimodule and I a subsemimodule of M. Define the Bourne relation ρ (cf. [8]) on M by $\rho = \{(x,y) \in M \times M : x + i = y + j$ for some $i, j \in I\}$. Then ρ is a congruence relation on the left S - semimodule M. Hence M/ρ can be made a left S-semimodule under $\oplus$ and $\odot$ given by

$$x\rho \oplus y\rho = (x + y)\rho \text{ and } a \odot x\rho = (ax)\rho.$$

This left S-semimodule is called the quotient semimodule of M modulo I and is denoted by M/I.

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We simply write $a(x\rho)$ in place of $a \odot x\rho$ and $(M/\rho, +)$ in place of $(M/\rho, \oplus)$ in our future discussion.

2. Semimodules over additive inverse semirings.

Let R be an additive inverse semiring and M a left R-semimodule. The zeroid $Z(M)$ of M is defined by $Z(M) = \{a \in M : a + x = x$ for some $x \in M\}$.

Theorem 2.1. If M is a k-simple R-semimodule, then M is either a group or M is a zeroid.

Proof. Let $a \in M$ and $r \in R$. Now $r + r' + r = r$, where r' is the additive inverse of r in R. Then $r + r'$ is an additive idempotent. Hence $(r + r')a + (r + r')a$

$= (r + r' + r + r')a = (r + r')a$.

This shows that for every $r \in R$, $(r + r')a$ is an idempotent of M.

Let $T = \{a \in M : a + a = a\}$.

Then $T \neq \phi$ and T is a subsemimodule of M.

Since M is k-simple, either $T = \{o\}$ or $\overline{T} = M$.

Case 1. Suppose $\overline{T} = M$. Let $m \in M$. Then $m + x = y$ for some $x, y \in T$. As $(M, +)$ is commutative, $m + x + x + y = x + y + y$. Then $m + (x + y) = x + y$. This shows that $m \in Z(M)$ Hence $M \subseteq Z(M)$. As $Z(M) \subseteq M$, it follows that $M = Z(M)$.

Case 2. Suppose $T = \{o\}$. Now as $RM \neq \{o\}$ there exist $r \in R$ and $a \in M$ such that $ra \neq o$. But $(r + r')a \in T$. Hence $(r + r')a = o$

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As $a \in M = \overline{Ra}$, there exist $r, t \in R$ such that $a + ra = ta$. Hence $a + (r + r')a = ta + r'a$ implies $a = ta + r'a$. Then $a + (r + t')a = (t + t')a + (r + r')a = o$ shows that $(M, +)$ is an additive group.

Corollary 2.2 If $(M, +)$ is an additive cancellative k-simple R-semimodule, then $(M, +)$ is an additive abelian group.

Proof. It follows immediately as $Z(M) = \{o\}$

Corollary 2.3. If M is an additive cancellative k-simple semimodule over the ring Z of integers, then M is a cyclic group of prime order.

Proof is immediate, since every simple abelian group is of prime order.

Definition 2.4. A left k-ideal I in an additive inverse semiring R is said to be modular iff there exists an element $e \in R$ such that for every $r \in R$, $re + r' \in I$, where r' is the additive inverse of r in R.

Theorem 2.5. A cancellative left semimodule $(M, +)$ over an additive inverse semiring R is k-simple iff M is isomorphic to the R-semimodule R/I, where I is a modular maximal left k-ideal of R.

Proof. Suppose M is k-simple. Since $(M, +)$ is cancellative, it follows from corollary 2.2 that $(M, +)$ is an abelian group. Let $o \neq a \in M$. As M is k-simple, $\overline{Ra} = M$. Let $b \in M$. Then there exist $r_i \in R$ such that $b + r_1a = r_2a$. This implies that $b = r_2a - r_1a =$

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$r_2a + r_1'a$, since $(r_1' + r_1)\ a = o$. Consequently, $b = ta$ for some $t = r_2 + r_1' \in R$.

