CHAPTER - II

SEMIRINGS OF FORMAL POWER SERIES AND LOCAL SEMIRINGS

In this chapter semirings of formal power series over semirings, in particular, over k-semifields are studied. Some properties of local semirings are also studied. It is shown that the semiring of formal power series over a k-semifield becomes a local semiring. Moreover the Jacobson radical of formal power series over a k-semifield is described.

1. Semirings of formal power series.

In this section we only consider semirings in which addition is commutative. Let S be a semiring with absorbing zero 0 and identity 1 and x an indeterminate. Let $S[[x]]$ denote the set of all expressions of the form

$$f = \sum_{i=o}^{\infty} a_i x^i, \ a_i \in S.$$

If $g = \sum_{i=o}^{\infty} b_i x^i$ is also an element of $S[[x]]$,

we define addition and multiplication in $S[[x]]$ as follows:

$$f + g = \sum_{i=o}^{\infty} (a_i + b_i) x^i \quad \text{and } fg = \sum_{i=o}^{\infty} d_i x^i, \text{ where}$$

$$d_i = \sum_{r+s=i} a_r b_s, \quad i = o, 1, 2, \ldots\ldots.$$

Then $S[[x]]$ is a semiring which is called the semiring of formal power series (in the indeterminate x) over S. We say that $f = g$ iff $a_i = b_i$ for all non-negative integers i, a_o is called the constant term of f.

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If $a_i = o$ for $i = o, 1, 2, \ldots\ldots, r-1$ but $a_r \neq o$, then f is said to be of order r.

Following usual convention, the Co-efficients not indicated are understood to be zero, the elements of S themselves may be considered to be elements of S[[x]]. That is, S[[x]] contains the semiring S as a subsemiring. In particular 1 is also identity and O is also absorbing zero of S[[x]]. Moreover, for example, if n is a positive integer, we may consider x^n to be an element of S[[x]]. We interpret it to be the formal power series with x^n having coefficient 1 and all other co-efficients being O.

Theorem 1.1. If S is a k-division semiring, then S[[x]] has no zero divisors.

Proof. : Let $f = \sum_{i=o}^{\infty} a_i x^i$ and $g = \sum_{i=o}^{\infty} b_i x^i$ be two non-zero elements of S[[x]]. Let r be the smallest index such that $a_r \neq O$ and t the smallest index such that $b_t \neq O$. Since S is a k-division semiring, then by using the proposition 5.8. of chapter I, we find that $a_r b_t \neq O$.

Now $fg = \sum_{n=o}^{\infty} d_n x^n$ where $d_n = \sum_{i=o}^{n} a_{n-i} b_i$. Then

$do = d_1 = \ldots\ldots = d_{r+t-1} = O$ and $d_{r+t} = a_r b_t \neq O$.

This shows that S[[x]] has no zero divisors.

Theorem 1.2. An element $f = \sum_{i=o}^{\infty} a_i x^i$ of S[[x]]

is semi-invertible in S[[x]] iff a_o is semi-invertible in S.

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Proof. Suppose f is semi-invertible in $S[[x]]$. Then there exist $g, h \in S[[x]]$ such that $1 + fg = fh$ and $1 + gf = hf$ by corollary 5.3 of chapter I.

Suppose $g = \sum_{i=0}^{\infty} b_i x^i$ and $h = \sum_{i=0}^{\infty} c_i x^i$. Then it follows that

$1 + a_o b_o = a_o c_o$ and $1 + b_o a_o = c_o a_o$. This shows that a_o is semi-invertible in S.

Conversely, suppose a_o is semi-invertible in S. Then there exist $b_o, c_o \in S$ such that

$1 + a_o b_o = a_o c_o$ and $1 + b_o a_o = c_o a_o$. We now look for

$g = \sum_{i=0}^{\infty} b_i x^i$ and $h = \sum_{i=0}^{\infty} c_i x^i$ such that

$1 + fg = fh$ and $1 + gf = hf$.

