CHAPTER - III

h - IDEALS OF SEMIRINGS

In the present chapter we study h-ideals of semirings $(S, +, \cdot)$ with commutative addition. Conditions are considered such that S has non-trivial h-ideals or maximal h-ideals, among others by the help of Iizuka congruence class semiring S/A defined by an ideal A of S.

It is well known that every h-ideal is a k-ideal but examples disprove the converse. But in this chapter it is shown that the concepts of h-ideals and k-ideals coincide in some class of semirings. It is also shown that the set of all h-ideals of an additive inverse semiring in which addition is commutative forms a complete modular lattice. Finally, certain types of ring congruences on an additive inverse semiring are characterized by the help of h-ideals.

1. h-ideals. Iizuka [20] defined a restricted class of ideals in semirings, which he called h-ideals. An h-ideal of a semiring S is an ideal A of S such that whenever $x + a + u = b + u$ where $x, u \in S$ and $a, b \in A$, then $x \in A$.

An h- ideal $A \subsetneq S$ is called a maximal h-ideal iff there is no h-ideal B satisfying $A \subsetneq B \subsetneq S$.

Moreover, each ideal A of S defines the Iizuka congruence [20] :

$\sigma_A = \{(x, y) \in S \times S : x + a + u = y + b + u$ for some $a, b \in A$ and $u \in S\}$.

Then σ_A is an additively cancellative (AC) congruence. The corresponding quotient semiring S/σ_A, consisting of the classes $x\sigma_A$, is also denoted

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by S/A.

Bourne and Zassenhaus [10] defined the zeroid Z(S) of a semiring S as

$$Z(S) = \{z \in S : z + x = x \text{ for some } x \in S\}.$$

If S has an additive idempotent, then $Z(S) \neq \emptyset$ and if S contains absorbing zero 0 and (S, +) is cancellative, then $Z(S) = \{0\}$.
Moreover Z(S) is an h-ideal of S and is contained in every h-ideal of S. Z (S) and S are called trivial h-ideals and other h-ideals (if they exist) are called proper h-ideals of S.

We introduce the notation

$S^* = S \setminus Z(S)$ if $Z(S) \neq \emptyset$

and $S^* = S$, otherwise.

The following results due to [20] and [26] are well known :

(i) If S has a non-empty zeroid Z(S), then the semiring S/σ_A determined by the Iizuka congruence σ_A on S has a zeroid equal to absorbing zero for any h-ideal A of S.

(ii) For an ideal A of S, if the semiring S/ρ_A determined by the Bourne congruence ρ_A is additively cancellative, then both the congruences ρ_A and σ_A on S coincide and moreover if A is a k-ideal, then it is also an h-ideal.

2. Maximal h-ideals.

Definition 2.1. Let S be a semiring. A pair of elements (e_1, e_2) with $e_1, e_2 \in S$ is said to be a left (right) identity pair iff $a+e_1a$

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$= e_2 a$ $(a + ae_1 = ae_2)$ for all $a \in S$. (e_1, e_2) is said to be an identity pair iff it is both a left and a right identity pair.

Example 2.2. Let $S = \{a \in N : a \geq 5\} \cup \{0\}$,

where N denotes the set of all non-negative integers. S is a semiring under usual addition and multiplication with integor 0 as the absorbing zero element. Here (6, 7), (7,8), (10, 11) ..... are identity pairs.

We assume that any semiring S with an identity pair (e_1, e_2) contains more than one element.

Definition 2.3. : Let S be a commutative semiring.

Then S is said to satisfy condition (C_1) iff for each $a \in S^*$ there exists an identity pair (e_1, e_2)(depending on a) such that

$e_1 + ra = e_2 + ta$ holds for some $r, t \in S$.

Lemma 2.4. If a commutative semiring S with zeroid Z(S) satisfies condition (C_1), then $ab \in Z(S)$ for. $a, b \in S$ implies $a \in Z(S)$ or $b \in Z(S)$.

Proof. By way of contradiction, assume $ab = z$ for some $z \in Z(S)$ and $a \notin Z(S)$, $b \notin Z(S)$. Then for $a \in S^*$ there exists an identity pair (e_1, e_2) such that $e_1 + ra = e_2 + ta$ for some $r, t \in S$. This yields $e_1 b + rab = e_2 b + tab$ i.e. $b + e_1 b + tz = e_1 b + rz$ ..... (1), as $b + e_1 b = e_2 b$. Since $z \in Z(S)$, $z + x = x$ for some $x \in S$. Then $b + e_1 b + tz + tx + rx = e_1 b + rz + rx + tx$ implies $b + e_1 b + tx + rx = e_1 b + tx + rx$. Hence $b \in Z(S)$, a contradiction.

Lemma 2.5. If a commutative semiring S satisfies condition (C_1) then $ab = ac$ for $a \in S^*$ and $b, c \in S$ implies $b + y = c + y$ for some $y \in S$.

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Proof. Let $a \in S^*$. Then according to condition (C_1) theree exists an identity pair (e_1, e_2) such that $e_1 + ra = e_2 + ta$ for some $r, t \in S$. Hence $e_2b + tab = e_1b + rab$ for each $b \in S$. Consequently,

$b + e_1b + tab = e_1b + rab$ ..... (1)

Similarly, $c + e_1c + tac = e_1c + rac$ ..... (2)

Hence (1) & (2) imply that

$c + e_1 (b + c) + tac + rab = b + e_1 (b + c) + tab + rac.$

Then $c + e_1 (b + c) + (t + r) ac = b + e_1 (b + c) + (t + r) ac$ implies that $c + y = b + y$ for some $y \in S$.

Lemma 2.6. If a commutative semiring S satisfies condition (C_1), then $ab + x = ac + x$ for $a \in S^*$, $b, c, x \in S$ implies $b + y = c + y$ for some $y \in S$.

