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ON MAXIMAL k-IDEALS OF SEMIRINGS

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ABSTRACT. For a semiring S with commutative addition, conditions are considered such that S has nontrivial k-ideals or maximal k-ideals, among others, by the help of the congruence class semiring S/A defined by an ideal A of S. Moreover, all maximal k-ideals of the semiring of nonnegative integers are described.

1. PRELIMINARIES

A *semiring* S is defined as an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$. A semiring S may have an *identity* 1 [a zero o], defined by $1a = a1 = a[o + a = a + o = a]$ for all $a \in S$. If there is an element $O \in S$ satisfying $Oa = aO = O$ for all $a \in S$, it is called *multiplicatively absorbing* or simply *absorbing*. Such an element satisfies $O + O = O$, but it need not be a zero of S, whereas a zero o of S need not even satisfy $oo = o$. Clearly, a semiring has an *absorbing zero* iff it has elements O and o which coincide.

A subset $A \neq \varnothing$ of a semiring S is called an *ideal* of S iff $a + b \in A$, $sa \in A$, and $as \in A$ hold for all $a, b \in A$ and all $s \in S$. An ideal A of S is called *proper* iff $A \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal A is called *maximal* iff there is no ideal B of S satisfying $A \subset B \subset S$. Obviously, a semiring S contains an ideal A consisting of one element iff S has an absorbing element O, and then $A = \{O\}$ is the only ideal of this kind. Finally, an ideal A of S is called *trivial*, iff $A = S$ holds or $A = \{O\}$, the latter clearly if S has an absorbing element. To deal with both cases simultaneously, we introduce the notion S' by $S' = S\backslash\{O\}$ if S has an absorbing element, and $S' = S$ otherwise.

In this paper we only consider semirings S for which $(S, +)$ is commutative. If also $(S, \cdot)$ is commutative, S is called a *commutative semiring*. Moreover, to avoid trivial exceptions, *each semiring S is assumed to have at least two elements.*

Using only commutativity of addition, the following concepts and statements, essentially due to [1, 2, 4], are well known. For each ideal A of a semiring S

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the *k-closure* $\overline{A}$ of A defined by

$$\overline{A} = \{\overline{a} \in S | \overline{a} + a_1 = a_2 \text{ for some } a_i \in A\}$$

is an ideal of S satisfying $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$. An ideal A of S is called a *k-ideal* of S iff $A = \overline{A}$ holds. Clearly, S is a k-ideal for each semiring S; however, if S has an absorbing element O, the ideal $\{O\}$ need not be a k-ideal of S. There are examples for $\{O\} \subset \overline{\{O\}} \subset S$ and $\{O\} \subset \overline{\{O\}} = S$, whereas $\{O\} = \overline{\{O\}}$ holds if O is an absorbing zero of S. A k-ideal $A \subset S$ is called a *maximal k-ideal* of S if there is no k-ideal B of S satisfying $A \subset B \subset S$. Note that a maximal k-ideal of S need not be a maximal ideal of S (cf. Remark 4.2).

Moreover, each ideal A of S defines a congruence ρ_A on $(S, +, \cdot)$ by

$$\rho_A = \{(x, y) \in S \times S | x + a_1 = y + a_2 \text{ for some } a_i \in A\}.$$

The corresponding congruence class semiring S/ρ_A, consisting of the classes $x\rho_A$, is also denoted by S/A. The k-closure $\overline{A}$ of A is such a congruence class, and $\overline{A}$ is the absorbing zero of S/A, regardless of whether S has a zero o or an absorbing element O (which implies $o\rho_A = \overline{A}$ or $O\rho_A = \overline{A}$, respectively). Moreover, ρ_A and $\rho_{\overline{A}}$, and hence S/A and $S/\overline{A}$ coincide.

2. Maximal k-ideals

Theorem 2.1. *Let S be a semiring such that $S = (a_1, \ldots, a_n)$ is a finitely generated ideal of S. Then each proper k-ideal A of S is contained in a maximal k-ideal of S.*

Proof. Let $\mathfrak{B}$ be the set of all k-ideals B of S satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i | i \in I\}$ in $\mathfrak{B}$. One easily checks that $B = \bigcup_{i \in I} B_i$ is a k-ideal of S, and $S = (a_1, \ldots, a_n)$ implies $B \neq S$, and hence $B \in \mathfrak{B}$. So by Zorn's lemma, $\mathfrak{B}$ has a maximal element as we were to show.

Corollary 2.2. *Let S be a semiring with identity 1. Then each proper k-ideal of S is contained in a maximal k-ideal of S.*

The proof is immediate by $S = (1)$.

Definition 2.3. A semiring S is said to satisfy condition (C) iff for all $a \in S'$ and all $s \in S$ there are $s_1, s_2 \in S$ such that

$$s + s_1 a = s_2 a$$

holds. Clearly, if S has an identity 1, then (C) is equivalent to the following condition (C′):

$$1 + s_1 a = s_2 a$$

holds for each $a \in S'$ and suitable $s_1, s_2 \in S$.

Example 2.4. Let P be the set of all nonnegative rational numbers. Then $(P, +, \cdot)$ with the usual operations, as well as $(P', +, \cdot)$, are semirings with 1 as identity satisfying condition (C′). The same is true, more generally, for each positive cone P of a totally ordered skew-field (cf. [3, Chapter VI]).

