CHAPTER- VII

APPLICATIONS OF YONEDA'S LEMMA IN SEMIRINGS

In this chapter we present two applications of Yoneda's Lemma; one yields a characterization of a class of semirings and the other yields a classification of F-vector bundles over a topological space, where F is the field of real numbers or complex numbers.

1. Yoneda's Lemma.

Let $\mathcal{C}$ be any category and $\mathcal{S}$ the category of sets and their functions. Then for each object $C \in \mathcal{C}$ we can define the covariant functor

$h_C : \mathcal{C} \longrightarrow \mathcal{S}$ by assigning

to an object A of $\mathcal{C}$ the object $h_C(A) = (C, A)$ of $\mathcal{S}$, where (C, A) denotes the set of all mormorphisms from C to A in $\mathcal{C}$,

and to a morphism $f : A \longrightarrow B$ in $\mathcal{C}$, the morphism

$h_C(f) : h_C(A) \longrightarrow h_C(B)$ given by

$h_C(f)(g) = fg$ for $g : C \longrightarrow A$ in $\mathcal{C}$, where the right composition is the composition of morphisms in $\mathcal{C}$. For each object $C \in \mathcal{C}$ we can define dually the contravriant functor $h^C : \mathcal{C} \longrightarrow \mathcal{S}$.

Yoneda's Lemma : 1.1. Let $\mathcal{C}$ be any category and T a covariant functor from $\mathcal{C}$ to $\mathcal{S}$. Then for any object $C \in \mathcal{C}$, there is a bijective correspondence: $\theta = \theta_{C,T} : (h_C, T) \longrightarrow T(C)$ such that θ is natural in C and T, where (h_C, T) is the set of all natural transformations from h_C to T. Dually, given $C \in \mathcal{C}$ and a centravariant functor $T : \mathcal{C} \rightarrow \mathcal{S}$, there is a bijective correspondence:

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$\theta = \theta^{C,T}$: $(h^C, T) \longrightarrow T(C)$ such that θ is natural in C and T.

2. Bitranslations on Semirings

Let S be a semiring such that (S, +) is commutative. S may or maynot contain an identity 1 or absorbing zero 0.

An analogue for semirings (with commutative addition) some relevant concept for rings [34] is introduced here.

Definition 2.1. A function 1 (respectively r) on a semiring S is said to be a left (respectively right) translation on S if 1 is written as a left (respectively right) operator and satisfies

$1(xy) = (1x)y$, $1(x + y) = 1x + 1y$,

(respectively $(xy)r = x(yr)$, $(x + y)r = xr + yr$)

for all $x, y \in S$. Moreover, the pair $(1, r)$ is linked if

$x(1y) = (xr)y$ $(x, y \in S)$, and is then called a bitranslation on S.

The set L(S) of all left translations (respectively R(S) of all right translations) on S with the operations of addition and multiplication defined by

$(1 + 1_1)x = 1x + 1_1x$, $(11_1)x = 1(1_1x)$

(respectively $x(r + r_1) = xr + xr_1$, $x(rr_1) = (xr)r_1$)

for all $x \in S$ is a semiring.

The set B(S) of all bitranslations on S with the operations of addition and multiplication defined by

$(1, r) + (1_1, r_1) = (1 + 1_1, r + r_1)$, $(1_1, r_1) = (11_1, rr_1)$ is a semiring and is called the translational hull of S.

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Two bitranslations (l,r) and (l_1, r_1) on S are said to be equal iff $l = l_1$ and $r = r_1$.

Definition 2.2. For any $s \in S$, the functions l_s and r_s given by

$l_s x = sx$, $xr_s = xs$ $(x \in S)$,

are called respectively the inner left, inner right translation on S induced by s ; the pair $T_s = (l_s, r_s)$ is called the inner bitranslation on S induced by s.

The set T(S) of all inner bitranslations on S is called the inner part of B(S). If $L_i(S)$ and $R_i(S)$ are respectively the set of all inner left or inner right translations on S, then

$T(S) \subset B(S)$, $L_i(S) \subset L(S)$, $R_i(S) \subset R(S)$.

Lemma 2.3. If $l \in L(S)$, $r \in R(S)$, $l_s \in L_i(S)$, $r_s \in R_i(S)$, $T_s \in T(S)$, $w = (l,r) \in B(S)$, then for every $s \in S$,

(i) $ll_s = l_{ls}$, $l_s l = l_{sr}$

(ii) $r_s r = r_{sr}$, $rr_s = r_{ls}$ and

(iii) $wT_s = T_{ls}$, $T_s w = T_{sr}$.

Proof. (i) $(ll_s)x = l(l_s x) = l(sx) = (ls)x = l_{ls}x$ for all $x \in S$ yields $ll_s = l_{ls} \in L_i(S)$.

Again $(l_s l)x = l_s(lx) = s(lx) = (sr)x = l_{sr}x$ for all $x \in S$ yields $l_s l = l_{sr} \in L_i(S)$.