Proof. For $a \in S^*$ there exists an identity pair (e_1, e_2) such that $e_1 + ra = e_2 + ta$ for some $r, t \in S$. Then $e_2b + tab = e_1b + rab$ for each $b \in S$.

Consequently, $b + e_1b + tab = e_1b + rab$ ..... (1)

Similarly, $c + e_1c + tac = e_1c + rac$ ...... (2)

Hence (1) & (2) imply that

$c + e_1 (b + c) + tac + rab + tx + rx = b + e_1(b+c) + tab + rac + tx + rx.$

Consequently, $c + e_1(b + c) + (t + r) ac + (t + r) x = b + e_1 (b + c) + (t + r) ac + (t + r) x$. This shows that $c + y = b + y$ for some $y \in S$.

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Lemma 2.7. Let S be a commutative semiring with an identity pair (e_1, e_2) and zeroid Z(S). If $\{e_1, e_2\} \subseteq Z(S)$, then $S = Z(S)$.

Proof. As $e_1, e_2 \in Z(S)$, then $e_1 + x = x$, $e_2 + y = y$ for some $x, y \in S$. Again $a + e_1 a = e_2 a$ for each $a \in S$. Now $e_1 a + ax = ax$ and $e_2 a + ay = ay$. Thus $a + e_1 a + ay = ay$ implies $a + (e_1 a + ax) + ay = ay + ax$. Consequently,

$a + ax + ay = ay + ax$.

This implies that $a \in Z(S)$. Hence $S = Z(S)$.

Note 2.8. If S contains an identity pair (e_1, e_2) and $Z(S) \neq S$, then $\{e_1, e_2\} \subsetneq Z(S)$.

Definition 2.9. The h-closure $\overline{A}^h$ of each ideal A of a semiring S is defined by

$\overline{A}^h = \{x \in S : x + a_1 + u = a_2 + u$ for some $a_1 \in A$ and $u \in S\}$.

We use the symbol $\overline{A}$ for $\overline{A}^h$ unless there is a scope for confusion.

Lemma 2.10. The h-closure $\overline{A}$ of an ideal A of a semiring S is an h-ideal of S and satisfies:

(i) $A \subseteq \overline{A}$ for any ideal A of S;

(ii) $A = \overline{A}$ iff A is an h-ideal of S;

(iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$ for ideals A and B of S;

(iv) $\overline{\overline{A}} = \overline{A}$ for each ideal A of S;

(v) $\overline{A}$ is the smallest h-ideal of S containing A.

Proof. Clearly, $\overline{A}$ is an h-ideal of S.

(i) Since $a + a + u = (a + a) + u$ for each $a \in A$, $a \in \overline{A}$. Consequently, $A \subseteq \overline{A}$.

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(ii) If $A = \bar{A}$, then A is an h-ideal. Next let A be an h-ideal. For $x \in \bar{A}$, there are $a_i \in A$ and $u \in S$ such that $x + a_1 + u = a_2 + u$. As A is an h-ideal, $x \in A$. Hence $\bar{A} \subseteq A$, Consequently $A = \bar{A}$ by (i).

(iii) Let $a \in \bar{A}$. Then $a + x + u \in A$ for some $x \in A$, $u \in S$. Hence $a + x + u \in B$ for some $x \in B$. As a result $\bar{A} \subseteq \bar{B}$.

(iv) Follows from (ii).

(v) Let B be any h-ideal containing A. Then $A \subseteq B$. This implies $\bar{A} \subseteq \bar{B} = B$ by (iii) & (ii). Consequently, $\bar{A}$ is the smallest h-ideal containing A.

Proposition 2.11. Let S be a commutative semiring with an identity pair (e_1, e_2). If A is an h-ideal containing $\{e_1, e_2\}$, then $A = S$.

Proof. As $e_1, e_2 \in A$, then $e_1 b$, $e_2 b \in A$ for each $b \in S$. Now $b + e_1b + u = e_2b + u$ for some $u \in S$ implies $b \in A$. This implies $A = S$.

Theorem 2.12. Let S be a commutative semiring with an identity pair (e_1, e_2). Then each proper h-ideal of S is contained in a maximal h-ideal of S.

Proof. Let A be a proper h-ideal of S and h(A) the set of all h-ideals B of S satisfying $A \subseteq B \subsetneq S$, partially ordered by inclusion. Consider a chain $\{B_i : i \in I\}$ in h(A). Then $B = \bigcup_{i \in I} B_i$ is an h-ideal of S. Now $B \neq S$, otherwise $e_1 \in B_i$ and $e_2 \in B_j$ for some $i, j \in I$. As $\{B_j\}$ forms a chain, either $B_i \subseteq B_j$ or $B_j \subseteq B_i$. For definiteness, we suppose $B_i \subseteq B_j$, so that both $e_1, e_2 \in B_j$. Consequently $B_j = S$ by

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proposition 2.11, which is a contradiction. Hence $B \in h(A)$. So by Zorn's lemma $h(A)$ has a maximal element as we were to show.

Theorem 2.13. Let S be a commutative semiring with zeroid $Z(S)$. Then the condition (C_1) implies that S contains only trivial h-ideals. The converse is true if the h-closure $\overline{Sa} = \overline{\{sa : s \in S\}} \neq Z(S)$ holds for all $a \in S^*$ and S contains an identity pair (e_1, e_2).

Proof. Assume that S satisfies (C_1). Let A be an h-ideal of S which contains at least one element $a \in S^*$. Then there exists an identity pair (e_1, e_2) such that $e_1 + ra = e_2 + ta$ for some $r, t \in S$ according to (C_1). Hence $e_1b + rab = e_2b + tab$ for each $b \in S$. Consequently, $b + e_1b + tab = e_1b + rab$. As $tab \in A$, $rab \in A$ and A is an h-ideal, it follows that $b \in A$. Consequently, $A = S$.

