CHAPTER - IV

SEMIRINGS WITH CHAIN CONDITIONS

In this chapter we undertake a study of semirings satisfying chain conditions on k-ideals and on h-ideals.

1. Semirings with chain conditions on k-ideals.

While pursuing k-ideal theory in semirings it is natural to study special classes of semirings with the help of k-ideals. In this section, the notion of a finitely generated k-ideal in hemirings is introduced and k-Noetherian hemirings are defined. These are required to satisfy a certain finiteness condition, namely every k-ideal should be finitely generated. In an analogue of rings, an equivalent formulation of the k-Noetherian requirement that k-ideals of the hemiring satisfy the ascending chain conditions is established. From this idea we are led in a natural way to obtain a generalization of Hilbert Basis Theorem for a class of semirings.

In this section H denotes a hemiring with multiplicative identity 1.

Let A be a non-empty subset of H. By the symbol $\langle A\rangle_k$ we mean the k-ideal $\langle A\rangle_k = \cap\{I : A \subseteq I,\ I \text{ is a k-ideal of } H\}$.
The collection of k-ideals which contain A is not empty, since H is itself a k-ideal containing any subset of H. Also the intersection of a family of k-ideals is a k-ideal by theorem 1.6 of chapter I. Thus

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$\langle A\rangle_k$ exists and satisfies the relation $A \subseteq \langle A\rangle_k$.

$\langle A\rangle_k$ is called the k-ideal generated by the set A.

If $A = \{a_1, a_2, \ldots\ldots, a_n\}$, then the k-ideal generated by $\{a_1, a_2 \ldots\ldots, a_n\}$ is denoted by $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k$.

Lemma 1.1. Let $a_1, a_2 \ldots., a_n$ be n elements of H, then $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k = \{a \in H : a + \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} t_i a_i, r_i, t_i \in H\}$.

Proof : Let $T = \{a \in H : a + \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} t_i a_i, r_i, t_i \in H\}$. We first show that

$$T \subseteq \langle a_1, a_2, \ldots\ldots, a_n\rangle_k.$$

For this, assume that $a \in T$. Then there exist $r_i, t_i \in H$ such that

$$a + \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} t_i a_i \ldots\ldots (1).$$

Now $\{a_1, a_2, \ldots\ldots, a_n\} \subseteq \langle a_1, a_2, \ldots., a_n\rangle_k$. Hence $\sum_{i=1}^{n} r_i a_i$ and $\sum_{i=1}^{n} t_i a_i$ belong to $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k$. Since $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k$ is a k-ideal, it follows from (1) that $a \in \langle a_1, a_2 \ldots\ldots, a_n\rangle_k$. As a result $T \subseteq \langle a_1, a_2, \ldots\ldots, a_n\rangle_k$.

To prove the converse we first note that

$$a_i + 0a_1 + \ldots. + 0a_n = 0a_1 + \ldots.. + 0a_{i-1} + 1a_i +$$

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$oa_{i+1} + \ldots\ldots + oa_n$. Hence $\{a_1, a_2, \ldots\ldots a_n\} \subseteq T$.

Then from the definition of $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k$ it follows that $\langle a_1, a_2, \ldots\ldots, a_n\rangle_k \subseteq T$. Consequently, $T = \langle a_1, a_2, \ldots\ldots, a_n\rangle_k$. Moreover, we find that T is a k-ideal.

This completes the proof.

Definition 1.2. The maximal condition for k-ideals in H is said to hold iff every non-empty set of k-ideals of H, partially ordered by set inclusion, has at least one maximal element, i.e. there exists a k-ideal which is not properly contained in any k-ideal of the set.

Definition 1.3. A hemiring H is said to satisfy the ascending chain condition (denoted by a.c.c.) on k-ideals, iff for any sequence of k-ideals $I_1, I_2, \ldots\ldots I_n \ldots$, of H with $I_1 \subseteq I_2 \subseteq \ldots\ldots \subseteq I_n \subseteq \ldots\ldots$ there exists an integer n (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$.