(ii) $x(r_s r) = (xr_s) r = (xs)r = x(sr) = xr_{sr}$ for all $x \in S$ yields $r_s r = r_{sr} \in R_i(S)$.

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Similarly, $rr_s = r_{1s} \in R_i(S)$.

By using (i) and (ii) it follows that

(iii) $wT_s = (1, r)\ (1_s, r_s) = (11_s, rr_s) = (1_{1s}, r_{1s}) = T_{1s} \in T(S)$

Similarly, $T_s w = T_{sr} \in T(S)$

Theorem 2.4. For a semiring S,

(i) $L_i(S)$ is an ideal of $L(S)$,

(ii) $R_i(S)$ is an ideal of $R(S)$

and (iii) $T(S)$ is an ideal of $B(S)$.

Proof. (i). Let $1_s, 1_t \in L_i(S)$. Then $(1_s + 1_t)\ x = 1_s x + 1_t x = sx + tx = (s + t)\ x = 1_{s+t} x$ for all $x \in S$ yields $1_s + 1_t \in L_i(S)$. Also by using lemma 2.3(i), 11_s, $1_s 1 \in L_i(S)$ for all $1 \in L(S)$. As a result $L_i(S)$ is an ideal of $L(S)$.

(ii) and (iii) follow by using lemmas 2.3(ii) and (iii) respectively.

Corollary 2.5. $L_i(S)$, $R_i(S)$ and $T(S)$ are respectively the subsemirings of $L(S)$, $R(S)$ and $B(S)$.

3. A characterization of Semirings

For the category theory we follow the terminology of [28]

Let S and W be semirings. A semiring homomorphism $f : S \longrightarrow W$ is a function such that

$f(s + t) = f(s) + f(t)$ and

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$f(st) = f(s) f(t)$

for all $s, t \in S$.

If in addition, f is bijective, f is said to be a semiring isomorphism.

It is easy to check that semirings and semiring homorphisms form a category denoted by S_r.

We note that for a semiring $S \in S_r$, $T(S)$ is also a semiring $\in S_r$.

For $f : S \longrightarrow W$. in S_r, we define

$f^* = T(f) : T(S) \longrightarrow T(W)$ by

$f_*(1_s, r_s) = (1_{f(s)}, r_{f(s)})$. Then for $g : W \longrightarrow X$ in S_r,

$(gf)^* : T(S) \longrightarrow T(X)$ in S_r is such that

$(gf)_* (1_s, r_s) = (1_{(gf)s}, r_{(gf)s}) = (1_{g(f(s))}, r_{g(f(s))})$

$= g_* (1_{f(s)}, r_{f(s)}) = g_* f_* (1_s, r_s)$

for all $(1_s, r_s) \in T(S)$, yielding $(gf)_* = g_* f_*$.

Also for the identity semiring homomorphism $1_S : S \longrightarrow S$ in S_r,

$1_S^* : T(S) \longrightarrow T(S)$ in S_r is such that

$1_S^* (1_s, r_s) = (1_{1_S s}, r_{1_S s}) = (1_s, r_s)$.

for all $(1_s, r_s) \in T(S)$ yielding 1_{S*} is the identity semiring homomorphism.

Hence we have the following theorem.

Theorem 3.1. $T : Sr \longrightarrow Sr$ is a covariant functor

Lemma 3.2. For a semiring S, the function $f : S \longrightarrow T(S)$ defined by $f(s)=T_s$ for $s \in S$ is a semiring homomorphism of S. onto $T(S)$.

Proof. The equalities

$f(s + t) = T_{s+t} = T_s + T_t$

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and $f(st) = T_{st} = T_s T_t$

for all $s, t \in S$ yield that

f is a semiring homomorphism.

Moreover, for any $T_s \in T(S)$, we find that

$f(s) = T_s$ for $s \in S$.

Hence f is a semiring homomorphism of S onto $T(S)$.

An analogue of a reductive semigroup [34] a reductive semiring is defined in [1].

Definition 3.3. A semiring S is said to be reductive iff for any $a, b \in S$, $ax = bx$ and $xa = xb$ for all $x \in S$ imply $a = b$.

Theorem 3.4. The function

$f : S \longrightarrow T(S)$ defined by

$f(s) = T_s$

is a semiring isomorphism iff S is reductive.

Proof. Suppose S is reductive and $f(s) = f(t)$ holds for $s, t \in S$. Then $T_s = T_t$ implies that $(l_s, r_s) = (l_t, r_t)$. Consequently $l_s = l_t$ and $r_s = r_t$ yield that $sx = tx$ and $xs = xt$ for all $x \in S$. As a result $s = t$. Hence by using lemma 3.2. it follows that f is a semiring isomorphism.

Conversely, let f be a semiring isomorphism and $ax = bx$, $xa = xb$ hold for $a, b \in S$ and all $x \in S$.

Then $l_a x = l_b x$ and $x r_a = x r_b$ hold for all $x \in S$.