For the converse assume that S has only trivial h-ideals and (e_1, e_2) is an identity pair. Then the h-closure $\overline{Sa}$ of Sa coincides with S for each $a \in S^*$. Hence $\overline{Sa} = S$ shows that $e_1 + s_1a + u = s_2a + u$ and $e_2 + t_1a + v = t_2a + v$ for some $s_1, t_1, u, v \in S$. Consequently, $e_1+(s_1+t_2)\, a + (u+v) = e_2 + (t_1 + s_2)\, a + (u+v)$. Now, $a + (e_1+u+v)\, a = (e_2 + u + v)\, a$ for all $a \in S$. This shows that $(e_1 + u + v, e_2+u+v)$ is also an identity pair. Consequently, $(e_1 + u + v) + (s_1 + t_2)\, a = (e_2 + u + v) + (t_1 + s_2)\, a$ implies that S satisfies condition (C_1).

Theorem 2.14. Let S be a commutative semiring with an identity pair (e_1, e_2) and A a proper h-ideal of S. Then A is a maximal h-ideal of S iff the semiring $S/A = S/\sigma_A$ satisfies the condition (C_1), where σ_A is the Iizuka congruence on S determined by A.

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Proof. Suppose A to be a maximal h-ideal of S. Then σ_A is an additively cancellative congruence on S. Hence $(S/A, +, .)$ is a commutative semiring under usual addition and multiplication of Iizuka congruence classes. Then A is the absorbing zero of S/A and $(e_1 \sigma_A, e_2\sigma_A)$ its identity pair and A its zeroid. Let $c\sigma_A \in (S/A)^*$. Then $c \notin A$ and the smallest ideal B of S containing c and A consists of all elements sc, a and sc+nc+a for $s \in S$, $n \in N$, and $a \in A$. Then $A \subsetneqq B$, otherwise $A = B$ would imply $e_2c + a = c + e_1c + a \in A$ and $e_1c + a \in A$. As A is also a k-ideal, it follows that $c \in A$. But this is a contradiction. If $\overline{B}$ is the h-closure of B, from $A \subsetneqq B$, it follows that $\overline{B} = S$ and hence $e_1 + b_1 + u = b_2 + u$ and $e_2 + b_3 + v = b_4 + v$ for suitable $b_i \in B$ and $u, v \in S$. Then $e_1 + (b_1 + b_4) + (u + v) = e_2 + (b_2 + b_3) + (u + v)$, where b_1+b_4, $b_2 + b_3 \in B$ and $u + v \in S$. Now we write $b_1 + b_4 = rc + nc + a$, $b_2 + b_3 = sc + mc + a_1$ and $u + v = x$ for suitable $r, s, x \in S$, $m, n \in N$, $a, a_1 \in A$. Hence $e_1 + rc + nc + a + x = e_2 + sc + mc + a_1 + x$. Now $c + e_1c = e_2c$ implies $nc + (ne_1)c = (n e_2)c$ and $mc + (me_1)c = (me_2)c$. Thus $e_1 + rc + nc + c + (ne_1)c + a + x + (me_1) c = e_2 + sc + mc + (me_1) c + a_1 + x + (ne_1)c$ yields $e_1 + (r + ne_2) c + a + x + (me_1)c = e_2 + (s + me_2) c + a_1 + x + (ne_1)c$. This implies that $e_1 \sigma_A + (r + ne_2 + me_1) \sigma_A (c\sigma_A) = e_2\sigma_A + (s + me_2 + ne_1) \sigma_A(c\sigma_A)$, since S/σ_A is addintively cancellative and $a, a_1 \in A$. This shows that S/A satisfies condition (C_1). Conversely, assume S/A satisfies (C_1). Let B be an h-ideal of S satisfying $A \subsetneqq B$. Then there is an element $c \in B \setminus A$ and

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$c\sigma_A \in (S/A)^*$. Hence there exists an identity pair $(e_3\sigma_A, e_4\sigma_A)$ in S/A such that $e_3\sigma_A + (r_1 \sigma_A) \ (c \sigma_A) = e_4\sigma_a + (r_2\sigma_A) \ (c\sigma_A)$, for suitable $r_1, r_2 \in S$. This implies that $e_3 + r_1c + a_2 + t = e_4 + r_2c + a_3 + t$ for some $a_2, a_3 \in A$ and $t \in S$ ..... (1).

Again, since $(e_3 \sigma_A, e_4 \sigma_A)$ is an identity pair in S/A for any $x \in S$, $x \sigma_A + (e_3\sigma_A) \ (x\sigma_A) = (e_4 \sigma_A) \ (x \sigma_A)$ holds. Thus there exist $a_4, a_5 \in A$ and $d \in S$ such that $x + e_3x + a_4 + d = e_4x + a_5 + d$. Now (1) implies that

$e_3x + r_1cx + a_2x + tx = e_4x + r_2cx + a_3x + tx$.

Thus $e_3x + r_1cx + a_2x + tx + a_5 + d$

$= (e_4x + a_5+d) + r_2 cx + a_3x + tx$

$= x + e_3x + a_4+d + r_2 cx + a_3x + tx$.

Hence $x + (r_2cx + a_3 x+a_4) + (e_3x + tx + d) = (r_1cx + a_2x + a_5) + (e_3x + tx + d)$ ..... (2).

As B is an h-ideal, $c \in B$ and $A \subsetneqq B$, then $r_2 cx + a_3x + a_4$, $r_1cx + a_2x + a_5 \in B$; it follows from (2) that $x \in B$. Hence $B = S$. This shows that A is a maximal h-ideal of S.

3. h-ideals in additive inverse semirings.

A semiring S is said to be additively regular iff for each $a \in S$, there exists an element $b \in S$ such that $a = a + b + a$. If in addition, the element b is unique and satisfies $b = b + a + b$, then S is called an additive inverse semiring and the unique inverse b of an element a is denoted by a'.