Hence $M = Ra$.

Define $f : R \longrightarrow M$ by $f(r) = ra$. Clearly, f is a homomorphism and ker f is a left k-ideal of R. Again $M = Ra$ implies that $a = ea$ for some $e \in R$.

Now for any $r \in R$, $(re + r')\ a = (rea + r'a)$ $= (ra + r'a) = o$ by theorem 2.1. Hence $re + r' \in \ker f = I$ (say) for all $r \in R$.

This implies that I is a modular left k-ideal of R. We claim that I is a maximal modular left k-ideal of R. Let J be a left k-ideal of R, such that $I \subsetneqq J$. Then there exist $r \in R$ such that $r \in J$ but $r \notin I$. Hence $ra \neq o$. Since M is k-simple, $M = R\ ra$. Then $a = tra$ for some $t \in R$. But $e = ea = tra$ implies $(tr + (tr)')a = o$. i.e. $ea + (tr)'a = o$ implies that $(e + (tr)'\ )\ a = o$. This shows that $e + (tr)' \in I$. Then $e + (tr)' \in J$. As J is a left k-ideal and $tr \in J$ and also $(tr)' \in J$, then $e \in J$.

Also $(r + r'e)\ a = ra + r'a = (r + r')a = o$ implies $r + r'e \in I \subsetneqq J$ for all $r \in R$. As J is a left k-ideal and $e \in J$, it follows that $r \in J$ for all $r \in R$. Consequently $J = R$ and hence I is a maximal modular left k-ideal.

Define $g : R / \ker f \longrightarrow M$ by

$g\ (r + \ker f) = f(r)$.

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Let $r + \ker f = t + \ker f$. Then $r + i_1 = t + i_2$ for some $i_1, i_2 \in \ker f = I$. Hence $ra + i_1a = ta + i_2a$ for all $a \in M$. But $i_1a = o = i_2a$ implies that $ra = ta$. Thus we find that g is well defined.

Clearly, g is an R-homomorphism.

Let $(r + \ker f) \in \ker g$. Then by definition of g,

$f(r) = o$ and $ra = o$. Hence $r \in \ker f$. Then $r + \ker f = \ker f$. This shows that g is a monomorphism.

Again as $M = Ra$, g is an epimorphism.

As a result g is an isomorphism.

Conversely, assume that I is a maximal modular left k-ideal of R

Now R/I is an R-semimodule. Let us show that R/I is k-simple.

For this, suppose $R(R/I) = \{0\}$. Then $r(t + I) = I$ for all $r \in R$.

This shows that $rt + I = I$. Hence $rt \in I$. As I is a modular left k-ideal, there exists $e \in R$ such that $r'e + r \in I$ for all $r \in R$.

Then $rt + r'et \in I$. As I is a left k-ideal and $rt \in I$ for all $r, t \in R$, $r'e \in I$. Again as I is a left k-ideal, then $r \in I$ for all $r \in R$. This contradicts the fact that I is a proper left k-ideal of R. Hence $R(R/I) \neq \{0\}$. Next let N be a k-subsemimodule of R/I.

Let $J = \{r \in R : r + I \in N\}$.

Now J is a left ideal of R such that $I \subseteq J$.

Let $r, r + t \in J$. Then $r + I \in N$ and $(r + I) + (t + I) \in N$. Since N is a k-subsemimodule of R/I, we find that $t + I \in N$. Hence $t \in J$.

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This implies that J is a left k-ideal. Since I is maximal we find that either J = I or J = R.

This shows that N is a trivial k-subsemimodule of R/I and hence R/I is k-simple.

3. AC-congruence free semimodules over hemirings.

Definition 3.1. A left S semimodule $M \neq \{o\}$ is said to satisfy condition (A) iff for any arbitrary fixed pair of elements $a, b \in M$ with $a \neq b$ and $x, y \in M$ there exist r_i (depending on x, y) $\in S$ such that $x + r_1 a + r_2 b = r_1 b + r_2 a + y$.