Example 2.5. Let $\mathbb{N}$ be the set of all nonnegative integers. Define $a + b = \max\{a, b\}$, and denote by $a \cdot b$ the usual multiplication. Then $(\mathbb{N}, +, \cdot)$ is a niring with 1 as identity, which satisfies (C′) since $1 + a = a$ holds for all $\in S'$.

nma 2.6. *If a semiring S with an absorbing zero O satisfies condition* (C), *n $ab = O$ for $a; b \in S$ implies $a = O$ or $b = O$.*

of. By way of contradiction, assume $ab = O$ and $a \neq O \neq b$. Then $s_1 a = s_2 a$, according to (C), yields $sb + s_1 ab = s_2 ab$, i.e., $sb = O$ for all S. Consequently, $x + s_3 b = s_4 b$ implies $x = O$ for all $s_3, s_4 \in S$, which tradicts (C) applied to the element $b \in S'$.

eorem 2.7. *Let S be a semiring. Then condition* (C) *implies that S contains y trivial k-ideals. The converse is true if $(S, \cdot)$ is commutative, and, provided t S has an absorbing element O, $Sa = \{sa | s \in S\} \neq \{O\}$ holds for all S'.*

of. Assume that S satisfies (C). Let A be a k-ideal of S which contains at t one element $a \in S'$. Then $s + s_1 a = s_2 a$, according to (C), implies $s \in A$ each $s \in S$, i.e., $A = S$. For the converse, our supplementary assumptions S yield that Sa is an ideal of S and that $Sa \neq \{O\}$ holds for each $a \in S'$ has an absorbing element O. Now assume that S has only trivial k-ideals. n the k-ideal $\overline{Sa}$ coincides with S for each $a \in S'$, regardless of whether as an element O or not. Now,

$$\overline{Sa} = \{s \in S | s + s_1 a = s_2 a \text{ for some } s_i \in S\} = S$$

es that S satisfies condition (C).

llary 2.8. *Let S be a commutative semiring with identity. Then S has only al k-ideals iff it satisfies condition* (C′).

f. It was already stated that (C′) is equivalent to (C) if S has an identity d $a = 1a \in Sa$ implies $Sa \neq \{O\}$ for all $a \in S'$ in the case that S has bsorbing element O. Hence the corollary follows from Theorem 2.7.

rem 2.9. *Let S be a commutative semiring with identity* 1 *and A a proper eal of S. Then A is maximal iff the semiring $S/A = S/\rho_A$ satisfies condi-* (C′).

f. Suppose A is a maximal k-ideal of S. Then A is the absorbing zero of and $1\rho_A$ is its identity. Consider any $c\rho_A \in (S/A)'$. Then $c \notin A$ holds, the smallest ideal B of S containing c and A consists of all elements a, and $sc + a$ for $s \in S$ and $a \in A$. From $A \subset B$ it follows $\overline{B} = S$, and e $1 + b_1 = b_2$ for suitable elements $b_1, b_2 \in B$. To avoid the discussion of ent cases, we add $1c + a$ with an arbitrary element $a \in A$ to $1 + b_1 = b_2$ obtain

$$1 + s_1 c + a_1 = s_2 c + a_2, \quad \text{i.e., } 1\rho_A + (s_1\rho_A)(c\rho_A) = (s_2\rho_A)(c\rho_A)$$

itable $s_i \in S$ and $a_i \in A$. This shows that S/A satisfies (C′).
nversely, assume (C′) for S/A, and let B be a k-ideal of S satisfy- $A \subset B$. Then there is an element $c \in B \setminus A$, and $c\rho_A \in (S/A)'$ yields $c)\rho_A = (s_2 c)\rho_A$ for suitable elements $s_i \in S$ by (C′). Hence $1 + s_1 c + a_1 = a_2$ holds for some $a_i \in A$, i.e., $1 + b_1 = b_2$ for $b_1, b_2 \in B$. This shows and that A is a maximal k-ideal of S.

3. Completely prime k-ideals

Recall that an ideal A of a semiring S is called *completely prime* (cf., e.g., [5]) iff $ab \in A$ implies $a \in A$ or $b \in A$ for all $a, b \in S$.

Proposition 3.1. *Let S be a commutative semiring with identity. Then each maximal k-ideal A of S is completely prime.*

Proof. By Theorem 2.9, the semiring S/A satisfies the condition (C′) and hence (C). Since S/A has A as its absorbing zero, we can apply Lemma 2.6 and obtain that S/A has no zero-divisors. Hence $a\rho_A \neq A$ and $b\rho_A \neq A$ imply $(ab)\rho_A \neq A$, i.e., $a \notin A$ and $b \notin A$ imply $ab \notin A$ as we were to show.

Concerning the converse of Proposition 3.1, we show that a completely prime ideal A of a commutative semiring S with identity need not be a k-ideal, and if it is one, A need not be a maximal k-ideal of S.

Example 3.2. Let S be the set of all real numbers a satisfying $0 < a \leq 1$, and define $a + b = a \cdot b = \min\{a, b\}$ for all $a, b \in S$. Then $(S, +, \cdot)$ is easily checked to be a commutative semiring with 1 as identity. Each real number r such that $0 < r < 1$ defines an ideal $A = \{a \in S | a \leq r\}$ of S which is obviously completely prime. However, $r + 1 = r$ together with $r \in A$ and $1 \notin A$ show that A is not a k-ideal of S. The same is true if one includes 0 in these considerations (in this case 0 is an absorbing element but not a zero of $S \cup \{0\}$), but also if one adjoins 0 as an absorbing zero to S (cf., e.g., [7, Lemma 1.3]).