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P. H. Karvellas [23] proved the following result :

Let S be an additive inverse semiring. Then

(i) $x = (x')'$, $(x+y)' = y' + x'$, $(xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$;

(ii) $E^{+}(S) = \{x \in S: x + x = x\}$ is an additively commutative semilattice and an ideal of S.

Clearly, every ring R is an additive inverse semiring, with $a' = -a$ for all $a \in R$.

Every h-ideal of a semiring S is a k-ideal but the converse is not true. The following theorem shows that in some class of semirings these two concepts coincide.

Theorem 3.1. Let S be an additive inverse semiring. Then a k-ideal A is an h-ideal iff $a + a' \in A$ for each $a \in S$.

Proof. Let A be a k-ideal of S such that A is also an h-ideal of S and a be an element of S. As A is an h-ideal, then $Z(S) \subseteq A$. Again $a + a' \in E^{+}(S) \subseteq Z(S)$ for each $a \in S$ implies that $a + a' \in A$ for each $a \in S$. Conversely, let A be a k-ideal of S such that $a + a' \in A$ for each $a \in S$. We claim that A is an h-ideal. Let $x + a + u = b + u$ holds for $a, b \in A$ and $u \in S$.

Then $x + a + u_1 + u' = b + u + u'$. Hence $x + p = v$ for some $p, v \in A$. As A is a k-ideal, $x \in A$. This implies A is an h-ideal.

Corollary 3.2. If A is an h-ideal of an additive inverse semiring S, then $a' \in A$ for each $a \in A$.

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Definition 3.3.[41] A left (right) k-ideal A of an additive inverse semiring S is said to be a full left (right) k-ideal iff $E^+(S) \subseteq A$.
A non-empty subset A of an additive inverse semiring S is called a full k-ideal iff it is both a left and a right full k-ideal.
For a full k-ideal A, $a + a' \in E^+(S) \subseteq A$ for all $a \in S$.
Clearly, every ideal of a ring R is a full k-ideal of R.
Example 3.4. In a distributive lattice with more than two elements, a proper ideal is a k-ideal but not a full k-ideal.
Example 3.5. Let $Z \times N = \{(a, b) : a, b$ are integers and $b > 0\}$.
Define $(a, b) + (c, d) = (a + c$, l. c. m. of b, d) and $(a, b)(c, d)$ = (ac, h. c. f of b, d).
Then $Z \times N$ becomes an additive inverse semiring in which addition is commutative. Let $A = \{(a, b) \in Z \times N : a = 0, b \in N\}$. Then A is a full k-ideal of $Z \times N$.
Proposition 3.6. Any full k-ideal of an additive inverse semiring S is an h-ideal and conversely.
Proof : Let A be a full k-ideal of S. Suppose
$x + a + t = b + t$ for some $x, t \in S$ and $a, b \in A$. Then $x + a + t + t' = b + t + t'$. As A is a full k-ideal, $t + t' \in E^+(S) \subseteq A$. Hence $x \in A$.
Consequently, A is an h-ideal.
Conversely, let A be an h-ideal. Since $E^+(S) \subseteq Z(S) \subseteq A$, it follows that A is a full k-ideal.

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Lemma 3.7. Let S be an additive inverse semiring and A, B be h-ideals of S. Then the k-closure $\overline{A + B}$ of A + B is an h-ideal of S such that

$$A \subseteq \overline{A + B} \quad \text{and} \quad B \subseteq \overline{A + B}.$$

Proof. As A + B is an ideal of S, $\overline{A + B}$ is a k-ideal. Let $a \in A$. Then $a = a + a' + a = a + (a' + a) \in A + B$ as $a' + a \in E^+(S) \subseteq B$.

This shows that $A \subseteq A + B \subseteq \overline{A + B}$.

Similarly, $B \subseteq \overline{A + B}$. As $E^+(S) \subseteq A$ and $E^+(S) \subseteq B$, $E^+(S) \subseteq A + B \subseteq \overline{A + B}$.

This implies that $\overline{A + B}$ is a full k-ideal. Hence $\overline{A + B}$ is an h-ideal by using lemma 3.6.

Theorem 3.8. If I(S) denotes the set of all h-ideals of an additive inverse semiring S, then I(S) is a complete modular lattice.

Proof. We first note that I(S) is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and hence the h-closure $\overline{A + B}$ of A + B belongs to I(S).

Define $A \wedge B = A \cap B$ and $A \vee B = \overline{A + B}$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A + B \subseteq C$ and $\overline{A + B} \subseteq \overline{C} = C$. Hence $\overline{A + B} \subseteq C$. As a result $\overline{A + B}$ is l. u. b of A, B. Thus we find that I(S) is a lattice. Note that $Z(S) \in I(S)$ and also $S \in I(S)$. Consequently, I(S) is a complete lattice.

Next suppose that $A, B, C \in I(S)$ such that $A \wedge B = A \wedge C$ and $A \vee B = A \vee C$ and $B \subseteq C$. Let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A + B}$. Hence there exist $a, a_1 \in A$, $b, b_1 \in B$ and $u \in S$ such that $x + a + b + u = a_1 + b_1 + u$.

Thus $x + a + a' + b + u = a_1 + b_1 + a' + u$ ..... (1). Now $x \in C$, $a + a' \in C$

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and b, $b_1 \in B \subseteq C$. Hence as C is an h-ideal, it follows from (1) that $a_1 + a' \in C$. Again $a' \in A$ by corollary 3.2 and hence $a_1 + a' \in A$. Thus $a_1 + a' \in A \cap C = A \cap B$. This shows that $a_1 + a' \in B$. As B is an h-ideal me find from (1) that $x \in B$ and hence $B = C$.

4. Congruences on semirings:

It is well known that there exists a one-one correspondence between the set of all ideals in a ring R and the set of all ring congruences on R. Examples show that this is not true for a semiring in general (cf. example 5.12 of Chapter - I), and even not true for arbitary additive inverse semirings (cf. example 4.1).