Proposition 3.2. If a left S-semimodule M satisfies condition(A) then S has only trivial k-subsemimodules.

Proof. Let $I \neq \{o\}$ be a k-subsemimodule of M.

Then there exists $o \neq a \in I$. By using condition (A) for the pair (a, o) we find that given $x, o \in M$ there exist $r_i \in S$ such that $x + r_1 a = r_2 a$. As $r_1 a$, $r_2 a \in I$ and I is a k-subsemimodule it follows that $x \in M$. Consequently, I = M.

Definition 3:3. A congruence ρ on a left S-semimodule M is called additively cancellative (AC) iff $(a + c, b + c) \in \rho$ implies $(a, b) \in \rho$ for some $a, b, c \in M$.

Lemma 3.4. A congruence ρ on a left S-semimodule M is AC iff $(a + c, b + d) \in \rho$ and $(c, d) \in \rho$ imply $(a, b) \in \rho$ for $a, b, c, d \in M$.

Proof. Let ρ be AC on M and $(a + c, b + d) \in \rho$ and $(c, d) \in \rho$ for

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some a, b, c, d $\in$ M. Then $(d,c) \in \rho$ and hence $(a + c + d, b + c + d) \in \rho$ implies $(a, b) \in \rho$.

Conversely, let ρ satisfy the given condition. Suppose $(a + c, b + c) \in \rho$ for some a, b, c $\in$ M. Now $(c, c) \in \rho$ for all c $\in$ M. Hence $(a, b) \in \rho$.

Note that identity congruence I_M and universal congruence M x M are AC - congruences on M and called trivial AC - congruences. Other AC - congruences (if they exist) on M are called proper or non-trivial AC - congruences.

Example 3.5: Let N be the set of all non-negative integers and N^- the set of all non-positive integers. Then N is a hemiring under usual addition and multiplication and N^- is a commutative semigroup with zero o under usual addition.

Now N^- is a semimodule over N under usual multiplication:

$N^- \times N \longrightarrow N^-$.

Clearly, N^- has no non-trivial subsemimodules but $\rho = \{(a,b) \in N^- \times N^- : ab > o\} \cup \{(o, o)\}$ is a non-trivial congruence.

Definition 3.6. Let ρ be a congruence on M. The closure $\bar{\rho}$ of ρ is defined by

$$\bar{\rho} = \{(a,b) \in M \times M : (a + x, b + x) \in \rho \text{ for some } x \in M\}.$$

Note that $\rho \subseteq \bar{\rho}$ for any congruence ρ on M.

Theorem 3.7. Let M be a left S-semimodule such that 1 $\in$ S and (M, +) is cancellative. If M has only trivial AC - congruences, then

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M satisfies condition (A).

Proof: Suppose a, b $\in$ M and a $\neq$ b. Define

$$\rho = \{ (x,y) \in M \times M : x + r_1 a + r_2 b = y + r_2 a + r_1 b$$

for suitable $r_i \in S \}$.

Clearly, ρ is a congruence on M. Then $\rho = I_M$ or $\rho = M \times M$.

To show that $\rho \neq 1_M$ we have to find p, q $\in$ M with p $\neq$ q such that (p, q) $\subset \rho$. Indeed, a $\neq$ b serves our purpose with $r_1 = o$ and $r_2 = 1$.

Hence $\rho = M \times M$. Then given x, y $\in$ M there exist $r_i \in S$ such that $x + r_1 a + r_2 b = y + r_2 a + r_1 b$ holds, which is the condition (A).

Theorem 3.8. Let M be a left S - semimodule such that 1 $\in$ S and M $\neq$ Z(M). Then M satisfies condition

(A) iff M has only trivial AC - congruences.