Example 3.3. The polynomial ring $\mathbb{Z}[x]$ over the ring $\mathbb{Z}$ of integers contains the subsemiring

$$S = \mathbb{N}[x] = \left\{ f(x) = \sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{N} \right\},$$

which is clearly commutative and has $1 \in \mathbb{N}$ as its identity. The ideal $A = (x)$ of S consists of all $f(x) \in S$ such that $a_0 = 0$ holds. Obviously, A is completely prime and a k-ideal of S. Now consider the set B consisting of all $f(x) \in S$ for which a_0 is divisible by 2. Clearly, B is a k-ideal of S, and $A \subset B \subset S$ shows that A is not a maximal k-ideal.

4. Maximal k-ideals of $\mathbb{N}$

In this section we consider the semiring $(\mathbb{N}, +, \cdot)$ of nonnegative integers with respect to their usual operations.

Proposition 4.1. *The semiring $\mathbb{N}$ has exactly the k-ideals $(a) = \{na | n \in \mathbb{N}\}$ for each $a \in \mathbb{N}$. Consequently, the maximal k-ideals of $\mathbb{N}$ are given by (p) for each prime number p.*

Proof. Obviously, each ideal (a) of $\mathbb{N}$ is a k-ideal. Now assume that $A \neq (0)$ is a k-ideal of $\mathbb{N}$. Let a be the smallest positive integer contained in A, and b any element of A. Then $b = qa + r$ holds for some $q \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfying $0 \leq r < a$. Since r belongs to the k-ideal A, it follows that $r = 0$, and, hence. $A = (a)$. The last statement follows since $(a) \subseteq (b)$ holds iff b divides a.

Remark 4.2. None of the maximal k-ideals (p) of $\mathbb{N}$ is a maximal ideal of $\mathbb{N}$. This follows since each ideal $A = (p)$ is properly contained in the proper ideal $B = \{b \in \mathbb{N} | b \geq p\}$ of $\mathbb{N}$.

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ON k-IDEALS OF SEMIRINGS

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ABSTRACT. Certain types of ring congruences on an additive inverse semiring are characterized with the help of full k-ideals. It is also shown that the set of all full k-ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular.

KEY WORDS AND PHRASES. Semiring, inverse semiring, k-ideals and ring congruence.
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1. PRELIMINARIES. A semiring is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

(i) S together with addition is a semigroup;

(ii) S together with multiplication is a semigroup; and

(iii) $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in S$.

A semiring S is said to be additively commutative if $a + b = b + a$ for all $a, b \in S$. A left (right) ideal of a semiring S is non-empty subset I of S such that

i) $a + b \in I$ for all $a, b \in I$; and

ii) $ra \in I (ar \in I)$ for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a non-empty subset I of S such that I is both a left and right ideal of S.

Henriksen [1] defined a more restricted class of ideals in a semiring, which he called k-ideals.

A left k-ideal I of a semiring S is a left ideal such that if $a \in I$ and $x \in S$ and if either $a + x \in I$ or $x + a \in I$, then $x \in I$.

Right k-ideal of a semiring is defined dually. A non-empty subset I of a semiring S is called a k-ideal if it is both a left k-ideal and a right k-ideal.

A semiring S is said to be additively regular if for each $a \in S$, there exists an element $b \in S$ such that $a = a + b + a$. If in addition, the element b is unique and satisfies $b = b + a + b$, then S is called an additively inverse semiring. In an additively inverse semiring the unique inverse b of an element a is usually denoted by a'. Karvellas [2] proved the following result:

Let S be an additively inverse semiring. Then

i) $x = (x')'$, $(x+y)' = y' + x'$, $(xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$.

ii) $E^+ = \{x \in S : x + x = x\}$ is an additively commutative semilattice and an ideal of S.

2. FULL k-IDEALS. In this section S denotes an additively inverse semiring in which addition is commutative and E^+ denotes the set of all additive idempotents of S.

A left k-ideal A of S is said to be full if $E^+ \subseteq A$. A right k-ideal of S is defined dually.

A non-empty subset I of S is called a full k-ideal if it is both left and a right full k-ideal.

EXAMPLE 1. In a ring every ring ideal is a full k-ideal.

EXAMPLE 2. In a distributive lattice with more than two elements, a proper ideal is a k-ideal but not a full k-ideal.

EXAMPLE 3. $Z \times Z^p = \{(a,b) : a, b \text{ are integers and } b > 0\}$. Define

$$(a,b) + (c,d) = (a+c, \text{l.c.m of } b,d) \text{ and } (a,b)(c,d) = (ac, h.c.f. \text{ of } b,d).$$

Then $Z \times Z^p$ becomes an additively inverse semiring in which addition is commutative.

Let $A = \{(a,b) \in Z \times Z^p : a = 0, b \in Z^p\}$. Then A is a full k-ideal of $Z \times Z^p$.

LEMMA 2.1. Every k-ideal of S is an additively inverse subsemiring of S.

PROOF. Let I be a k-ideal of S. Clearly I is a subsemiring of S. Let $a \in I$. Then

$$a + (a' + a) = a \in I.$$

Since I is a k-ideal, it follows $a' + a \in I$. Again this implies that $a' \in I$. Hence the lemma.

LEMMA 2.2. Let A be an ideal of S. Then

$$\overline{A} = \{a \in S : a + x \in A \text{ for some } x \in A\} \text{ is a } k\text{-ideal of } S.$$

PROOF. Let $a, b \in \overline{A}$. The $a + x, b + y \in A$ for some $x, y \in A$. Now

$$a + x + b + y = (a+b) + (x+y) \in A.$$

As $x + y \in A, a + b \in \overline{A}$. Next let $r \in S$, $ra + rx = r(a+x) \in A$.
As $rx \in A$, $ra \in \overline{A}$. Similarly, $ar \in \overline{A}$. As a result $\overline{A}$ is an ideal of S. Next, let c and $c + d \in \overline{A}$. Then there exists x and y in A such that $c + x \in A$ and $c + d + y \in A$.
Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$

Hence $d \in \overline{A}$ and $\overline{A}$ is a k-ideal of S. Since $a + a' \in A$ for all $a \in A$, it follows that $A \subseteq \overline{A}$.