This shows that $l_a = l_b$ and $r_a = r_b$ yielding

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$f(a) = T_a = (1_a, r_a) = (1_b, r_b) = T_b = f(b)$.

Hence $a = b$. Consequently S is reductive.

Corollary. If S is a multiplicatively cancellative semiring, then : $S \longrightarrow T(S)$ defined by

$f(s) = T_s$ is a semiring isomorphism.

For a given semiring $S \in S_r$, the set of all semiring homomorphisms $F_S(X)$ from S to X in S_r forms a semiring $\in S_r$ for each $X \in S_r$ under usual addition and multiplication of functions.

We also define for each semiring homomorphism $f : X \longrightarrow Y$ in S_r, the semiring homomorphism $f^* = F_S(f) : F_S(X) \longrightarrow F_S(Y)$ by $f^*(g) = fg$ for $g : S \longrightarrow X$ in S_r.

Then it follows that

Lemma 3.5. F_S is a covariant functor from the category S_r to itself.

Let (F_S, T) denote the set of all natural transformations from the covariant functor F_S to the covariant functor T.

Theorem 3.6. For each semiring $S \in S_r$, the set (F_S, T) admits a semiring structure such that (F_S, T) is semiring isomorphic to $T(S)$.

Proof. Using Yoneda Lemma for the covariant functors F_S and T, we find that there is a bijective correspondence

$f_S : T(S) \longrightarrow (F_S, T)$

for every $S \in S_r$.

As $T(S)$ is a semiring by corollary 2.5, f_S induces compositions on the set (F_S, T)

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such that (F_S, T) admits a semiring structure yielding (F_S, T) semiring isomorphic to $T(S)$.

Theorem 3.7 For a semiring S with commutative addition, S is semiring isomorphic to the semiring (F_S, T) iff S is reductive.

Proof. The theorem is immediate by using theorems 3.4 and 3.6.

4. A Classification of Vector Bundles

The concept of vector bundles arose from the study of tangent vector fields to smooth geometric objects, e. g. spheres, projective spaces, and more generally, manifolds. Vector bundle is a bundle with an additional vector space structure on each fibre. We follow the terminology of [19].

Let F denote the field of real numbers or complex numbers.

For each topological space X, let $Vect_F(X)$ denote the set of all isomorphism classes of F-vector bundles over X.

A classification of $Vect_F(X)$ by Yoneda's lemma and some algebraic structural results on $Vect_F(x)$ are obtained.

Let Top denote the category of topological spaces and continuous maps and $\mathcal{S}$ denote the category of sets and functions

We define $Vect_F : Top \longrightarrow \mathcal{S}$ by

$X \longrightarrow Vect_F(X)$ for $X \in Top$

and for $f : Y \longrightarrow X$ in Top,

$Vect_F(f) : Vect_F(X) \longrightarrow Vect_F(Y)$ by

$Vect_F(f)\ (\beta) = f^*(\beta)$, an F-vector bundle over Y induced by f.

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(Here β denotes both a vector bundle and its isomorphism class).

Then $Vect_F$ is a contravariant functor.

For a given space $X \in Top$, let $H^X(Y)$ denote the set of all continuous maps from Y to X in Top.

We define for each $f : M \longrightarrow N$ in Top,

$H^X(f) = f^* : H^X(N) \longrightarrow H^X(M)$ by

$f^*(g) = gf$ for $g : N \longrightarrow X$ in Top.

Then it follows that

Lemma 4.1. H^X is a contravariant functor from the category Top to the category $\mathcal{S}$ of sets.

It is known in (cf. 19, page 102) that $Vect_F(X)$ admits a semiring structure with commutative addition under the addition function

$(\alpha, \beta) \longrightarrow \alpha \oplus \beta$

and the multiplication function

$(\alpha, \beta) \longrightarrow \alpha \otimes \beta$

where $\oplus$ and $\otimes$ denote respectively Whitney sum and tensor product of two vector bundles over X.

Theorem 4.2. For each space $X \in Top$, the set $(H^X, Vect_F)$ of natural transformations between contravariant functors H^X and $Vect_F$ admits semiring structure such that $(H^X, Vect_F)$ is semiring isomorphic to $Vect_F(X)$.

Proof. Using Yoneda Lemma for the contravariant functors H^X and $Vect_F$.

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We find that there is a bijection

$f_X : Vect_F(X) \longrightarrow (H^X, Vect_F)$

for every $X \in$ Top.

As $Vect_F(X)$ is a semiring, f_X induces compositions on the set $(H^X, Vect_F)$ such that $(H^X, Vect_F)$ admits a semiring structure with addition commutative and yielding $(H^X, Vect_F)$ semiring isomorphic to $Vect_F(X)$.

Remark 4.3. The theorem yields a representation of the contravariant functor $Vect_F$. In this way the problem of classifying vector bundles, i.e. of computing $Vect_F(X)$ has been reduced to the determination of the sets of the natural transformations $(H^X, Vect_F)$.

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