Proof: Suppose M satisfies condition (A) and ρ is an AC - congruence on M. If $\rho \neq I_M$, then we claim that $\rho = M \times M$. As $\rho \neq I_M$, there exist (a,b) $\in \rho$ such that a $\neq$ b. Then given x, y $\in$ M, there exist $r_i \in S$ such that

$x + r_1 a + r_2 b = y + r_2 a + r_1 b$. Hence $(x + r_1 a + r_2 b, y + r_2 a + r_1 b) \in \rho$.

Now (a, b) $\in \rho$ implies $(r_1 a, r_1 b) \in \rho$ and

and $(r_2 b, r_2 a) \in \rho$ so that $(r_1 a + r_2 b, r_2 a + r_1 b) \in \rho$, implying (x, y) $\in \rho$. Consequently, $\rho = M \times M$.

Conversely, suppose M has only trivial AC - congruences.

We claim that M satisfies condition (A). As $\bar{I}_M$ is an AC - congruence.

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$\overline{I}_M = M \times M$ or $\overline{I}_M = I_M$. If $\overline{I}_M = M \times M$, then for any $x \in M$, $(x, o) \in \overline{I}_M$.

Hence $(x + y, y) \in I_M$ for some $y \in M$.

Consequently, $x + y = y$, implying $x \in Z(M)$.

As a result, $M = Z(M)$ a contradiction. This implies that $\overline{I}_M = I_M$.

Suppose $x + c = x + d$ for some $x, c, d \in M$.

Then $(x + c, x + d) \in I_M$. Again $(x, x) \in I_M$. Consequently, $(c,d) \in I_M$.

Hence $c = d$. This shows that $(M, +)$ is cancellative.

Hence the theorem follows by theorem 3.7.

4. Exact sequences of semimodules.

Let A and B be S - semimodules and $f : A \longrightarrow B$ a semimodule homomorphism. Define

$K_f = \{(a, b) \in A \times A : f(a) + x = f(b) + x$ for some $x \in B\}$.

$I_f = \{(c,d) \in B \times B : c + f(a) = d + f(b)$ for some $a, b \in A\}$.

f is said to be a monic iff $K_f = \overline{\Delta}_A$, where

$\Delta_A = \{(a, a) : a \in A\}$ and $\overline{\Delta}_A = \{(a, b) \in A \times A : a + x = b + x$ for some $x \in A\}$

f is said to be an epic iff for any $b \in B$, there exist some $a_i \in A$ and $x \in B$ such that $b + f(a_1) + x = f(a_2) + x$. f is said to be an equivalence iff f is both an epic and a monic.

Theorem 4.1. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be S-semimodule homomorphisms. Then

(a) $K_{gf} \supseteq K_f$, equality holds if g is a monic;

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(b) $I_{gf} \subseteq I_g$;

(c) If gf is a monic, then f is a monic;

(d) If gf is an epic, then g is an epic.

Proof. (a) Let $(a, b) \in K_f$. Then $f(a) + x = f(b) + x$ for some $x \in B$ and $g(f(a) + x) = g(f(b) + x)$.

Hence, $gf(a) + g(x) = gf(b) + g(x)$ shows that $(a,b) \in K_{gf}$ and hence $K_f \subseteq K_{gf}$.

Suppose g is a monic and let $(a, b) \in K_{gf}$.

Then $gf(a) + y = gf(b) + y$ for some $y \in C$.

Hence, $g(f(a)) + y = g(f(b)) + y$ shows that $(f(a), f(b)) \in K_g$. Since g is a monic, $K_g = \bar{\Delta}_B$.

Hence $f(a) + t = f(b) + t$ for some $t \in B$.

Consequently $(a, b) \in K_f$ and hence $K_{gf} \subseteq K_f$.

Consequently, $K_f = K_{gf}$

(b) Let $(a,b) \in I_{gf}$. Then $a + gf(c) = b + gf(d)$ for some $(c,d) \in A \times A$. Hence $a + g(f(c)) = b + g(f(d))$, where $(f(c), f(d)) \in B \times B$. This shows that $(a,b) \in I_g$ and hence $I_{gf} \subseteq I_g$.