Example 4.1. Let Q^+ be the set of all non-negative rational numbers. Define addition and multiplication in Q^+ by $a + b = \max\{a, b\}$ and ab = usual multiplication. Then Q^+ is an additively inverse semiring. The relation

$$\rho = \{(a, b) \in Q^+ \times Q^+ : ab \neq o\} \cup \{(o, o)\}$$ is a congruence relation on Q^+ such that $\rho \neq Q^+ \times Q^+$ and $\rho \neq$ identity congruence. This shows that there exist at least three congruences on Q^+ but Q^+ has only trivial ideals $\{0\}$ and Q^+.

Definition 4.2. A congruence relation ρ on a semiring S is called a ring congruence iff the quotient semiring S/ρ is a ring.

There always exists at least one ring congruence on a semiring S, namely, the universal congruence.

Definition 4.3. A ring congruence ρ on a semiring S is said to be the

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minimum ring congruence on S iff σ is any ring congruence on S, then $\rho \subseteq \sigma$.

The following theorem gives the minimum ring congruence on an additive inverse semiring with absorbing zero o.

Theorem 4.4. If S is an additive inverse semiring with absorbing zero o, then the relation $\sigma = \{(a,b) \in S \times S : e + a = e + b$ for some $e \in E^+(S)\}$ is the minimum ring congruence on S.

Proof. It is known in [18] that σ is the minimum group congruence on the inverse semigroup (S, +). Let us show that $(a, b) \in \sigma$ implies $(ca, cb) \in \sigma$ and $(ac, bc) \in \sigma$ for all $c \in S$. Since $(a, b) \in \sigma$, there exists $e \in E^+(S)$ such that $a + e = b + e$.

Then $ac + ec = bc + ec$ and $ca + ce = cb + ce$ for all $c \in S$. Since $E^+(S)$ is an ideal of S and $e \in E^+(S)$, ec and $ce \in E^+(S)$. Hence (ac, bc), and $(ca, cb) \in \sigma$. Consequently, σ is a congruence on the semiring S. Hence σ is a ring congruence on S. Let δ be a ring congruence on S. Then δ is a group congruence on (S, +). Hence $\sigma \subseteq \delta$.

We now characterize those ring congruences ρ on an additive inverse semiring S such that $-(a\rho) = a'\rho$, where $-(a\rho)$ denotes the additive inverse of $a\rho$ in the ring S/ρ.

Theorem 4.5. Let A be an h-ideal of an additive inverse semiring S. Then the relation

$\rho_A = \{(a, b) \in S \times S : a + b' \in A\}$ is a ring congruence on S such that $-(a\rho_A) = a'\rho_A$.

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Proof. Since $a + a' \in E^+(S) \subseteq Z(S) \subseteq A$ for all $a \in S$; it follows that ρ_A is reflexive. Let $a + b' \in A$. Now from corollary 3.2, we find that $(a + b)' \in A$. Then $b + a' = (b')' + a' = (a + b')' \in A$. Hence ρ_A is symmetric. Let $a + b' \in A$ and $b + c' \in A$. Then $a + b + b' + c' \in A$. Also $b + b' \in E^+(S) \subseteq Z(S) \subseteq A$. Since A is also a k-ideal, we find that $a + c' \in A$. Hence ρ_A is an equivalence relation. Let $(a, b) \in \rho_A$ and $c \in S$. Then $a + b' \in A$. Since $(c + a) + (c + b)' = c + a + b' + c' = (a+b') + (c + c') \in A$, $ca + (cb)' = ca + cb' = c(a+b') \in A$, $ac + (bc)' = ac + b'c = (a+b')\, c \in A$, it follows that ρ_A is a congruence relation on S. So we obtain the quotient semiring S/ρ_A whose addition and multiplication are defined by $a\rho_A + b\,\rho_A = (a+b)\rho_A$ and $(a\rho_A)\,(b\rho_A) = (ab)\rho_A$.
Now $a\rho_A + b\rho_A = (a + b)\,\rho_A = (b + a)\rho_A = b\,\rho_A + a\,\rho_A$.
Let $e \in E^+(S)$ and $a \in S$. Now $(e + a) + a' = e + (a + a') \in E^+(S)$. We find that $(e + a)\rho_A = a\rho_A$ and hence $e\,\rho_A + a\,\rho_A = a\,\rho_A$. Also $a\rho_A + a'\rho_A = (a + a')\,\rho_A = e\rho_A$.

Hence $e\rho_A$ is the zero element and $a'\rho_A$ is the negative element of $a\rho_A$ in the ring S/ρ_A.

Theorem 4.6. Let ρ be a congruence on an additive inverse semiring S such that S/ρ is a ring and $-(a\rho) = a'\rho$. Then there exists an h-ideal A of S such that $\rho_A = \rho$.

Proof. Let $A = \{a \in S : (a, e) \in \rho$ for some $e \in E^+(S)\}$. Since ρ is reflexive, it follows that $E^+(S) \subseteq A$. Then $A \neq \phi$, since $E^+(S) \neq \phi$.
Let $a, b \in A$. Then there exist $e, f \in E^+(S)$ such that $(a,e) \in \rho$ and

Cont... 83.