Now $a_1 + a_o b_o a_1 = a_o c_o a_1$. Then $a_1 b_o + a_o b_o a_1 b_o$

$= a_o c_o a_1 b_o$ and $a_1 c_o + a_o b_o a_1 c_o = a_o c_o a_1 c_o$.

Hence $a_1 b_o + a_o b_o a_1 b_o + a_o c_o a_1 c_o = a_1 c_o + a_o b_o a_1 c_o + a_o c_o a_1 b_o$.

Consequently,

$a_o (b_o a_1 b_o + c_o a_1 c_o) + a_1 b_o = a_o (b_o a_1 c_o + c_o a_1 b_o) + a_1 c_o$.

Hence there exist $b_1 = b_o a_1 b_o + c_o a_1 c_o$ and $c_1 = b_o a_1 c_o + c_o a_1 b_o$ $\in S$ such that $a_o b_1 + a_1 b_o = a_o c_1 + a_1 c_o$.

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We now prove the theorem by induction.

Suppose there exist $b_1, b_2, \ldots\ldots b_m$; $c_1, c_2, \ldots\ldots, c_m$ in S such that

$$a_ob_m + a_1b_{m-1} + \ldots\ldots + a_mb_o = a_oc_m + a_1c_{m-1} + \ldots + a_mc_o.$$

Now $1 + a_ob_o = a_oc_o$ implies

$a_n + a_ob_oa_n = a_oc_oa_n$, for $n = 1, 2, 3, \ldots\ldots$

Hence $a_nb_t + a_ob_oa_nb_t = a_oc_oa_nb_t$ and

$a_nc_t + a_ob_oa_nc_t = a_oc_oa_nc_t$, for $t = o. 1, 2, \ldots\ldots, m.$

Consequently,

$$a_nb_t + a_ob_oa_nb_t + a_oc_oa_nc_t = a_nc_t + a_ob_oa_nc_t + a_oc_oa_nb_t.$$

Thus taking $n = m+1, m, \ldots\ldots 1$; $t = o, 1, 2, \ldots\ldots, m$ we have

$$a_{m+1}b_o + a_ob_oa_{m+1}b_o + a_oc_oa_{m+1}c_o = a_{m+1}c_o + a_ob_oa_{m+1}c_o + a_oc_oa_{m+1}b_o.$$

$$a_mb_1 + a_ob_oa_mb_1 + a_oc_oa_mc_1 = a_mc_1 + a_ob_oa_mc_1 + a_oc_oa_mb_1.$$

$$\ldots\ldots$$

$$a_1b_m + a_ob_oa_1b_m + a_oc_oa_1c_m = a_1c_m + a_ob_oa_1c_m + a_oc_oa_1b_m.$$

Hence $a_{m+1}b_o + a_mb_1 + \ldots\ldots + a_1b_m + a_o(b_oa_{m+1}b_o + b_oa_mb_1 + \ldots + b_oa_1b_m + c_oa_{m+1}c_o + c_oa_mc_1 + \ldots\ldots + c_oa_1c_m)$

$= a_{m+1}c_o + a_mc_1 + \ldots\ldots + a_1c_m + a_o(b_oa_{m+1}c_o + b_oa_mc_1 + \ldots\ldots + b_oa_1c_m + c_oa_{m+1}b_o + c_oa_mb_1 + \ldots\ldots + c_oa_1b_m).$

Consequently, there exist $b_{m+1}, c_{m+1} \in S$ such that

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$a_o b_{m+1} + a_1 b_m + \ldots + a_m b_1 + a_{m+1} b_o = a_o c_{m+1} + a_1 c_m + \ldots +$
$a_m c_1 + a_{m+1} c_o$.

Thus we find $g, h \in S[[x]]$ such that $1 + fg = fh$.

We can also show that $1 + gf = hf$.

As a result, f is semi-invertible in $S[[x]]$.