(c) Suppose gf is a monic, Then $K_{gf} = \bar{\Delta}_A$.

Let $(a, b) \in K_f$. Then $f(a) + x = f(b) + x$ for some $x \in B$ and $gf(a) + g(x) = gf(b) + g(x)$. Hence $(a,b) \in K_{gf} = \bar{\Delta}_A$. This implies $K_f \subseteq \bar{\Delta}_A$ and hence $K_f = \bar{\Delta}_A$.

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Consequently f is a monic.

(d) Suppose gf is an epic. Then for $c \in C$, there exist $a_i \in A$ and $x \in C$ such that $c + gf(a_1) + x = gf(a_2) + x$. Then $c + g(f(a_1))+x = g(f(a_2)) + x$ for some $f(a_1), f(a_2) \in B$. This shows that g is an epic.

Lemma 4.2. If $f : A \longrightarrow B$ be an S-semimodule homomorphism; then,

(i) K_f is an AC - congruence on A.

(ii) I_f is a congruence on B.

Proof (i) Clearly, $(a, a) \in K_f$ for all $a \in A$ and if $(a,b) \in K_f$, then $(b,a) \in K_f$. If (a,b) and $(b,c) \in K_f$ then there exist $x, y \in B$ such that $f(a) + x = f(b) + x$ and $f(b) + y = f(c) + y$. Hence $f(a) + f(b) + x + y = f(c) + f(b) + x + y$ shows that $(a,c) \in K_f$. Again $sf(a) + sx = sf(b) + sx$ implies $f(sa) + sx = f(sb) + sx$ for all $s \in S$.

Also, $f(a) + f(c) + x = f(b) + f(c) + x$ implies that $f(a+c) + x = f(b+c) + x$ for all $c \in A$. Consequently K_f is a congruence on A. Finally, let, $(a+c, b+c) \in K_f$. Then $f(a+c) + m = f(b + c) + m$ for some $m \in B$. Hence $f(a) + f(c) + m = f(b) + f(c) + m$ shows that $(a, b) \in K_f$. Consequently K_f is an AC - Congruence on A.

(ii) Clearly, $(b,b) \in I_f$ for all $b \in B$ and if $(c, d) \in I_f$ then $(d, c) \in I_f$. Let $(c, d), (d, e) \in I_f$. Then there exist $a, b, x, y \in A$ such that $c + f(a) = d + f(b)$ and $d + f(x) = e + f(y)$. Hence $c + f(a)+f(x)$

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$= d + f(b) + f(x) = e + f(b) + f(y)$. Consequently, $c + f(a + x) = e + f(b+y)$. Hence $(c,e) \in I_f$. Clearly I_f is a congruence on B from the definition of I_f.

We now prove an analogue of fundamental homomorphism theorem.

Theorem 4.3. Let A and B be S-semimodules and $f: A \longrightarrow B$ is an epimorphism, then

$$A/K_f \cong B/\bar{\Delta}_B$$

Proof. Define $g : A/K_f \longrightarrow B/\bar{\Delta}_B$ by

$$g(aK_f) = f(a)\bar{\Delta}_B.$$

Then g is well defined. Suppose $aK_f, bK_f \in A/K_f$ and $aK_f = bK_f$. Then $(a,b) \in K_f$ and hence $f(a) + x = f(b) + x$ for some $x \in B$. This implies that $(f(a), f(b)) \in \bar{\Delta}_B$. Hence $f(a)\bar{\Delta}_B = f(b)\bar{\Delta}_B$. This shows that g is well defined.

Also, $g(aK_f + bK_f) = g((a + b) K_f) = (f(a + b))\bar{\Delta}_B$

$= (f(a) + f(b))\bar{\Delta}_B = g(a K_f) + g(bK_f)$;

$g(s(ak_f)) = g((sa) K_f) = f(sa)\bar{\Delta}_B = sf(a)\bar{\Delta}_B$

$= sg(aK_f)$.

