INTRODUCTION

H.S. Vandiver gave the first formal definition of a semiring and developed the theory of a special class of semirings in 1934 [46]. Theory of semirings can be treated as a common generalization of the theory of associative rings and theory of distributive lattices. A semiring S is defined as an algebra $(S, +, .)$ such that $(S, +)$ and $(S, .)$ are semigroups connected by $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$.

The set N of all non-negative integers with usual addition and multiplication of integers is an example of a semiring, called the semiring of non-negative integers.

A semiring S may have an additive zero o defined by $o + a = a + o = a$ for all $a \in S$ or a multiplicative zero 0 defined by $0a = a0 = 0$ for all $a \in S$. S may contain both o and 0 but they may not coincide. Consider the semiring $(N, +, .)$, where N is the set of all non-negative integers;

$$a + b = \{\text{l. c. m. of } a \text{ and } b \text{ when } a \neq o, b \neq o\};$$

$$= 0, \text{ otherwise };$$

and $a . b$ = usual product of a and b.

Then the integer 1 is the additive zero and integer 0 is the multiplicative zero of $(N, +, .)$.

In general, the concept of multiplicative zero 0 and additive zero o may not coincide in a semiring. H. J. Weinert [47] proved that both these concepts coincide if $(S, +)$ is a cancellative semigroup. Clearly, S has an absorbing zero (or a zero element) iff it has elements 0 and o which coincide [48]. Thus an absorbing zero of a semiring S is an element 0 such that $0a = a0 = 0$ and $0 + a = a + 0 = a$ for all $a \in S$.

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A semiring S may have an identity 1 defined by $1a = a1 = a$ for all $a \in S$.

A semiring S is said to be additively commutative iff $a + b = b + a$ for all $a, b \in S$. An additively commutative semiring with an absorbing zero 0 is called a hemiring. An additively cancellative hemiring is called a halfring [49]. A multiplicatively cancellative commutative semiring with identity 1 is called a semidomain [14]. In this thesis we only consider semirings S for which $(S, +)$ is commutative. If $(S, \circ)$ is also commutative, S is called a commutative semiring. Moreover, to avoid trivial exceptions, each semiring S is assumed to have at least two elements.

A subset $A \neq \emptyset$ of a semiring S is called an ideal (left, right) of S iff $a + b \in A$, $sa \in A$ and $a s \in A$ ($s a \in A$, $a s \in A$) hold for all $a, b \in A$ and all $s \in S$. An ideal A of S is called proper iff $A \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal A is called maximal iff there is no ideal B of S satisfying $A \subset B \subset S$ An ideal A of S is called trivial iff $A = S$ holds or $A = \{0\}$ (the latter clearly holds if S has an absorbing zero 0).

Many results in rings related to ideals have no analogues in semirings. As noted by Henriksen [17], it is not the case that any ideal A of a semiring S is the kernel of a homomorphism. To get rid of these difficulties, he defined in 1958 [17] a more restricted class of ideals in a semiring which he called k-ideals. A k-ideal of a semiring S is an ideal of S such that whenever $x + a \in A$, where $a \in A$ and $x \in S$, then $x \in A$. Iizuka [20] defined a still more restricted class of ideals in semirings, which he called h-ideals. An h-ideal A of a semiring S is an ideal of S such that if $x + a + u = b + u$, where $x, u \in S$ and

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a, b $\in$ A, then x $\in$ A. It is clear that every h-ideal is a k-ideal. But examples disapprove the converse. Using only commutativity of addition, the following concepts and statements essentially due to Bourne [8], Bourne and Zassenhaus [10] and Henriksen [17], are well known. For each ideal A of a semiring S the k-closure $\bar{A}$ of A defined by $\bar{A} = \{a \in S : a + a_1 = a_2$ for some $a_1 \in A\}$ is a k-ideal of S satisfying $A \subseteq \bar{A}$ and $\bar{\bar{A}} = \bar{A}$. Clearly, an ideal A of S is a k-ideal of S iff $A = \bar{A}$ holds. A proper k-ideal A of S is called a maximal k-ideal of S iff there is no k-ideal B of S satisfying $A \subset B \subset S$.