Theorem 1.4. The following three conditions on k-ideals of H are equivalent.

(i) H satisfies the ascending chain condition on k-ideals.

(ii) The maximal condition for k-ideals hold in H.

(iii) Every k-ideal in H is finitely generated.

Proof: The proof of (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is similar to that of the ring. So it remains to prove that (iii) $\Rightarrow$ (i) :

Let (iii) hold in H. We take an ascending chain of k-ideals in H :

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$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots \subseteq I_n \subseteq \ldots\ldots$

Let $I = \cup I_i$, $i = 1, 2, 3, \ldots\ldots$ Let $a, b \in I$.

Then there exist positive integers i, j such that $a \in I_i$ and $b \in I_j$.

Let $r = \max(i,j)$. Then $a, b \in I_r$. Since I_r is a k-ideal, $a+b$, ax, $xa \in I_r \subseteq I$ for all $x \in H$. Also $a + y \in I_r$, $a \in I_r$ and $y \in H$ imply $y \in I_r \subseteq I$. Hence I is a k-ideal of H. By (iii), $I = \langle a_1, a_2, \ldots, a_t \rangle_k$ where $a_i \in I_{n_i}$ for some n_i, $i = 1, 2, \ldots\ldots, t$.

Let $m = \max(n_1, n_2, \ldots\ldots, n_t)$. Then $a_1, a_2, \ldots\ldots, a_t \in I_m$. We claim that $I = I_m$. Since $I = \bigcup_i I_i$, $I_m \subseteq I$. We next prove that $I \subseteq I_m$. We observe that I_m is a k-ideal containing $\{a_1, a_2 \ldots\ldots, a_t\}$.

So $I_m \supseteq \langle a_1, a_2, \ldots\ldots, a_t \rangle_k = I$ by lemma 1.1. Hence $I = I_m$. This shows that every ascending chain of k-ideals of H terminates. This implies (i).

Definition 1.5. A hemiring H with 1 satisfying any one of the three equivalent conditions of the theorem 1.4 is called a k-Notherian hemiring.

Definition 1.6. A hemiring H is said to be additively cancellative iff $a + b = x + b$ implies $a = x$ for arbitary elements $a, b, x \in H$. If H is additively cancellative, then H[x], the hemiring of polynomials over H is also additively cancellative.

Definition 1.7. A halfring (cf. eg [45]) is a triple (H, +, .), where (H,+) is a commutative cancellative semigroup with zero o and (H,.)

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is a semigroup whose multiplication distributes over the addition from both sides.

Thus an additively cancellative hemiring is a halfring. A k-Noetherian hemiring H is said to be a k-Noetherian halfring iff H is a halfring. The following construction shows how to construct a ring $\widetilde{R}$ from any given additively cancellative hemiring with 1 ; i.e. from a halfring with 1: Let H be an additively cancellative hemiring with 1 and let S = H x H. Define operations of addition and multiplication on S by (a, b) + (c, d) = (a + c, b + d) and (a, b) (c, d) = (ac + bd, ad + bc) for all a, b, c, d, ∈ H. These operations turn S into a halfring with absorbing zero (o,o) and identity (1,o). If H is commutative, S is so.
Now D = {(a, a) : a ∈ H}. Clearly D is a k-ideal of S.
Then S/D is a hemiring and in this case S/D is in fact a ring, called the ring of differences of the halfring H and denoted by $\widetilde{H}$. As noted in [50], $\widetilde{H}$ is the smallest ring (unique upto isomorphism) in which H can be embedded: H consists of all differences a - b for a, b ∈ H, where elements of the image of H under the isomorphism of embedding are identified with the elements of H.

Example 1.8. [45]. Let R be the ring of all functions of the non-negative integers N into the rationals Q with the operation of point wise addition and multiplication and let C be the subsemiring of R, consisting of zero function and all functions f satisfying the condition f(i) > o for all i ∈ N. Then $\widetilde{C}$ = R. Now each non-zero element of C has a multiplicative

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inverse, so that C is a division semiring (halfring) and thus C has no non-trivial ideals. But R is a complete direct product of a countably infinite family of copies of the rationals. Hence R does not even satisfy a chain comdition on k-ideals.