Hence g is an S - semimodule homomorphism.

$g(aK_f) = g(bK_f)$ implies $f(a)\bar{\Delta}_B = f(b)\bar{\Delta}_B$. This shows that $(f(a), f(b)) \in \bar{\Delta}_B$ so that, $f(a) + b' = f(b) + b'$ for some $b' \in B$, which implies $(a, b) \in K_f$ and hence $a K_f = bK_f$ proves that g is a monomorphism. Finally, since f is an epimorphism,

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for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.

Hence for every $b\overline{\Delta}_B \in B/\overline{\Delta}_B$ there exists $aK_f \in A/K_f$ such that

$g(aK_f) = f(a)\overline{\Delta}_B = b\overline{\Delta}_B$. Consequently, g is an isomorphism.

Definition 4.4. Let A, B be S - semimodules. Then an S-semimodule homomorphism $f : A \longrightarrow B$ is said to be a Z - homomorphism iff for each $a \in A$ there exists $x \in B$ such that $f(a) + x = x$.

Note that

f is a Z - homomorphism iff $f(a) \in Z(B)$ for each $a \in A$.

Lemma 4.5. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be S - semimodule homomorplisms. Then

$gf : A \longrightarrow C$ is a Z - homomorphism iff $\overline{I}_f \subseteq K_g$.

Proof. Suppose $gf : A \longrightarrow C$ is a Z-homomorphism.

Let $(a,b) \in \overline{I}_f$. Then there exists $x \in A$ such that $(a + x, b + x) \in I_f$.

Hence there exist $c, d \in A$ such that $a + x + f(c) = b + x + f(d)$.

Then $g(a) + g(x) + gf(c) = g(b) + g(x) + gf(d)$ ..... (1).

As gf is a Z-homomorphism, there exist $r, t \in C$ such that $gf(c) + r = r$ and $gf(d) + t = t$. From (1) we have, $g(a) + g(x) + r + t + gf(c)$ $= g(b) + g(x) + r + t + gf(d)$. Then $g(a) + g(x) + r + t$ $= g(b) + g(x) + r + t$. Hence $(a, b) \in K_g$ and then $\overline{I}_f \subseteq K_g$.

Conversely, suppose $\overline{I}_f \subseteq K_g$.

Let $a \in A$ and $f(a) = x$. Since $x + f(o) = o + f(a)$, we find that $(x, o) \in I_f \subseteq K_g$. Then $g(x) + t = g(o) + t$ for some $t \in C$.

Consequently, $gf(a) + t = t$ shows that gf is a Z-homomorphism.

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Definition 4.6. The short sequence of S-semimodules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is said to be exact iff f is a monic and g is an epic and $\overline{I}_f = K_g$.

Definition 4.7. A k-subsemimodule N of a semimodule M is said to be a j-subsemimodule iff for each $x \in M$ there exists some $y \in M$ such that $x + y \in N$.

Example 4.8. The short sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\pi} B/A \longrightarrow 0$ of S-semimodules and their homomorphisms is exact, where f is the inclusion map and A is a j-subsemimodule of B.