An ideal A of a semiring S is called completely prime (cf e.g. [21]) iff a b $\in$ A implies a $\in$ A or b $\in$ A for all a, b $\in$ S.

Let S and T be hemirings. A map $f : S \longrightarrow T$ is said to be a hemiring homomorphism iff $f(s_1 + s_2) = f(s_1) + f(s_2)$; $f(s_1 s_2) = f(s_1) f(s_2)$ and f(o) = f(o).

A hemiring homomorphism $f : S \longrightarrow T$ is said to be an N - homomorphism [32] iff whenever f(x) = f(y) for some x, y $\in$ S, there exist $r_i \in$ ker f such that $x + r_1 = y + r_2$ holds.

An equivalence relation ρ on a semiring S is called a congruence relation iff (a, b) $\in \rho$ implies (a + c, b + c) $\in \rho$, (ca, cb) $\in \rho$ and (ac, bc) $\in \rho$ for all c $\in$ S.

A congruence relation ρ on a semiring S is said to be additively cancellative (AC) iff (a + c, b + d) $\in \rho$, and (c, d) $\in \rho$ imply (a,b) $\in \rho$.

Each ideal A of S defines a congruence relation ρ_A on (S, + ,.)
$\rho_A = \{(x,y) \in S \times S : x + a_1 = y + a_2$ for some $a_i \in A\}$; this is known as Bourne congruence [8]. The corresponding class semiring S/ ρ_A, consisting of the classes x ρ_A, is also denoted by S/A.

Each ideal A of S defines another type of congruence relation known

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as Iizuka congruence [20]:

σ_A = { (x, y) ∈ S x S : x + a + u = y + b + u for some a, b ∈ A and u ∈ S }. The corresponding class semiring S/σ_A consists of the classes $x\sigma_A$, x ∈ S.

Bourne and Zassenhaus [10] defined the zeroid Z (S) of a semiring S as Z(S) = { z ∈ S : z + x = x for some x ∈ S }. Clearly, Z(S) is an h-ideal of S and is contained in every h-ideal of S. Z(S) and S are called trivial h- ideals and other h-ideals (if they exist) are called proper h-ideals of S. A proper h-ideal A of S is called a maximal h-ideal iff there is no h-ideal B of S satisfying $A \subset B \subset S$.

The h-closure A^{-h} of each ideal A of a semiring S defined by, A^{-h} = {x ∈ S : x + a_1 + u = a_2 + u for some a_i ∈ A and u ∈ S} is the smallest h-ideal of S containing A.

A semiring S is said to be additively regular iff for each a ∈ S, there exists an element b ∈ S such that a = a + b + a. If in addition, the element b is unique and satisfies b = b + a + b, then S is called an additively inverse semiring.

A semiring S is said to be semisubtractive (cf.e.g. [32]) iff for each pair a, b in S at least one of the equations a + x = b or b + x = a is solvable in S.

Let S be a semiring with absorbing zero o. A left semimodule over S is a commutative additive semigroup M with a zero element O

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together with an operation

$$S \times M \longrightarrow M; \quad (a, x) \longrightarrow ax$$

called the scalar multiplication, such that

for all $a, b \in S$, $x, y \in M$, $a(x+y) = ax + ay$,

$(a + b) x = ax + bx$, $(ab) x = a(bx)$, $ox = 0$.

A right S - semimodule is defined in an analogous manner.

Let H and T be hemirings. An (H,T) bi - semimodule M is both a left semidodule over H and a right semimodule over T such that $(bx) a = b(xa)$ for all $x \in M$, $b \in H$ and $a \in T$.

Let R be an arbitary hemiring. Let S be also a hemiring.