$(b,f) \in \rho$. Then $(a+b, e+f) \in \rho$. But $e + f \in E^+(S)$. Hence $a + b \in A$. Again for any $r \in S$, $(ra, re) \in \rho$ and $(ar, er) \in \rho$. But re and $e\, r \in E^+(S)$. Hence A is an ideal of S. Let $a + b \in A$ and $b \in A$. Then there exist $e, f \in E^+(S)$ such that $(a+b, f) \in \rho$ and $(b, e) \in \rho$. Hence $f\rho = (a+b)\rho = a\rho + b\rho = a\rho + e\rho$. But $f\rho$ and $e\rho$ are additive idempotents in the ring S/ρ. Hence $e\rho = f\rho$ is the zero element of S/ρ. As a result, $a\rho$ is the zero element of S/ρ. Then $a\rho = e\rho$. This implies $a \in A$. So we find that A is a full k-ideal of S. Hence by using proposition 3.6, A is an h-ideal of S. Now consider the congruences ρ_A and ρ. Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$. But $b + b' \in E^+(S)$. Hence $a + b' \in A$ and $(a,b) \in \rho_A$. Conversely, suppose that $(a, b) \in \rho_A$. Then $a + b' \in A$. Hence $(a + b', e) \in \rho$ for some $e \in E^+(S)$. As a result, $e\rho = a\rho + b'\rho = a\rho - b\rho$ holds in the ring S/ρ. But $e\rho$ is the zero element of S/ρ. Consequently, $a\rho = b\rho$. This shows that $(a,b) \in \rho$ and hence $\rho_A = \rho$.

5.h - Noetherian additive inverse hemirings and Cohen's theorem.

In this section H denotes a hemiring with identity 1.

Definition. 5.1. The maximal condition for h-ideals in H, is said to hold iff every non-empty set of h-ideals of H, partially ordered by set inclusion, has at least one maximal element, i.e. there exists an h-ideal, which is not properly contained in any h-ideal of the set.

Definition 5.2. A hemiring H satisfies the ascending chain condition (denoted by a.c.c.) on h-ideals, iff for any sequence of h-ideals I_1, I_2, ..., I_n, ... , of H with $I_1 \subseteq I_2 \subseteq \ldots\ldots \subseteq I_n \subseteq$ .... there

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exists an integer n (depending on the sequence) such that $I_m = I_n$ for all $m \geq n$.

The h-ideal generated by $c_1, c_2, \ldots, c_n \in H$ denoted by $\langle c_1, c_2 \ldots, c_n \rangle_h$ is defined by $\langle c_1, c_2, \ldots, c_n \rangle_h = \left\{ x \in H : x + \sum_{i=1}^{n} r_i c_i + d = \sum_{i=1}^{n} t_i c_i + d,\ r_i, t_i, d \in H \right\}$.

Theorem 5.3. The following three conditions on h-ideals of H are equivalent :

(i) H satisfies the ascending chain condition on h-ideals.

(ii) The maximal condition for h-ideals hold in H.

(iii) Every h-ideal in H is finitely generated.

Proof. The proof is similar to that of the ring.

Definition 5.4. An additive inverse hemiring with 1 satisfying any one of the three equivalent conditions of the theorem 5.3 is called an h-Noetherian additive inverse hemiring.

I.S. Cohen [12] proved that a commutative ring R with multiplicative identity is Noetherian iff every prime ideal of R is finitely generated. P. J. Allen [6] showed that Cohen's theorem is valid for the class of M - semirings.

We present the following generalization of Cohen's theorem in the class of additive inverse commutative hemirings with multiplicative identity in terms of h-ideals.

Theorem 5.4. : An additive inverse commutative hemiring H with 1 is h-Noetherian iff every completely prime h-ideal of H is finitely generated.

Cont... 85.

Proof: If H is h-Noetherian, then every h-ideal of H is finitely generated by theorem 5.3. Consequently every completely prime h-ideal of H is finitely generated. Conversely, suppose that every completely prime h-ideal of H is finitely generated but H is not h-Noetherian. This assures that the collection σ of all h-ideals of H which are not finitely generated is non-empty. Appealing to Zorn's lemma, σ must contain a maximal element, call it I. Let (I:b) be the k-ideal defined by $(I:b) = \{ x \in H : x\, b \in I \}$. Since I is an h-ideal, $a + a' \in I$ and hence $(a + a')\, b \in I$ for each $a \in S$. Consequently, $a + a' \in (I:b)$ for each $a \in S$. By theorem 3.1, it follows that (I:b) is an h-ideal. By virtue of our assumption I cannot be a completely prime h-ideal of H. Consequently, there exist elements $a, b \in H$ such that $a \notin I$, $b \notin I$ but $ab \in I$. Now $b \in (I, b)_h$, the smallest h-ideal containing I and b where $b \notin I$. Also $a \in (I : b)$ but $a \notin I$. Then both the h-ideals $(I, b)_h$ and $(I : b)$ properly contain I. By the maximality of I in σ, it follows that the two h-ideals $(I, b)_h$ and $(I : b)$ are finitely generated. For definiteness, let us suppose that

$(I, b)_h = \langle c_1, c_2 \ldots\ldots , c_n \rangle_h$, where

$$\langle c_1, c_2 \ldots\ldots c_n \rangle_h = \{ x \in H,\ x + \sum_1^n r_i c_i + d = \sum_{i=1}^n t_i c_i + d \text{ for some } r_i, t_i, d \in S \};$$

and $(I : b) = \langle d_1, d_2 \ldots\ldots , d_m \rangle_h$.

Cont... 86.

Since $c_i \in (I, b)_h$, $c_i + a_i + r_i b + t = u_i + p_i b + t$, where $a_i, u_i \in I$ and $r_i, p_i, t \in H$, $i = 1,2, \ldots\ldots, n$.

We claim that $(I,b)_h = \langle a_1, a_2, \ldots\ldots, a_n ; u_1, u_2, \ldots, u_n ; b \rangle_h$. Let $x \in \langle a_1, a_2 \ldots\ldots, a_n ; u_1, u_2, \ldots\ldots, u_n ; b \rangle_h$.