COROLLARY. Let A be an ideal of S. Then $\overline{A} = A$ iff A is a k-ideal.

LEMMA 2.3. Let A and B be two full k-ideals of S, then $\overline{A+B}$ is a full k-ideal of S such that

$$A \subseteq \overline{A+B} \text{ and } B \subseteq \overline{A+B}.$$

PROOF. It can be shown that $A + B$ is an ideal of S. Then from Lemma 2.2, we find $\overline{A+B}$ is a k-ideal and $A + B \subseteq \overline{A+B}$. Now $E^+ \subseteq A, B$. Hence $E^+ \subseteq A + B \subseteq \overline{A+B}$. This implies that $\overline{A+B}$ is a full k-ideal. Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + B \text{ as } a' + a \in E^+ \subseteq B.$$

Hence $A \subseteq \overline{A+B}$ and similarly $B \subseteq \overline{A+B}$.

THEOREM 2.4. If $I(S)$ denotes the set of all full k-ideals of S, then $I(S)$ is a complete lattice which is also modular.

PROOF. We first note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and from Lemma 2.3, $\overline{A+B} \in I(S)$. Define $A \wedge B = A \cap B$ and $A \vee B = \overline{A+B}$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A+B \subseteq C$ and $\overline{A+B} \subseteq \overline{C}$. But $\overline{C} = C$. Hence $\overline{A+B} \subseteq C$. As a result $\overline{A+B}$ is the l.u.b. of A, B. Thus we find that $I(S)$ is a lattice. Now E^+ is an ideal of S. Hence $\overline{E}^+ \in I(S)$ and also $S \in I(S)$; consequently $I(S)$ is a complete lattice. Next suppose that $A, B, C \in I(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A+B}$. Hence there exists $a+b \in A+B$ such that $x+a+b = a_1+b_1$ for some $a_1 \in A$, $b_1 \in B$.
Then

$$x+a+a'+b = a_1+b_1+a'.$$

Now $x \in C, a+a' \in C$ and $b \in B \subseteq C$. Hence $a_1+b_1+a' \in C$. But $b_1 \in C$. Consequently, $a_1+a' \in C \cap A = C \cap B$. Hence $a_1+a' \in B$. So from $x+a+b = a_1+b_1$ we find that $x+a+a'+b = a_1+a'+b \in B$. But $(a+a')+b \in B$ and B is a k-ideal. Hence $x \in B$ and $B = C$. This proves that $I(S)$ is a modular lattice.

3. RING CONGRUENCES.

A congruence ρ on a semiring S is called a ring congruence if the quotient semiring S/ρ is a ring.

In this section we assume S is an additively inverse semiring in which addition is commutative. We want to characterize those ring congruences on S such that - $(a\rho) = a'\rho$ where a' denotes the inverse of a in S and $-(a\rho)$ denotes the additive inverse of $a\rho$ in the ring S/ρ.

THEOREM 3.1. Let A be a full k-ideal of S. Then the relation

$\rho_A = \{(a,b) \in S \times S : a+b' \in A\}$ is a ring congruence on S such that $-(a\rho_A) = a'\rho_A$.

PROOF. Since $a+a' \in E^+ \subseteq A$ for all $a \in S$, it follows that ρ_A is reflexive. Let $a+b' \in A$. Now from Lemma 2.1, we find that $(a+b')' \in A$. Then $b+a' = (b')'+a' = (a+b')' \in A$. Hence ρ_A is symmetric. Let $a+b' \in A$ and $b+c' \in A$. Then $a+b+b'+c' \in A$. Also $b+b' \in E^+ \subseteq A$. Since A is a k-ideal, we find that $a+c' \in A$. Hence ρ_A is an equivalence relation. Let $(a,b) \in \rho_A$ and $c \in S$. Then $a+b' \in A$. Since

$$(c+a)+(c+b)' = c+a+b'+c' = (a+b')+(c+c') \in A,\ ca+(cb)' = ca+cb' = c(a+b') \in A,$$

$$ac+(bc)' = ac+b'c = (a+b')c \in A,$$

it follows that ρ_A is a congruence on S. So we obtain the quotient semiring where addition and multiplication are defined by

$$a\rho_A + b\rho_A = (a+b)\rho_A \text{ and } (a\rho_A)(b\rho_A) = (ab)\rho_A.$$

Now

$$a\rho_A + b\rho_A = (a+b)\rho_A = (b+a)\rho_A = b\rho_A + a\rho_A.$$

Let $e \in E^+$ and $a \in S$. Now $(e+a)+a' = e+(a+a') \in E^+$.
We find that $(e+a)\rho_A = a\rho_A$. Then $e\rho_A + a\rho_A = a\rho_A$.

Also

$$a\rho_A + a'\rho_A = (a+a')\rho_a = e\rho_A.$$

Hence $e\rho_A$ is the zero element and $a'\rho_A$ is the negative element of $a\rho_A$ in the ring S/ρ_A.

THEOREM 3.2. Let ρ be a congruence on S such that S/ρ is a ring and $-(a\rho) = a'\rho$ there exists a full k-ideal A of S such that $\rho_A = \rho$.