R is said to be an S -semialgebra iff R is a bi-semimodule over S such that $(ax)b = a(xb)$ for all $a, b \in R$ and $x \in S$.

An h-ideal A of the S - semialgebra R is said to be a left modular h-ideal iff there exists an element $e \in R$ such that

(i) each $x \in R$ satisfies $ex + a + z = x + b + z$ for some $z \in R$ and some $a, b \in A$;

(ii) each $s \in S$ satisfies $se + c + h = es + d + h$ for some $h \in R$ and some $c, d \in A$.

In this case e is called a left unit modulo A.

A pair (e_1, e_2) of elements of a semiring S is called an identity pair [42] iff $a + e_1 a = e_2 a$ and $a + ae_1 = ae_2$ for all $a \in S$.

We define a right modular h-ideal in a similar manner.

A congruence ρ on a semiring S is called a ring congruence [41]

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iff the quotient semiring S/ρ is a ring.

A semiring S with o and 1 is said to be h-Noetherian (k - Noetherian) iff it satisfies any one of the following three equivalent conditions:

(i) S satisfies the ascending chain condition on h-ideals (k-ideals):

(ii) The maximal condition for h-ideals (k-ideals) hold in S;

(iii) Every h-ideal (k-ideal) in S is finitely generated.

We recall Yoneda's Lemma (cf. eg. [28]) :

Let $\mathcal{C}$ be any category and T a covariant functor from $\mathcal{C}$ to $\mathcal{S}$ (category of sets and functions). Then, for any object $C \in \mathcal{C}$, there is a bijective correspondence :

$$\theta \ : \ (h_C, T) \longrightarrow T(C)$$

where (h_C, T) is the class of natural transformations from the set valued functor h_C to the set valued functor T such that θ is natural in C and T.

The dual result for contravarient functors is analogous.

The concept of vector bundles arose from the study of tangent vector fields to smooth geometric objects, e.g. spheres, projective spaces, and more generally, manifolds. Vector boundle is a bundle with an additional vector space structure on each fibre.

This thesis consists of seven chapters : (1) k-ideals of semirings ; (II) Semirings of formal power series ; (III) h-ideals of semirings; (IV) Semirings with chain conditions; (V) Semimodules over semirings;

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(VI) Structure spaces of semialgebras; (VII) Applications of Yoneda's Lemma in semirings. Each chapter is divided into some sections. Chapter I is devoted to a study of k-ideals of semirings. In the ring theory, ideals play a very important role and it is therefore natural to investigate k-ideals in the semiring theory. In section 1 we develop some basic properties of k-ideals. In section 2, for a semiring S we consider conditions such that S has non-trivial k-ideals or maximal k-ideals among others. We characterize maximal k-ideals A of S by the help of the Bourne congruence class semiring S/A defined by A. The Fundamental homomorphism theorem for rings is not generally applicable in semiring theory. In this section we show that for N-homomorphism of a class of semirings the Fundamental homomorphism theorem is valid. In section 3 we study completely prime k-ideals of semirings. We show that each maximal k-ideal of a commutative semiring with 1 is completely prime but examples disapprove the converse. In section 4 we describe all maximal k-ideals of the semiring N of all non-negative integers. In section 5 we introduce the notions of k-division semirings, k-semifields, 0-congruence free semirings and j-ideals of semirings and investigate their properties. We completely characterize 0-congruence free semirings and also j-ideals. In section 6 we extend the concept of divisibility, irreducibility and g. c. d property in the semidomain N of non-negative integers to an arbitary semidomain. We introduce the notions of principal k-ideal semidomains and k-irreducible elements. We characterize a

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principal k-ideal semidomain with the help of its k-irreducible elements and also g.c.d. property of a semidomain with the help of its k-ideals. In section 7 we define direct k-summends of semirings. We show that if every left k-ideal of a hemiring H with 1 is a direct k-summand then (i) H is regular in the sense of Von Neumann (cf. [8]). (ii) for each left k-ideal A and a right k-ideal B of H, $A \cap B = \overline{BA}$ where $\overline{BA}$ is the k-closure of BA; (iii) the set k(H) of all k-ideals of H form a complete distributive lattice.