Then $x + \sum_{i=1}^{n} (x_i a_i + y_i u_i) + rb + d$

$$= \sum_{i=1}^{n} (l_i a_i + m_i u_i) \; sb + d \ldots\ldots (1),$$

where $x_i, y_i, l_i, m_i, r, s, d \in H$. Since

$\sum_{i=1}^{n} (x_i a_i + y_i u_i)$ and $\sum_{i=1}^{n} (l_i a_i + m_i u_i) \in I$,

it follows that $x \in (I,b)_h$. This implies that

$\langle a_1, a_2, \ldots\ldots, a_n ; u_1, u_2, \ldots\ldots, u_n ; b \rangle_h \subseteq (I, b)_h \ldots (2)$.

Conversely, let $x \in (I, b)_h$. Then

$x + \sum_{i=1}^{n} q_i c_i + p = \sum_{i=1}^{n} f_i c_i + p$, where

$q_i, f_i, p \in H$, $i = 1, 2, \ldots\ldots n$.

Hence $x + \sum_{i=1}^{n} q_i (c_i + a_i + r_i b + t) + \sum_{i=1}^{n} f_i a_i +$

$\sum_{i=1}^{n} f_i (r_i b + t) = \sum_{i=1}^{n} q_i a_i + \sum_{i=1}^{n} q_i (r_i b + t) +$

Cont... 87).

$\sum_{i=1}^{n} f_i (c_i + a_i + r_i b + t)$. Then

$$x + \sum_{i=1}^{n} q_i (u_i + p_i b + t) + \sum_{i=1}^{n} f_i a_i + \sum_{i=1}^{n} f_i (r_i b + t) =$$

$$\sum_{i=1}^{n} q_i a_i + \sum_{i=1}^{n} f_i (u_i + p_i b + t) + \sum_{i=1}^{n} q_i (r_i b + t).$$ Thus

$$x + \sum_{i=1}^{n} (f_i a_i + q_i u_i) + \sum_{i=1}^{n} (q_i p_i + f_i r_i) b$$

$$+ \sum_{i=1}^{n} (q_i + f_i) t = \sum_{i=1}^{n} (q_i a_i + f_i u_i) +$$

$$\sum_{i=1}^{n} (q_i r_i + f_i p_i) b + \sum_{i=1}^{n} (q_i + f_i) t.$$

This shows that $x \in \langle a_1, a_2, \ldots, a_n ; u_1, u_2, \ldots, u_n; b \rangle_h$.

Hence $(I, b)_h \subseteq \langle a_1, a_2, \ldots, a_n; u_1, u_2 \ldots, u_n ; b \rangle_h$ ... (3)

Consequently, $\langle I, b \rangle_h = \langle a_1, a_2, \ldots, a_n ; u_1, u_2, \ldots, u_n ; b \rangle_h$

Thus we have,

$(I, b)_h = \langle c_1, c_2 \ldots, c_m \rangle_h = \langle a_1, a_2, \ldots, a_n, u_1, u_2,$
$u_n; b \rangle_h$.

Let $J = \langle a_1, a_2 \ldots, a_n, u_1, u_2 \ldots, u_n ; bd_1,$
$bd_2, \ldots, bd_m; b + b' \rangle_h$. As $b + b' \in E^+(S) \subseteq Z(S) \subseteq I$ and
$a_i, u_i, bd_j \in I$, $i = 1, 2, \ldots n$, $j = 1, 2, \ldots, m$,
it follows that $J \subseteq I$.

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On the otherhand, for $x \in I \subseteq (I, b)_h$, we find that $x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) +$

$+ tb + d = \sum_{i=1}^{n} (f_i a_i + g_i u_i) + sb + d$, where

$s_i, t_i, s, t, f_i, g_i, d \in H$.

Then $x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) + (t + s') b + d =$

$$\sum_{i=1}^{n} (f_i a_i + g_i u_i) + (s + s') b + d \quad \ldots\ldots (4).$$

Since $\sum_{i=1}^{n} (t_i a_i + s_i u_i) \in I$, $x \in I$ and

$\sum_{i=1}^{n} (f_i a_i + g_i u_i) + (s + s') b \in I$, as $s + s' \in E^{+}(S)$

$\subseteq Z(S) \subseteq I$ and I is an h-ideal, it follows that

$(t + s') b \in I$ and hence $t + s' \in (I : b)$.

Then there exist $l_i, v_i, u \in H$ such that

$$t + s' + \sum_{i=1}^{m} l_i d_i + u = \sum_{i=1}^{m} v_i d_i + u \quad \ldots\ldots (5).$$

Hence from (4) it follows that

$$x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) + (t + s') b + \sum_{i=1}^{m} l_i d_i b + d + ub$$

$$= \sum_{i=1}^{n} (f_i a_i + g_i u_i) + (s + s') b + d + \sum_{i=1}^{m} l_i d_i b + ub.$$

By using (5), this implies that

Cont... 89.

$$x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) + \sum_{i=1}^{m} v_i d_i b + ub + d$$

$$= \sum_{i=1}^{n} (f_i a_i + g_i u_i) + (s + s') b + \sum_{i=1}^{m} 1_i d_i b + ub + d.$$

As $(s + s') + (s + s') = s + s'$, we have

$$x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) + \sum_{i=1}^{m} v_i d_i b + (s + s') b + ub + d$$

$$= \sum_{i=1}^{n} (f_i a_i + g_i u_i) + (s + s') b + \sum_{i=1}^{m} 1_i d_i b + ub + d \; \ldots \; (6).$$

Since H is an additive inverse commutative semiring,

$(s + s') b = sb + s'b = b's + bs = (b+b')s$.

Hence from (6) it follows that

$$x + \sum_{i=1}^{n} (t_i a_i + s_i u_i) + \sum_{i=1}^{m} v_i bd_i + (b + b') s$$

$$+ ub + d = \sum_{i=1}^{n} (f_i a_i + g_i u_i) + \sum_{i=1}^{m} 1_i bd_i$$

$$+ (b + b')s + ub + d.$$

This shows that $x \in J$. Thus $I \subseteq J$.