PROOF. Let $A = \{a \in S: (a,e) \in \rho$ for some $e \in E^+\}$. Since ρ is reflexive, it follo $E^+ \subseteq A$. Then $A \neq \phi$, since $E^+ \neq \phi$. Let $a, b \in A$. Then there exist $e, f \in E^+$ su $(a,e) \in \rho$ and $(b,f) \in \rho$. Then $(a+b, e+f) \in \rho$. But $e+f \in E^+$. Hence $a+b \in A$. Again $r \in S$, $(ra, re) \in \rho$ and $(ar, er) \in \rho$. But re and er $\in E^+$. Hence A is an ideal of S.

Let $a+b \in A$ and $b \in A$. Then there exist $e, f, \in E^+$ such that $(a+b, f) \in \rho$ and $(b$ Hence $f\rho = (a+b)\rho = a\rho + b\rho = a\rho + e\rho$. But $f\rho$ and $e\rho$ are additive idempotents in the ri Hence $e\rho = f\rho$ is the zero element of S/ρ. As a result, $a\rho$ is the zero element of S/ρ. $a\rho = e\rho$. This implies $a \in A$. So we find that A is a full k-ideal of S. Consider n congruences ρ_A and ρ. Let $(a,b) \in \rho$. Then $(a+b', b+b') \in \rho$. But $b+b' \in E^+$. Hence $a+$ and $(a,b) \in \rho_A$. Conversely suppose that $(a,b) \in \rho_A$. Then $a+b' \in A$. Hence $(a+b', e) \in \rho$ fo $e \in E^+$. As a result, $e\rho = a\rho + b'\rho = a\rho - b\rho$ holds in the ring S/ρ. But $e\rho$ is the zero ele S/ρ. Consequently $a\rho = b\rho$. This show that $(a,b) \in \rho$ and hence $\rho_A = \rho$.

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ON DIVISION HEMIRINGS*

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Abstract : In this paper the concept of division hemirings is introduced in more general sense and different interesting properties of a division hemiring are studied. Also introducing the notion of left k-Artinian semiring it is shown that an additively cancellative hemiring H with more than one element is a multiplicatively cancellative left k-Artinian hemiring iff H is a division hemiring.

1. **Preliminaries.** A semiring (Vandivar (1934) is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

(*i*) S together with addition is a semigroup.

(*ii*) S together with multiplication is a semigroup.

(*iii*) $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a, b, c \in S$.

A zero element of a semiring S is an element 0 such that $0x=x0=0$ and $0+x=x+0=x$ for all $x \in S$.

A hemiring H is a semiring with 0 such that $a+b=b+a$ for all $a, b \in H$.

A hemiring H is said to be commutative if $ab=ba$ for all $a, b \in H$.

The concept of ideal in semirings that is found most often in the current literature is the following :

A left (right) ideal of a semiring S is a non-empty subset A of S such that

(*i*) $a+b \in A$ for all $a, b \in A$.

(*ii*) $ra \in A(ar \in A)$ for all $r \in S$ and $a \in A$.

An ideal of a semiring S is a non-empty subset A of S such that A is both right and left ideal of S.

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Although ideals in semirings are useful for many purposes, they do not in general coincide with the usual ring ideals and for this reason, this definition is somewhat limited in trying to obtain analogues of ring theorems for semirings.

A more restricted class of ideals in semiring is defined by Henriksen (1958) which he called k-ideals.

A left k-ideal A of a semiring S is a left ideal such that for $a \in A$ and $x \in S$, if either $a+x \in A$ or $x+a \in A$, then $x \in A$.

Right k-ideals of a semiring are defined dually.

A k-ideal A of a semiring S is such that A is both a left k-ideal and a right k-ideal.

2. k-Closure Of An Ideal.

Definition 2. 1. (Bourne and Zassenhans 1958). *If A is a left (right) ideal of a hemiring H, then the k-closure of A denoted by $\bar{A}$ defined by $\bar{A}=\{a \in H : a+x \in A$ for some $x \in A\}$ is a left (right) k-ideal of H. If A is an ideal of H, $\bar{A}$ is a k-ideal of H.* The k-closure $\bar{A}$ of an ideal (left, right) A satisfies the following properties.

(*i*) $A \subseteq \bar{A}$ for any ideal (left. right) A of H.

(*ii*) $A=\bar{A}$ iff A is a k-ideal (left. right) of H.

(*iii*) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$ for ideals (left, right) A and B of H.

3. Division hemirings.

In this section we introduce the notion of a division hemiring in a more general way and study some interesting properties of it.

Definition 3. 1. *Let H be a hemiring. A pair of elements (e_1, e_2) with $e_1, e_2 \in H$ is called a left (right) identity pair if $a+e_1a=e_2a$ $(a+ae_1=ae_2)$ for all $a \in H$.*

Definition 3. 2. *A pair of elements (e_1, e_2), $e_1, e_2 \in H$ is called an identity pair of hemiring H if it is both a left and a right identity pair.*

Example 3. 3. Let $H=\{a \in Z^+ : a \geqslant 5\} \cup \{0\}$ where Z^+ denotes the set of all non negative integers, is a hemiring with integer 0 as the zero element. Here, (6, 7), (7, 8) (10, 11)……are identity pairs.

Note 1. If (0, 0) is an identity pair of H, then $H=\{0\}$. Hence if H contains an identity pair (e_1, e_2) and $H \neq \{0\}$, then $(e_1, e_2) \neq (0, 0)$.

So in this paper we assume that any hemiring with an identity pair (e_1, e_2) contains more than one element.