Proof. We claim that $K_f = \overline{\Delta}_A$. Let $(a,b) \in K_f$. Then $f(a) + x = f(b) + x$ for some $x \in B$. As f is the inclusion map, $a + x = b + x$. Since A is a j-subsemimodule of B, for $x \in B$ there exists some $y \in B$ such that $x + y \in A$, Hence $a + x + y = b + x + y$ implies $(a, b) \in \overline{\Delta}_A$. This shows that f is a monic. Again for $b\rho \in B/A$, $\pi(b) = b\rho$ shows that $b\rho + \pi(0) + x\rho = \pi(b) + x\rho$ for some $x\rho \in B/A$. Hence π is an epic. Next we prove that $\overline{I}_f = K_\pi$. Let $(a,b) \in \overline{I}_f$. Then $(a + x, b + x) \in I_f$ for some $x \in B$. Then there exist $a_i \in A$ such that $f(a_1) + a + x = f(a_2) + b + x$. Hence $a_1 + a + x = a_2 + b + x$. Then $\pi(a_1) + \pi(a) + \pi(x) = \pi(a_2) + \pi(b) + \pi(x)$. Since $a_1, a_2 \in A$, $\pi(a_1) = \pi(a_2)$ and hence $\pi(a) + y\rho = \pi(b) + y\rho$ for some $y\rho \in B/A$. Hence $(a, b) \in K_\pi$. Thus $\overline{I}_f \subseteq K_\pi$.

Conversely, let $(a, b) \in K_\pi$. Then $\pi(a) + y\rho = \pi(b) + y\rho$ for some

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$y\rho \in B/A$. Hence $a\rho + y\rho = b\rho + y\rho$ yields $a + y + b_1 = b + y + b_2$ for some $b_1, b_2 \in A$. Consequently, $a + f(b_1) + y = b + f(b_2) + y$. This shows that $(a, b) \in \bar{I}_f$. Hence $K_\pi \subseteq \bar{I}_f$.

Consequently $\bar{I}_f = K_\pi$.

Lemma 4.9. (The short five lemma for semimodules)

Let S be a hemiring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of S-semimodules and their homomorphisms such that each row is a short exact sequence. Then

(i) If α, γ are monic, β is also a monic.

(ii) If α, γ are epic, β is also an epic.

(iii) If α, γ are equivalences, β is also an equivalence.

Proof (i) Suppose α and γ are monic and $(b_1, b_2) \in K_\beta$. Then $\beta(b_1) + b' = \beta(b_2) + b'$ for some $b' \in B'$. Hence $g'\beta(b_1) + g'(b')$ $= g'\beta(b_2) + g'(b')$. Since $g'\beta = \gamma g$, $\gamma g(b_1) + g'(b') = \gamma g(b_2) + g'(b')$. This shows that $(g(b_1), g(b_2)) \in K_\gamma$; since γ is a monic, $g(b_1) + c = g(b_2) + c$ for some $c \in C$. This shows that $(b_1, b_2) \in K_g = \bar{I}_f$. Hence $b_1 + b_3 + f(a_1) = b_2 + b_3 + f(a_2)$ for some $b_3 \in B$ and $a_1 \in A$ ... (1)

Then $\beta(b_1) + \beta(b_3) + \beta f(a_1)$

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$= \beta(b_2) + \beta(b_3) + \beta f(a_2)$. Consequently, $\beta(b_1) + \beta(b_3) + f'\alpha(a_1) + b' = \beta(b_2) + \beta(b_3) + b' + f'\alpha(a_2)$.

This implies $(\alpha(a_1), \alpha(a_2)) \in K_{f'}$,

since $\beta(b_1) + b' = \beta(b_2) + b'$. As f' is a monic, $K_{f'} = \Delta_A$.

Hence $\alpha(a_1) + a' = \alpha(a_2) + a'$ for some $a' \in A'$.

Then, $(a_1, a_2) \in K_\alpha$. Again as α is a monic, $a_1 + a = a_2 + a$ for some $a \in A$. Now $f(a_1) + f(a) = f(a_2) + f(a)$ gives $f(a_1) + b_1 + b_3 + f(a) = f(a_2) + f(a) + b_1 + b_3$. Consequently, by using (1), $b_2+b_3+f(a_2)+f(a) = f(a_2) + f(a) + b_1 + b_3$. This shows that $(b_1, b_2) \in \Delta_B$. Hence $K_\beta \subseteq \Delta_B$. But $\Delta_B \subseteq K_\beta$, Hence $K_\beta = \Delta_B$ implies that β is a monic.