Definition 1.9. [45] A halfring H is called (right, left) initial iff H is a ring with (right, left) identity. In the same paper, H. E. Stone proved the following generalization of the Hilbert Basis Theorem. Let H be an initial halfring. Then $H[x]$ is right Noetherian iff $\widetilde{H}$ is right Noetherian.

The example 1.8 shows that an analogue of Hilbert Basis theorem does not hold for all halfrings. We now show that it does hold for a class of halfrings.

Definition 1.10. A hemiring H is said to be semi-subtractive iff for every pair of elements a and b in H at least one of the equations $a + x = b$ or $b + x = a$ is solvable in H.

We now prove the following generalization of the Hilbert Basis Theorem:

Theorem 1.11. If H is a k-Noetherism halfring and $H[x]$, the halfring of polynomials over H is semi-subtractive, then $H[x]$ is k-Noetherian.

Proof. Let I be an arbitrary non-zero k-ideal of $H[x]$. We claim that I is finitely generated. For each interger $r \geq 0$, we construct the set

$$I_r = \{h \in H : a_1x + a_2x^2 + \ldots\ldots + a_{r-1}x^{r-1} + hx^r \in I\} \cup \{0\}.$$

Then I_r is an ideal of H such that

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$I_r \subseteq I_{r+1}$, $r = o, 1, 2, 3, \ldots\ldots$. As a result we have

$I_o \subseteq I_1 \subseteq I_2 \ \ldots\ldots \subseteq I_n \subseteq \ldots\ldots$

This gives an ascending chain of k-ideals:

$\bar{I}_o \subseteq \bar{I}_1 \subseteq \bar{I}_2 \subseteq \ldots\ldots \subseteq \bar{I}_n \subseteq \ldots\ldots$ in H, where $\bar{I}_n$ is the k-closure of I_n in H. As H is k-Noetherian, there exists an integer n such that $\bar{I}_r = \bar{I}_n$ for all $r \geq n$. Moreover, each k-ideal $\bar{I}_r$ is finitely generated.

Let $\bar{I}_r = \langle a_{r1}, a_{r2}, \ldots\ldots, a_{rm_r} \rangle_k$,

$r = o, 1, 2, \ldots\ldots, n$, where a_{rj} is the leading co-efficient of $f_{rj}(x)$, a polynomial of degree r in I. We claim that $m_o + m_1 + \ldots + m_n$ polynomials f_{ij} generate I. Let they generate the k-ideal J. Then

$J = \langle f_{o1}(x), \ldots, f_{om_o}(x), \ldots\ldots, f_{n1}(x), \ldots, f_{nm_n}(x) \rangle_k \subseteq I$, by our choice of f_{ij}.

Conversely, let $f(x) \in I$ and $f(x)$ is of degree r. Let $f(x) = b_o + b_1 x + \ldots\ldots + b_{r-1}x^{r-1} + bx^r$ ($b \neq o$). We claim that $f(x) \in J$. To prove this, we apply induction on r. If $r = o$, we have $f(x) = b_o \in I_o \subseteq \bar{I}_o \subseteq J$.

Next we assume inductively that any polynomial of degree $<r$ lying in I belongs to J.

Case 1. Let $r > n$. Then the leading co-efficient $b \in \bar{I}_r = \bar{I}_n$. So we have

$b + s_1 a_{n1} + s_2 a_{n2} + \ldots\ldots + s_{m_n} a_{nm_n}$

Cont. 100.

$= c_1 \; a_{n1} + c_2 \; a_{n2} + \ldots\ldots + c_{m_n} \; a_{nm_n}$ ..... (2)

for some $s_1, s_2, \ldots\ldots, s_{m_n}$; $c_1, c_2, \ldots\ldots, c_{m_n} \in H$.