DEFINITION 3. 4. *Let H be a hemiring with an identity pair $(e_1, e_2) \neq (0, 0)$. An element $0 \neq a \in H$ is called left (right) invertible with respect to (e_1, e_2) if there exist r_1, r_2 s_1, s_2 in H such that $e_1+r_1a=e_2+r_2a$ $(e_1+as_1=e_2+as_2)$.*

LEMMA 3. 5. *Let H be an additively cancellative hemiring with an identity pair $(e_1, e_2) \neq (0, 0)$. If r_1, r_2 and s_1, s_2 be four elements of H such that*

$e_1+r_1a=e_2+r_2a$ and $e_1+as_1=e_2+as_2$, then there exist $r_3, r_4 \in H$ such that

$e_1+r_3a=e_2+r_4a$ and $e_1+ar_3=e_2+ar_4$.

Proof. Suppose $e_1+r_1a=e_2+r_2a$ and $e_1+as_1=e_2+as_2$ for some $r_1, r_2, s_1, s_2 \in H$.

Then $e_1s_1+r_1as_1=e_2s_1+r_2as_1$ and $s_1+e_1s_1=e_2s_1$.

Hence $r_1as_1=s_1+r_2a\,s_1$ ...(1)

Again $r_1e_1+r_1a\,s_1=r_1e_2+r_1as_2$ and $r_1+r_1e_1=r_1e_2$

Hence $r_1a\,s_1=r_1+r_1a\,s_2$ ...(2)

Thus from (1) and (2) it follows that $r_1+r_1a\,s_2=s_1+r_2a\,s_1$ ...(3)

Again $e_1s_2+r_1a\,s_2=e_2s_2+r_2a\,s_2$ and $s_2+e_1s_2=e_2s_2$

Hence $r_1a\,s_2=s_2+r_2a\,s_2$ ...(4)

Similarly, $r_2a\,s_1=r_2+r_2a\,s_2$ ...(5)

Thus from (3), (4) and (5) it follows that $r_1+s_2=s_1+r_2$ ...(6)

and $r_2=r_1a\,s_2=s_2+r_2a\,s_1$ ...(7)

Let $r_3=r_1+s_2=s_1+r_2$ and $r_4=r_2+s_2$.

Then $e_1+(r_1+s_2)\,a=e_1+r_1a+s_2a=e_2+r_2a+s_2a=e_2+(r_2+s_2)a$.

Thus $e_1+r_3a=e_2+r_4a$.

Similarly, $e_1+ar_3=e_2+ar_4$

This completes the proof of the lemma.

DEFINITION 3. 6. *Let H be a hemiring and $0 \neq a \in H$ and $(e_1, e_2) \neq (0, 0)$ be an identity pair of H. A pair $(c, d) \in HxH$ is called an inverse relative to the identity pair (e_1, e_2) if $e_1+ca=e_2+da$ and $e_1+ac=e_2+ad$.*

Let $V_{e_1}^{e_2}(a)$ denote the set of all inverses of a in H with respect to the identity pair (e_1, e_2).

LEMMA 3. 7. *Let H be an additively cancellative hemiring and (e_1, e_2) and (f_1, f_2) be two identity pairs of H.*

Then $V_{e_1}^{e_2}(a)\neq\phi$ *iff* $V_{f_1}^{f_2}(a)\neq\phi$. *In other words, the element* $0\neq a\in H$ *has an inverse with respect to* (e_1, e_2) *iff a has also an inverse with respect to* (f_1, f_2).

Proof. Let $V_{e_1}^{e_2}(a)\neq\phi$. Then there exists a pair $(c, d)\in V_{e_1}^{e_2}(a)$.

Hence $e_1+ca=e_2+da$ ...(1)

Since (f_1, f_2) is an identity pair, $x+xf_1=xf_2$ and $x+f_1x=f_2x$ for all $x\in$ H,

Then $e_1+e_1f_1=e_1f_2$ and $e_2+e_2f_1=e_2f_2$

Also since (e_1, e_2) is an identity pair, then $f_1+f_1e_1=f_1e_2$ and $f_2+f_2e_1=f_2e_2$.

Now $f_1e_1+f_1ca=f_1e_2+f_1da$ by (1). Consequently, $f_1ca=f_1da+f_1$ ...(2)

Also $f_2ca+f_2e_1=f_2da+f_2e_2$ by (1). Consequently, $f_2ca=f_2da+f_2$ ...(3)

Hence $f_1+f_1da+f_2ca=f_2+f_2da+f_1ca$ by (2) and (3).

Then $f_1+(f_1d+f_2c)a=f_2+(f_2d+f_1c)a$.

Hence $f_1+xa=f_2+ya$ for some $x, y\in H$.

Similarly we can show that $f_1+ar=f_2+as$ for some $r, s\in H$.

Hence from lemma 3. 5 it follows that $V_{f_1}^{f_2}(a)\neq\phi$.

This converse part follows similarly.

DEFINITION 3. 8. *A non-zero element* $a\in H$ *is said to be invertible if there exists an identity pair* (e_1, e_2) *in H such that* $V_{e_1}^{e_2}(a)\neq\phi$.

DEFINITION 3. 9. *An additively cancellative hemiring H is called a division hemiring, if H contains an identity pair and every non-zero element of H is invertible in H.*

DEFINITION 3. 10. *A multiplicatively commutative division hemiring is called a hemifield.*

THEOREM 3. 11. (*a*) *A division hemiring H has no zero divisors.*

(*b*) *A division hemiring H is multiplicatively cancellative.*

Proof. (*a*) Let $a, b\in H$ such that $ab=0$. Let $a\neq 0$. Then there exists an identity pair (e_1, e_2) such that $V_{e_1}^{e_2}(a)\neq\phi$. So there exist $c.\ d\in H$ such that $e_1+ca=e_2+da$. Then $e_1b+cab=e_2b+dab$. Hence $e_1b=e_2b$. But $b+e_1b=e_2b$. Hence $b=0$.