(ii) To prove that β is an epic, let us suppose that, $b' \in B'$.

Then $g'(b') \in C'$. As γ is an epic there exist $c_i \in C$ such that

$g'(b') + \gamma(c_1) + c' = \gamma(c_2) + c'$ for some $c' \in C'$..... (2)·

Since $c_i \in C$ and g is an epic, there exist $b_i \in B$ such that $c_1 + g(b_1) + x = g(b_2) + x$ and $c_2 + g(b_3) + y = g(b_4) + y$ for some $x, y \in C$. Hence $c_1 + g(b_1 + b_4) + (x + y) = c_2 + g(b_2+b_3) + x + y$.

This implies that $c_1 + g(b_5) + c = c_2 + g(b_6) + c$ for some $b_5, b_6 \in B$ and $c \in C$. Then $\gamma(c_1) + \gamma g(b_5) + \gamma(c) = \gamma(c_2) + \gamma g(b_6) + \gamma(c)$. Hence $\gamma(c_1) + \gamma(c) + g'\beta(b_5) + c' = \gamma(c_2) + \gamma(c) + g'\beta(b_6) + c' = g'(b') + \gamma(c_1) + c' + \gamma(c) + g'\beta(b_6)$ by using (2).

This shows that, $(\beta(b_5), b' + \beta(b_6)) \in K_{g'} = I_{f'}$.

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Then there exist a_1', a_2' $\in A'$ and b_1' $\in$ B' such that

$\beta(b_5) + f'(a_1') + b_1' = b' + \beta(b_6) + f'(a_2') + b_1'$ ..... (3)

Since α is an epic, there exist $\bar{a}_1$, $\bar{a}_2$ $\in A$ and $\bar{x} \in A'$ such that

$a_1' + \alpha(\bar{a}_1) + \bar{x} = \alpha(\bar{a}_2) + \bar{x}$. Similarly, there exist $\bar{a}_3$, $\bar{a}_4$ $\in A$ and $\bar{y} \in A'$

such that $a_2' + \alpha(\bar{a}_3) + \bar{y} = \alpha(\bar{a}_4) + \bar{y}$. Then, $a_1' + \alpha(\bar{a}_1) + \alpha(\bar{a}_4)$

$+ \bar{x} + \bar{y} = \alpha(\bar{a}_2) + \alpha(\bar{a}_3) + \bar{x} + \bar{y} + a_2'$. Hence, $a_1' + \tilde{x} + \alpha(a_1) = a_2' +$

$+ \tilde{x} + \alpha(a_2)$, where $\tilde{x} = \bar{x} + \bar{y} \in A'$, $a_1 = \bar{a}_1 + \bar{a}_4$, $a_2 = \bar{a}_2 + \bar{a}_3 \in A$.

Consequently, $f'(a_1') + f'(\tilde{x}) + f'\alpha(a_1) = f'(a_2') + f'(\tilde{x})$

$+ f'\alpha(a_2)$ gives $f'(a_1') + f'(\tilde{x}) + \beta f(a_1) = f'(a_2') + f'(\tilde{x}) + \beta f(a_2)$

by commutativity of the diagram. Then, $\beta(b_5) + f'(a_1') + b_1' + f'(\tilde{x})$

$+ \beta f(a_1) = \beta(b_5) + f'(a_2') + b_1' + f'(\tilde{x}) + \beta f(a_2)$. Hence by (3), we have,

$b' + \beta(b_6) + f'(a_2') + b_1' + f'(\tilde{x}) + \beta f(a_1)$

$= b_1' + \beta(b_5 + f(a_2)) + f'(a_2') + f'(\tilde{x})$. Then $b' + \beta(b_6 + f(a_1))$

$+ f'(\tilde{x} + a_2') + b_1' = \beta(b_5 + f(a_2)) + f'(\tilde{x} + a_2') + b_1'$.

This shows that β is an epic.

(iii) It follows from (i) and (ii).