As $H[x]$ is semisubtractive, for the pair of polynomials $F(x) = f(x) + x^{r-n} [s_1 f_{n1}(x) + s_2 f_{n2}(x) + \ldots \ldots + s_{m_n} f_{nm_n}(x)]$ and $G(x)$ $x^{r-n}[c_1 f_{n1}(x) + c_2 f_{n2}(x) + \ldots\ldots + c_{m_n} f_{nm_n}(x)]$ in $H[x]$, there exists a polynomial $f_1(x)$ or $g_1(x) \in H[x]$ such that $f_1(x) + G(x) = F(x)$ or $F(x) + g_1(x) = G(x)$.

Without loss of generality we assume that

$f_1(x) + G(x) = F(x)$.

i.e. $f_1(x) + x^{r-n} [c_1 f_{n1}(x) + c_2 f_{n2}(x) + \ldots\ldots + c_{m_n} f_{nm_n}(x)]$

$= f(x) + x^{r-n} [s_1 f_{n1}(x) + s_2 f_{n2}(x) + \ldots\ldots + s_{m_n} f_{nm_n}(x)]$ .....(3).

The co-efficient of x^r in the second part of the l.h.s. of (3) is

$\sum_{i=1}^{m_n} c_i a_{ni}$ and that of x^r in the right hand side of (3) is

$b + \sum_{i=1}^{m_n} s_i \; a_{ni}$. They are equal by relation (2). As $H[x]$ is additively cancellative, the terms containing x^r in the second part of the l.h.s. and these of the r.h.s of (3) cancel out leaving $f_1(x)$ + (some polynonial of degree $<r$ lying in I) = some polynomial of degree

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<r lying in I. .... (4).

As I is a k-ideal, then by (4), it follows that $f_1(x) \in I$. The degree of $f_1(x) \not> r$, otherwise we have a contradiction by (4). Consequently $f_1(x)$ is a polynomial of degree <r lying in I. Hence by inductive hypothesis, $f_1(x) \in J$. Thus by (3), f(x) + (some polynomial $\in$ J) = some polynomial $\in$ J. As J is a k-ideal, $f(x) \in J$.

Case 2. Let $r \leq n$. Then the co-efficient b of f(x) lies in $\overline{I}_r$. So we find elements $d_1, d_2, \ldots, d_{m_r}$; $s_1, s_2, \ldots, s_{m_r}$ in H such that

$b + s_1 a_{r1} + s_2 a_{r2} + \ldots + s_{m_r} a_{rm_r} = d_1 a_{r_1} + d_2 a_{r2} + \ldots + d_{m_r} a_{rm_r}$ ..... (5) by lemma 1.1.

As H [x] is semisubtractive, without loss of generality there exists a polynomial $f_2(x) \in H[x]$ such that

$f_2(x) + [d_1 f_{r1}(x) + d_2 f_{r2}(x) + \ldots + d_{m_r} f_{rm_r}(x)] = f(x) + [s_1 f_{r1}(x) + s_2 f_{r2}(x) + \ldots + s_{m_r} f_{rm_r}(x)]$ ..... (6).

The co-efficient of x^r in the second part of the l.h.s of (6) is the same as that of x^r in the r.h.s. of (6) by the relation (5). As H[x] is additively cancellative, the above terms cancel out leaving $f_2(x)$ + (some polynomial of I of degree <r) = some polynomial of I of degree <r. Proceeding as in the case 1, it follows that $f(x) \in J$. Thus in either case, $I \subseteq J$. Hence I = J. This completes the proof of the theorem.

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2. Semirings with chain Conditions on h-ideals

Definition 2.1. A commutative semiring S is said to satisfy the descending chain condition (d.c.c.) on ideals (h-ideals) iff given any descending chain of ideals (h-ideals) of S:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots \supseteq I_m \supseteq \cdots\cdots\cdots ,$$

there exists a positive integer n such that

$$I_m = I_n \text{ for all } m \geq n.$$

Remark 2.2. The descending chain condition on ideals of a semiring S implies d.c.c. on h-ideals but the converse is not true.