Corollary 1.3. If S is a k-division semiring, then the semi-invertible elements of $S[[x]]$ are precisely those formal power series with non-zero constant terms and every non-zero element of $S[[x]]$ of order r is of the form $x^r u$ such that $u \in S[[x]]$ is semi-invertible in $S[[x]]$.

Proof. The first statement follows from theorem 1.2.

For the second statement let f be a non-zero element of order r in $S[[x]]$.

Suppose $f = \sum_{i=r}^{\infty} a_i x^i$ where $a_r \neq 0$. Then $f = x^r (a_r + a_{r+1} x +$

$$+ a_{r+2} x^2 + \ldots + \ldots) = x^r u \text{ where } u = a_r + a_{r+1} x + a_{r+2} x^2 +$$

$\ldots \in S[[x]]$ and as $a_r \neq 0$, u is semi-invertible in $S[[x]]$ by theorem 1.2.

Definition 1.4. A k-ideal A of a semiring S is said to be principal iff A is generated by an element $a \in S$ i.e. $A = \langle a \rangle_k$ for some $a \in S$, where $\langle a \rangle_k = \{x \in S : x + s_1 a = s_2 a,\ s_i \in S\}$.

Theorem 1.5. If S is a k-division semiring, then every k-ideal of $S[[x]]$ is principal.

Proof. Let A be a k-ideal of $R = S[[x]]$. Suppose $A \neq \{0\}$. Then A contains

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some non-zero element. Let $P = \{n : n \text{ is the order of some } f \in A\}$. Then P is a collection of some positive integers. By well ordering principle, P has at least one element. Let r be the least element of P. Then there exists some $f \in A$ such that order of f denoted by $O(f)$ is r. Then we can write $f = x^r u$ for some $u \in R$ such that u is semi-invertible in R. Then there exist $h_1, h_2 \in R$ such that $1 + uh_1 = uh_2$ Hence $x^r + x^r uh_1 = x^r uh_2$. This implies $x^r + fh_1 = fh_2$. As A is a k-ideal and $f \in A$, it follows that $x^r \in A$. Let g be a non-zero element of A. Now $O(g) \geq r$. Let $O(g) = r + t$ $(t \geq o)$. So we can write $g = x^r g_1$ for some $g_1 \in R$. Hence it follows that each k-ideal $A = \langle x^r \rangle_k$ for some positive integer r (depending on A).

2. Local semirings:

While considering k-ideals of a semiring S, it is natural to consider conditions such that S has the unique maximal k-ideal.

In this section S denotes a commutative semiring with absorbing zero 0 and identity 1 and M denotes the set of all non semi-invertible elements of S (including 0)

Theorem 2.1. The set M is precisely the union of all proper k-ideals of S which in turn is also the union of all maximal k-ideals of S.

Proof: Let P be a proper k-ideal of S. Then $a \in P$ implies $a \in M$. Otherwise, $a \notin M$ would imply $1 + ra = sa$ for suitable $r, s \in S$, yielding $1 \in P$, as P is a k-ideal and $ra, sa \in P$. This shows that $P = S$, a contradiction. As a result $P \subseteq M$ and hence $\bigcup P_i \subseteq M$, where $\{P_i, i \in I\}$ is

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the collection of all proper k-ideals of S. On the otherhand, if $0 \neq x \in M$, $1 + rx = sx$ holds for no r, $s \in S$. Hence $1 \notin \overline{Sx}$ implies $\overline{Sx}$ is a proper k-ideal of S. As a result $x \in \overline{Sx} \subseteq P_i$ for some $i \in I$. Hence $M = \bigcup_i P_i$.

The second part is immediate by corollary 2.7 of chapter I.

Theorem 2.2. If M is a k-ideal of S, then M is the unique maximal k-ideal of S.