(*b*) Let $a, x, y \in H$ such that $ax = ay$ and $a \neq 0$, Let (c, d) be an inverse of a in H ith respect to the identity pair (e_1, e_2) of H. Now $cax = cay$, $e_1x + cax = e_2x + dax$ and $x + e_1x = e_2x$.

Hence $cax = x + dax$ ...(1)

Similarly, $cay = y + day$ ...(2)

Hence $x = y$.

Similarly we can prove the right cancellative property in H.

EXAMPLE 3.12. Let $H = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \text{ are real numbers and } a, d \geqslant 5\right\} \cup \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$, nder usual addition and multiplication of matrices H is an additively cancellative miring with $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Now $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ are members of H.

Clearly, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

for all $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H$.

Hence $\left(\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \right)$ is a left identity pair.

Let $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a \geqslant 5$, $d \geqslant 5$.

Now $\begin{pmatrix} 5+\frac{1}{a} & -\frac{b}{ad} \\ 0 & 5+\frac{1}{d} \end{pmatrix} \in H$. and $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 5+\frac{1}{a} & -\frac{b}{ad} \\ 0 & 5+\frac{1}{d} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$= \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

This shows that $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is an invertible element in H.

Hence H is a division hemiring. This is not a hemifield, since

$$\begin{pmatrix} 7 & 2 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 0 & 8 \end{pmatrix} \neq \begin{pmatrix} 5 & 6 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 0 & 9 \end{pmatrix}.$$

Note that H is a hemiring without a multiplicative identity.

Hence $(H \setminus \{0\}, .)$ is not a multiplicative group.

EXAMPLE 3.13. Let $Q^+(\sqrt{2})=\{a+b\sqrt{2}$; a, b are non-negative rational number Then with respect to usual addition and multiplication $Q^+(\sqrt{2})$ becomes a hemiring wi integer 0 as the zero element and (0, 1) as an identity pair. Let $a+b\sqrt{2}$ be a non ze element of $Q^+(\sqrt{2})$. Then a, b can not be both zero. Hence a^2-2b^2 is either a posit integer or a negative integer If $a^2-2b^2>0$, then

$$\frac{a}{a^2-2b^2},\ \frac{b\sqrt{2}}{a^2-2b^2}\ \text{are elements of } Q^+(\sqrt{2}) \text{ and}$$

$$0+(a+b\sqrt{2})\ \frac{a}{a^2-2b^2}=1+(a+b\sqrt{2})\ \frac{b\sqrt{2}}{a^2-2b^2}.$$

If $a^2-2b^2<0$, then $2b^2-a^2>0$ and $\frac{a}{2b^2-a^2}$,

$\frac{b\sqrt{2}}{2b^2-a^2}\in Q^+(\sqrt{2})$. In this case we can write

$$0+(a+b\sqrt{2})\ \left(\frac{b\sqrt{2}}{2b^2-a^2}\right)=1+(a+b\sqrt{2})\ \frac{a}{2b^2-a^2}.$$

Hence $a+b\sqrt{2}$ is invertible in $Q^+(\sqrt{2})$ and $Q^+(\sqrt{2})$ is a division hemiri As $Q^+(\sqrt{2})$ is multiplicatively commutative, it is a hemifield.

In this example we notice that $Q^+(\sqrt{2})$ contains the multiplicative identity, but $1+\sqrt{2}\in Q^+(\sqrt{2})$ there does not exist any element $a+b\sqrt{2}$ in $Q^+(\sqrt{2})$ such that $(a+b$ $(1+\sqrt{2})=1$. Hence $(Q^+(\sqrt{2})\setminus\{0\}, .)$ is not a group.

THEOREM 3.14. *A division hemiring H has only two left (right) k-ideals.*

Proof. Clearly H and $\{0\}$ are left k-ideals of H. Let $A\neq\{0\}$ be a left k-ideal of Then there exists an element $0\neq a\in H$. Since H is a division hemiring, a must be inverti Hence there exist $r_1, r_2\in H$ such that $e_1+r_1a=e_2+r_2a$. Let $0\neq b\in H$. Then be_1+br $=be_2+br_2a$. Since (e_1, e_2) is an identity pair, $b+be_1=be_2$. Hence $be_2+br_1a=b+be_2+br$ Then from the additive cancellative property of H, it follows that $br_1a=b+br_2a$. Si $a\in A$, we find that br_1a, br_2a, $\in A$. Hence $br_2a\in A$ and $b+br_2a\in A$. Since A is a k-ideal, we find that $b\in A$. Hence $H\subseteq A$. Also $A\subseteq H$. Consequently, $H=A$.

THEOREM 3.15. *Let H be an additively cancellative hemiring with an identity p (e_1, e_2). If H has no right k-ideals excepting H and $\{0\}$, then H is a division hemiring.*

Proof. Let $0 \neq a \in H$. Then $\overline{aH} \neq \{0\}$ and hence $\overline{aH} = H$. Since $e_1, e_2 \in H$, there exist $v_1, u_2, v_2 \in H$, such that $e_1 + au_1 = av_1$ and $e_2 + au_2 = av_2$. Hence $e_1 + au_1 + av_2 = av_1 + e_2$ $+au_2$. Then $e_1 + ax = e_2 + ay$...(1) for $x = u_1 + v_2$, $y = u_2 + v_1 \in H$.