Consequently, $I = J$. Hence one concludes that I is itself finitely generated, an impossibility, since $I \in \sigma$.

This contradiction completes the proof.

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Theorem 6.5. Every homorphic image of a commutative h-Noetherian additive inverse hemiring with identity is also a commutative h-Noetherian additive inverse hemiring with identity.

Proof. Let $f : S \longrightarrow T$ be an epimorphism of a commutative h-Noetherian additive inverse hemiring S with identity onto a hemiring T with identity. Let $b \in T$. Then $f(a) = b$ for some $a \in S$. Hence $a = a + a' + a$. Then $f(a) = f(a) + f(a') + f(a)$. This shows that $f(a')' = f(a')$. Hence T is an additive inverse hemiring. Moreover, as S is commutative, T is also commutative.

Consider any ascending chain of h-ideals of T :

$J_1 \subseteq J_2 \subseteq \ldots\ldots \subseteq J_n \subseteq \ldots\ldots$ (1)

Put $I_r = f^{-1} (J_r)$ for $r = 1, 2, 3, \ldots\ldots$

Then $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots\ldots$ (2)

forms an ascending chain of ideals of S.

Moreover, if $a + b, a \in I_r$, then $f(a + b) = f(a) + f(b) \in J_r$ and $f(a) \in J_r$ for $r = 1, 2, 3, \ldots\ldots$

As each J_r is also a k-ideal, $f(b) \in J_r$. This implies $b \in I_r$. Consequently, each I_r is a k-ideal.

Now for any $a \in S$, $f(a + a') = f(a) + f(a') = f(a) + f(a')'$ $= t + t'$ for $t = f(a) \in T$. As each J_r is an h-ideal, $t + t' \in J_r$ for each r.

Consequently, $f^{-1}(t + t') \in I_r$. This shows that $a + a' \in I_r$ for each $a \in S$. Consequently, each I_r is an h-ideal by theorem 3.1.

Hence according to our hypothesis for the chain (2), there is some

Cont... 91.

index n such that $I_m = I_n$ for all $m \geq n$.

Since f is an epimorphism, $f(I_r) = J_r$ for each r. Hence $J_m = J_n$ whenever $m \geq n$ so that the original chain (1) also stablizes at some point.

Consequently, T is a commutative h-Noetheriam additive inverse hemiring with identity.

Theorem 6.6. Let H be a commutative additive inverse hemiring with identity and I an h-ideal of H. If I and H/I are h-Noetherian, then H is also h-Noetherian.

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots$ be an ascending chain of h-ideals of H. Then $I \cap I_1 \subseteq I \cap I_2 \subseteq \ldots\ldots$ (1) is also an ascending chain of h-ideals of I. Define J_r by

$J_r = \{a\rho_I \in H/I : a\rho_I = b\rho_I$ for some $b \in I_r\}$

for $r = 1, 2, 3, \ldots\ldots$, where ρ_I is the Iizuka congruence on H determined by I.

Let $a\rho_I$, $b\rho_I \in \in J_r$. Then there exist $x, y \in I_r$ such that $a\rho_I = x\rho_I$ and $b\rho_I = y\rho_I$. Hence $x + y \in I_r$ and $sx, xs \in I_r$ for all $s \in H$. This shows that J_r is an ideal of H/I.

Next, let $a\rho_I + b\rho_I \in J_r$ and $a\rho_I \in J_r$.

Hence $(a+b)\rho_I = d\rho_I$ for some $d \in I_r$ and $a\rho_I = x\rho_I$ for some $x \in I_r$. Hence $a + b + i_1 + u = d + i_2 + u$ and $a + i_3 + p = x + i_4 + p$ for some $i_t \in I$ and $p \in H$. Hence $a + i_3 + b + i_1 + u + p = d + i_2 + u + i_3 + p$ implies $b + x + i_4 + i_1 + u + p = d + i_2 + i_3 + u + p$. Then $b + x + x' + i_4 + i_1 + u + p$

Cont... 92.

$= d + x' + i_2 + i_3 + u + p$ implies $b\rho_I = (d + x')\rho_I$.

Since $x \in I_r$, then $x' \in I_r$ and hence $d + x' \in I_r$. As a result $b\rho_I \in J_r$

So we find that J_r is a k-ideal of H/I for r = 1, 2, 3, ...

Since $(a + a') \in I$ for each $a \in S$, then

$(a + a')\rho_I = e\rho_I$ for some $e \in E^+(S) \subseteq I_r$.

This implies that each J_r is an h-ideal of H/I for r = 1, 2, 3, ...

Also we have

$J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots\ldots$ (2),

an ascending chain of h-ideals of H/I.

Since H/I and I are h-Noetherian, there exists a positive integer m such that

$$J_m = J_{m+i} \text{ and } I \cap I_m = I \cap I_{m+i} \text{ for all } i \geq 0.$$

We claim that $I_{m+i} = I_m$ for all $i \geq 0$.

Let $a \in I_{m+i}$. Then $a\rho_I \in J_{m+i} = J_m$ for $i \geq 0$.

Hence there exists some $b \in I_m$ such that $a\rho_I = b\rho_I$.

Now $a + i_1 + v = b + i_2 + v$ for some $i_1, i_2 \in I$ and $v \in H$.

Then $a + b' + i_1 + v = b + b' + i_2 + v$. Hence as $b + b' \in I$ and I is an h-ideal, then $a + b' \in I$. But $b \in I_m$ implies $b' \in I_m$ and hence $b' \in I_{m+i}$ for all $i \geq 0$.

Thus $a + b' \in I_{m+i}$. So we find that $a + b' \in I \cap I_{m+i}$ $= I \cap I_m$. Hence $a + b' \in I_m$. But $b' \in I_m$ and I_m is also a k-ideal.

Consequently, $a \in I_m$. This implies that $I_m = I_{m+i}$ for all $i \geq 0$.

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