Proof. Let M be a k-ideal of S. Then M is an absorbing zero of the semiring S/M defined by the Bourne congruence ρ_A on S. Suppose $a\rho_A \neq M$. Then $a \notin M$. Hence there exist suitable s_1, $s_2 \in S$ such that

$1 + s_1 a = s_2 a$. Then

$1\rho_M + (s_1 \rho_M)(a \rho_M) = (s_2 \rho_M)(a \rho_M)$ yields that S/M satisfies condition (C) (of. definition 2.8 of Chapter I). Hence M is a maximal k-ideal of S by theorem 2.13 of chapter I.

To show the uniqueness of the maximal k-ideal, let A be a proper k-ideal of S which is maximal. Then by using theorem 2.13 of chapter - I we find that S/A satisfies the condition (C). As $A \neq S$, there is no element $0 \neq a \in A$ satisfying $1 + ra = sa$ for some r, $s \in S$. Otherwise we would reach a contradiction. Then clearly every element of A is non semi-invertible in S. Hence $A \subseteq M$. As A and M are both maximal k-ideals of S, $A = M$.

Definition 2.3. A commutative semiring S with absorbing zero o and identity 1 is said to be a local semiring iff S has unique maximal k-ideal.

Clearly, every k-semifield is a local semiring by corollary 5.11 of Chapter-I.

Theorem 2.4. If S is a commutative semiring with absorbing zero o and

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identity 1, then the following conditions are equivalent.

(i) S is a local semiring.

(ii) M is a k-ideal of S.

Proof (i) $\Longrightarrow$ (ii) Let S be a local semiring with the maximal k-ideal A. For $x \in M$, x is not semi-invertible in S. Hence $\overline{Sx}$ is a proper k-ideal of S. Now by Corollary 2.7 of Chapter I, $\overline{Sx}$ is contained in a maximal k-ideal of S. But (i) implies that this maximal k-ideal is A. Hence $\overline{Sx} \subseteq A$. As a result, $x \in A$. This shows that $M \subseteq A$. But by theorem 2.1, $A \subseteq M$. Hence $M = A$.

This shows that M is the maximal k-ideal of S.

(ii) $\Longrightarrow$ (i) by theorem 2.2.

Corollary 2.5. If S is a local semiring with M its k-ideal of non semi-invertible elements of S, then S/M is a k-semifield.

Proof. As S is a local semiring, M is the maximal k-ideal of S by theorem 2.4. Clearly, M is a proper k-ideal of S. Hence by theorem 2.13 of chapter I, S/M satisfies the condition (C). Consequently, S/M is a k-semifield by using theorem 2.12 of chapter I and Corollary 5.11 of Chapter - I

Theorem 2.6. If S is a k-semifield, then the semiring of formal power series $S[[x]]$ over S is a local semiring.

Proof. Since S is a commutative semiring with o and 1, we find that $S[[x]]$ is also a commutative semiring with o and 1. Now $x \in S[[x]]$. Suppose $R = S[[x]]$. Consider the k-ideal $\overline{Rx}$. We claim that $\overline{Rx}$ is the unique

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maximal k-ideal of $S[[x]]$. From theorem 1.2 it follows that a formal power series $f \in S[[x]]$ is non semi-invertible in $S[[x]]$ iff the constant term of f is o. Hence $f \in \overline{Rx}$ iff f is non semi-invertible in $S[[x]]$. Consequently, $\overline{Rx}$ is the k-ideal of all non semi-invertible elements of $S[[x]]$. Then by using theorem 2.4 it follows that $S[[x]]$ is local semiring.

Theorem2.7. A commutative semiring S with 0 and 1 is a local semiring iff for a, b, r, $s \in S$, $1 + a + r = b + s$ implies one of a, r, b, s is semi-invertible in S.

Proof. Suppose S is a local semiring and let A be the unique maximal k-ideal of S. Suppose $1 + a + r = b + s$ for some a, b, r, $s \in S$. If all of a, b, r, s are non semi-invertible elements of S, then a, r, b, $s \in A$ as A contains all non semi-invertible elements of S. Hence $a + r$, $b + s \in A$. As A is a k-ideal, $1 \in A$ and hence $A = S$, a contradiction.