In chapter II we study semirings of formal power series and local semirings. In section 1 we develop some basic properties of semirings of formal power series. In section 2 we consider conditions such that a semiring S has the unique maximal k-ideal, leading to the concept of local semirings. We also investigate formal power series over k-division semirings and determine the Jacobson radical of formal power series over a k-semifield.

In Chapter III we study h-ideals of semirings. In section 1 we discuss some basic properties of h-ideals. In section 2 we consider conditions such that a semiring S has non-trivial h-ideals or maximal h-ideals, among others with the help of the Iizuka congruence class semiring S/A defined by an ideal A of S. In section 3 we investigate h-ideals of an additive inverse semiring. It is well known that every h-ideal is a k-ideal but the converse is not true. But we prove in this secti-on that in some class of semirings these two concepts coincide. We also

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prove that the set of all h-ideals of an additive inverse semiring forms a complete distributive lattice. In section 4, we characterize certain types of ring congruences on an additive inverse semiring with the help of its h-ideals and also determine the minimum ring congruence. In section 5 we find a generalization of Noetherian ring and also of Cohen's theorem in a class of additive inverse semirings.
In Chapter IV we study semirings with chain conditions. In section 1 we introduce the notion of a finitely generated k-ideal in hemirings and define k-Noetherian hemirings. As an analogue of rings, we establish an equivalent formulation of the k- Noetherian requirement that the k-ideals of the hemiring satisfy the ascending chain condition. H.E. Stone showed in 1972 [45] that an analogue of the Hilbert Basis Theorem does not hold for all halfrings. In this section we show that it does hold for a class of halfrings, yielding a generalization of the Hilbert Basis Theorem.
In section 2 we study semirings with chain conditions on h-ideals.
In chapter V, We study semimodules over semirings. In section 1 we study some basic properties of semimodules and k-simple semimodules over hemirings.
In section 2 we study semimodules M over additive inverse semirings R and characterize cancellative k-simple left semimodules M over R with the the help of quotient R-semimodules R/I for some suitable modular maximal left k-ideal I of R. In section 3 we consider additive cancellative (AC) congruence free semimodules. We investigate conditions such that a semimodule M has only trivial AC-congruences. In section 4 we prove

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an analogue of the fundamental homomorphism theorem and study exact sequences of semimodules and their homomorphisms. Finally we prove the short fixe lemma for semimodules.

In Chapter VI we study structure spaces of semialgebras. In section 1 we study the structure spaces of semirings S, formed by the class of completely prime k-ideals of S. The properties mainly, separation axioms, compactness and connectedness in these structure spaces are investigated. These properties for the semiring of non-negative integers are examined. In section 2 we introduce the concept of modular h-ideals of semialgebras and show that the set M(R) of all maximal modular h-ideals of S-semialgebra R depends only on the hemiring structure of R. This result motivates to define in section 3 a Galois connection between the partially ordered set of subsets of R and the partially ordered set of the subsets of M(R). This leads to the Stone- Jacobson - Zarisky topology on M(R).M(R) endowed with this topolggy is called a structure space of the S-semialgebra R. Conditions are considered such that M(R) is a Hausdorff space. In chapter VII we give some applications of Yoneda's lemma in semirings. In section 1 we discuss Yoneda's lemma. In section 2 we define bitranslations of semirings and study their properties. In section 3 we characterize certain class of semirings with the help of Yoneda's lemma. In section 4, F denotes the field of real numbers or complex numbers and $Vect_F(X)$ denotes the set of all isomorphism classes of F-vector bundles over a topological space X; we find a representation of the contravariant functor $Vect_F$ by Yoneda's lemma.

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Finally, we reduce the problem of classifying vector bundles to determination of suitable semirings of natural transformations.

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