Hence a is left invertible, Let $a \neq 0$ and $ba = ca$ with $a, b, c, \in H$. Now from (1), we find $be_1 + bax = be_2 + bay$. But $b + be_1 = be_2$.

Hence $bax = b + bay$. Similarly, $cax = c + cay$. Since $ba = ca$, we have $b + bay = c + cay$. This implies $b = c$. Similarly, $ab \neq ac$ implies $b = c$.

Again $a + e_1 a = e_2 a$. Hence from (1), $e_1 a + axa = e_2 a + aya$.

Then $axa = a + aya$. Consequently, $axa + ae_1 = ae_2 + aya$.

Hence $e_1 + xa = e_2 + ya$. Consequently a is invertible in H.

Hence H is a division hemiring.

4. **K-Artinian Hemiring.** In this section we define a k-Artinian hemiring and find its relation with a division hemiring.

DEFINITION 4. 1. *A hemiring H is said to satisfy the descending chain condition for left ideals (k-ideals) of H if given any descending chain of left ideals (k-ideals) of H*

$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \ldots,$

there exists a positive integer n such that $I_m = I_n$ for all $m \geq n$.

DEFINITION 4. 2. *A hemiring H is said to be a left Artinian (k-Artinian) hemiring if H satisfies the descending chain condition for left ideals (k-ideals) of H.*

NOTE. From the definition every left Artinian hemiring is always a left k-Artinian hemiring. The following example shows that the converse is not true.

EXAMPLE. From example 3.13, we find that $H = Q^+(\sqrt{2}) = \{a + b\sqrt{2} : a, b$ are non negative rational numbers$\}$ is a division hemiring. Hence by theorem 3.14 we find that $\{0\}$ and H are the only two k-ideals of H. Hence $Q^+(\sqrt{2}) = H$ is a left k-Artinian hemiring. Let $x = 1 + \sqrt{2}$. Then $Hx^n = \{(a + b\sqrt{3})(1 + \sqrt{2})^n : a, b$ are non-negative rational numbers$\}$ is a left ideal of H for positive integers n. Suppose $Hx = H$. Then $1 = (a + b\sqrt{2})(1 + \sqrt{2})$ for some $a, b \in Q^+$. Hence $1 = a + 2b$ and $a + b = 0$. But it is not possible. Hence $H \supsetneq Hx$.

Again $Hx \supsetneq Hx^2$. Otherwise $Hx = Hx^2$ implies $1 + \sqrt{2} = (a + b\sqrt{2})(1 + \sqrt{2})^2$ for some $a, b \in Q^+$. This shows $1 = (a + b\sqrt{2})(1 + \sqrt{2})$ which is not possible. Proceeding in this way we have a strictly descending chain of left ideals of infinite length in H:

$$H \supsetneq Hx \supsetneq Hx^2 \supsetneq Hx^3 \supsetneq \cdots$$

Consequently the hemieing $Q^+(\sqrt{2})$ is left k-Artinian but not left Artinian.

THEOREM 4.3. *Let H be an additively cancellative hemiring with more than one eleme Then H is multiplicatively cancllative k-Artinian, iff H is a division hemiring.*

PROOF. Suppose H is a division hemiring. Then from theorems 3.11 (b) and 3.14 find that H is a multiplicatively cancellative left k-Artinian hemiring. Conversely assu that H is multiplicatively cancellative left k-Artinian. Let $0 \neq a \in H$. Define $I_n = \{ra^n + m$ $r \in H$ and $m \in Z^+\}$ where Z^+ is the set of all non-negative integers, is a left ideal of H generated by a^n. We can show that $I_n \supseteq I_{n+1} \supseteq I_{n+2} \supseteq \cdots$.

Then we have a descending chain of left k-ideals $\bar{I}_1 \supseteq \bar{I}_2 \supseteq \bar{I}_3 \cdots$ in H. Since H is k-Artinian there exists a positive integer n such that $\bar{I}_n = \bar{I}_{n+1}$. Hence $a^n + (ra^{n+1} m_1 a^{n}$ $= (ta^{n+1} + m_2 a^{n+1})$ where $r, t \in H$ and $m_1, m_2 \in Z^+$. Since H is multiplicatively cancellati $a \neq 0$ implies $a^i \neq 0$ for any positive integer i. Hence from the cancellative property, find that $a + (ra + m_1 a)a = (ta + m_2 a)$ a. Let $ra + m_1 a = e_1$ and $ta + m_2 a = e_2$. Her $a + e_1 a = e_2 a$. This shows that $(e_1, e_2) \neq (0, 0)$.

Let $b \in H$. Then $ba + be_1 a = be_2 a$. By cancelative property, $b + be_1 = be_2$ for $b \in H$. Hence (e_1, e_2) is a right identity pair. Now for every $c \in H$ and $0 \neq b \in$ $bc + be_1 e = be_2 c$. By cancellative property of $(H, .)$ $c + e_1$ $c = e_2 c$. Conseqently we find t (e_1, e_2) is an identity pair. Since H contains the identity pair we find that $\{ra^n + m$ $r \in H, m \in Z^+\} = \{ra^n : r \in H\}$. Hence $\overline{\{ra^{n+1} : r \in H\}} = \overline{\{ra^{n+1} : r \in H\}}$. Then $a^n + xa^{n+1} = ya$ for some $x, y \in H$. But $a^n + e_1 a^n = e_2 a^n$. Hence $e_2 a^n + xa^{n+1} = ya^{n+1} + e_1 a^n$. By cancellat property, we have $e_2 + xa = ya + e_1$. Since $(H, .)$ is cancellative, it now follows tha invertible. Hence H is a division hemiring.

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