Conversely, assume that the given condition holds in S. We claim that S is a local semiring. If a is not semi-invertible in S, then $\overline{Sa}$ is a proper k-ideal of S. Hence by corollary 2.7 of Chapter I, $\overline{Sa}$ is contained in a maximal k-ideal A(say) of S. We claim that A contains all non semi-invertible elements of S. Suppose x is a non semi-invertible element of S such that $x \notin A$. Then $\overline{(A,x)} = S$ by the maximality of the k-ideal A of S. This implies $1 + a_1 + r_1x = a_2 + r_2x$ for some $r_i \in S$, $a_i \in A$. Since a_1, $a_2 \in A$, they cannot be semi-invertible in S. Hence by hypothesis, either r_1x or r_2x is semi-invertible in S. If r_1x is semi-invertible in S, then $1 + t_1 r_1x = t_2r_1x$ for some $t_1, t_2 \in S$. This shows that x is also

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semi-invertible in S. Similarly, if r_2x is semi-invertible in S, x is also semi-invertible in S. But this is a contradiction. Hence A consists of all non semi-invertible elements of S. As a result S is a local semiring by theorem 2.4.

Theorem 2.8. Let S be a commutative semiring with 0 and 1. If S is a local semiring, then S[[x]] is also a local semiring.

Proof: Let $f = \sum_{i=0}^{\infty} a_i x^i$, $g = \sum_{i=0}^{\infty} b_i x^i$,

$$h = \sum_{i=0}^{\infty} c_i x^i \text{ and } t = \sum_{i=0}^{\infty} d_i x^i \in S[[x]]$$

such that $1 + f + g = h + t$. Then

$1 + a_0 + b_0 = c_0 + d_0$. Hence by theorem 2.7 it follows that one of a_0, b_0, c_0, d_0 is semi-invertible in S. Then theorem 1.2 implies that one of f, g, h, t is semi-invertible in S[[x]]. Moreover S[[x]] is a commutative semiring with 0 and 1. Hence theorem 2.7 again implies that S[[x]] is a local semiring.

Remarks 2.9. As every k-semifield is a local semiring, the theorem 2.6 also follows from theorem 2.8.

The Jacobson radical of a hemiring was introduced by Bourne [8] and was subsequently studied by Bourne and Zassenhaus [10], Iisuka [20].

Definition 2.10. Let S be a commutative semiring with o and 1. The Jacobson radical of S, denoted by rad S is the set

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red $S = \cap \{M : M$ is a maximal k-ideal of $S\}$.

The Jacobson radical of S always exists by corollary 2.7 of Chapter I. Clearly, red S forms a k-ideal of S and is contained in each maximal k-ideal of S.

Let us determine the Jacobson radical of $S[[x]]$ over a k-semifield S.

Proposition 2.11. If S is a k-semifield, then red $(S[[x]])$

$$= \{ f = \sum_{i=0}^{\infty} a_i x^i \in S[[x]] : a_0 = 0 \}.$$

Proof. Let $A = \{ f = \sum_{i=0}^{\infty} a_i x^i \in S[[x]] : a_0 = 0 \}$.

Then A is a k-ideal of $S[[x]]$. As $1 \in S[[x]]$ but $1 \notin A$, A is a proper k-ideal of $S[[x]]$. From theorem 1.2 we find that each element of A is non semi-invertible. Now let $f \in S[[x]]$ be such that

$f = \sum_{i=0}^{\infty} a_i x^i$ is not semi-invertible in $S[[x]]$. Then from the theorem 1.2 it follows that $a_0 = 0$. Hence $f \in A$. So we find that A is the set of all non semi-invertible elements of $S[[x]]$. As S is a k-semifield, $S[[x]]$ is a local semiring by theorem 2.6. Hence the non semi-invertible elements of $S[[x]]$ forms the only maximal k-ideal. As a result red $(S[[x]]) = A$.

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