Example 2.3. Consider $S = Q^+(\sqrt{2}) = \{a + b\sqrt{2} : a, b$ are non-negative rational numbers$\}$. Then under usual addition and multiplication, S is a commutative semiring with zeroid $Z(S) = \{0\}$.

For $a + b\sqrt{2} \in S^* = S \setminus Z(S)$, there exists an identity pair (0, 1) such that

$$0 + \frac{a}{a^2 - 2b^2}(a + b\sqrt{2}) = 1 + \frac{b\sqrt{2}}{a^2 - 2b^2}(a + b\sqrt{2}) \text{ if}$$

$a^2 - 2b^2 > 0$ and

$$0 + \frac{b\sqrt{2}}{2b^2 - a^2}(a + b\sqrt{2}) = 1 + \frac{a}{2b^2 - a^2}(a + b\sqrt{2}) \text{ if}$$

$2b^2 - a^2 > 0$. Hence S satisfies condition (C_1) (c.f. definition 2.3

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of chapter III).

As a result $Z(S) = \{0\}$ and S are the only h-ideals of S by theorem 2.13 of Chapter III. Consequently, S satisfies d.c.c. on h-ideals of S. Next suppose $x = 1 + \sqrt{2}$. Then $x \in S$. Now we have a strictly descending chain of ideals of infinite length in S :

$$S \supsetneqq Sx \supsetneqq Sx^2 \supsetneqq Sx^3 \supsetneqq \ldots\ldots$$

Definition 2.3. Let S be a commutative semiring with zeroid Z(S). Then S is said to satisfy:

(a) the condition (D) iff (i) $a\, b \in Z(S)$ implies either $a \in Z(S)$ or $b \in Z(S)$ and (ii) $ab + x = ac + x$ for $a \in S^*$, $b, c, x \in S$ implies $b + y = c + y$ for some $y \in S$.

(b) the condition (E) iff every descending chain of h-ideals of S terminates.

Theorem 2.4. Let S be a commutative semiring with an identity pair (e_1, e_2). Then S satisfies condition (C_1) of Chapter III iff S satisfies the condition (D) and (E).

Proof. Suppose S satisfies condition (C_1). Then it satisfies both the conditions (D) and (E) by lemmas 2.4 and 2.6 and by theorem 2.13 of Chapter III.

Conversely, let S satisfy conditions (D) and (E).

Suppose $a \in S^*$. Define

$I_n = \{ra^n : r \in S\}$.

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Then I_n is an ideal of S such that

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots\ldots$

Hence S has a descending chain of h-ideals:

$\overline{I}_1 \supseteq \overline{I}_2 \supseteq \overline{I}_3 \ldots\ldots \supseteq \overline{I}_n \supseteq \overline{I}_{n+1} \supseteq \ldots\ldots,$

where $\overline{I}_r$ is the h-closure of I_r in S.

Using condition (E), we find that $\overline{I}_n = \overline{I}_{n+1}$ for some positive integer n. As $a^n \in \overline{I}_{n+1}$ there exist some r, t, d $\in$ S such that

$$a^n + r a^{n+1} + d = t a^{n+1} + d.$$

Using condition (D), we find that $a^n \notin Z(S)$ as $a \notin Z(S)$. As (e_1, e_2) is an identity pair, $a^n + e_1 a^n = e_2 a^n$ implies $e_2 a^n + r a^{n+1} + d_1 = t a^{n+1} + e_1 a^n + d_1$ for some $d_1 \in$ S. Hence $a^n (e_2 + ra) + d_1 = a^n (e_1 + ta) + d_1$ (by using commutativity in S) implies by condition D(ii) that $e_2 + ra + y = e_1 + ta + y$ for some $y \in S$..... (1).

As $(e_1 + y, e_2 + y)$ is also an identity pair, it follows from (1) that S satisfies condition (C_1).