CHAPTER - VI

STRUCTURE SPACES OF SEMIALGEBRAS

In this chapter we study the structure spaces of semirings, formed by the class of completely prime k-ideals. Also a Galois connection is defined between the partially ordered set of subsets of M(R) of all maximal modular h-ideals of an S-semialgebra R and the partially ordered subsets of R. This leads to define the Stone-Jacobson - Zarisky topology on M(R). We also study the structure space M(R) endowed with this topology.

1. Structure spaces of semirings.

Let S be a commutative semiring with absorbing 0 and identity 1 and $\mathcal{A}$ the collection of completely prime k-ideals of S.

Then the routine computations prove the following:

For any subset A of $\mathcal{A}$, define

$$\bar{A} = \left\{ I \in A : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I \right\}.$$ Then

Proposition 1.1

(1) $A \subseteq \bar{A}$

(2) $\bar{A} = \bar{\bar{A}}$

(3) $A \subseteq B \Longrightarrow \bar{A} \subseteq \bar{B}$

and (4) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

for all subsets A, B of $\mathcal{A}$.

Remark $A \longrightarrow \bar{A}$ is a closure operator defining some topology $\tau_{\mathcal{A}}$

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called the hull-kernel topology on $\mathcal{A}$. Now let $\Delta(a) = \{I \in \mathcal{A} : a \in I$ and $C\,\Delta(a) = \mathcal{A} \setminus \Delta(a) = \{I \in \mathcal{A} : a \notin I\}$, $a \in S$.

Proposition 1.2: $\{C\,\Delta(a) : a \in S\}$ forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.

Proposition 1.3. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 - space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Corollary 1.4. Let M be the set of all maximal k-ideals of S. Then (M, τ_M) is a T_1 - space, where τ_M is the induced topology on M from $(\mathcal{A}, \tau_{\mathcal{A}})$.

Proof follows immediately, since $M \subseteq \mathcal{A}$ by proposition 3.2 of Chapter I.

Let I be any k-ideal of S, we define

$$\Delta(I) = \{I' \in \mathcal{A} : I \subseteq I'\}.$$

Proposition 1.5. Any closed set in $\mathcal{A}$ is of the form $\Delta(I)$, where I is a k-ideal of S.

Proof. Let $\bar{A}$ be any closed set in $\mathcal{A}$, where $A \subseteq \mathcal{A}$. Let $A = \{I_\alpha : \alpha \in \Lambda\}$ and $I = \bigcap_{\alpha \in \Lambda} I_\alpha$. Then $\bar{A} = \Delta(I)$.

Theorem 1.6. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space if and only if for any distinct pair of elements I, J of $\mathcal{A}$, there exist $a, b \in S$ such that $a \notin I$, $b \notin I$ and there does not exist any element K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.

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Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be Hausdorff. Then for any pair of distinct elements I, J of $\mathcal{A}$ there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \phi$. It follows that $a \notin I$, $b \notin J$ and for any completely prime k-ideal K of S for which $a, b \notin K$ implies $K \in C\Delta(a) \cap C\Delta(b)$, a contradiction, since $C\Delta(a) \cap C\Delta(b) = \phi$.

Conversely let the given condition hold and $I, J \in \mathcal{A}$, $I \neq J$. Let $a, b \in S$ be such that $a \notin I$, $b \notin J$ and there does not exist any K of $\mathcal{A}$ such that $a \notin K$, $b \notin K$. Then $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \phi$ which proves that $(\mathcal{A}, \tau_{\mathcal{A}})$ is Hausdorff.

Corollary 1.7. If $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_2 - space, then no proper prime k-ideal contains any other proper completely prime k-ideal. If $(\mathcal{A}, \tau_{\mathcal{A}})$ contains more than one element, then there exist $a, b \in S$ such that $\mathcal{A} = C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$, where I is the k-ideal generated by a, b.

Proof. Suppose $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_2 - space. Since every T_2 space is a T_1 - space, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 - space. Hence by proposition 1.3, no proper k-ideal contains any other proper completely prime k-ideal. Now let $I, J \in \mathcal{A}$ where $I \neq J$. Then by Theorem 1.6 there exist a, b in S such that $a \neq b$, $a \notin I$, $b \notin J$ and $C\Delta(a) \cap C\Delta(b) = \phi$. Let I be the k-ideal generated by a, b. Then since $C\Delta(a) \cap C\Delta(b) = \phi$, any element of $\mathcal{A}$ belongs to $C\Delta(a)$ or $C\Delta(b)$ or $\Delta(I)$, therefore $C\Delta(a) \cup C\Delta(b) \cup \Delta(I) = \mathcal{A}$.

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Theorem 1.8. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist a k-ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\ \Delta\ (a)$.

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a regular space. Then for any $I \in \mathcal{A}$ and any closed set $\Delta(J)$ not containing I, there exist disjoint open sets U, V such that $I \in U$ and $\Delta(J) \subseteq V$. If $a \notin I$ then $I \in C\Delta(a)$ and $\mathcal{A} \setminus C\ \Delta\ (a)$ is a closed set not containing I. Let U, V be disjoint open sets containing I and $\mathcal{A} \setminus C\ \Delta(a)$ respectively. Then there exists $b \in S$ such that $I \in C\ \Delta(b) \subseteq U$ and $\mathcal{A} \setminus V = \Delta(J)$ (say) is a closed set satisfying the relation $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\ \Delta(a)$, where J is a k-ideal of S.

Conversely, let the given condition hold. Let $I \in \mathcal{A}$ and $\Delta(K)$ be any closed set not containing I. Let $a \notin I$, $a \in K$. Then by the given condition, there exist ideal J and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\ \Delta(a)$. Obviously, $C\ \Delta\ (a) \cap \Delta(K) = \phi$. So $C\ \Delta(b)$ and $A \setminus \Delta(J)$ are two disjoint open sets containing I and $\Delta(K)$ respectively. Consequently, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space.

Theorem 1.9. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space if and only if there exist finite member of elements $a_1, a_2, \ldots, a_r$ in S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a compact space. Then the open cover $\{C\ \Delta(a) : a \in S\}$ has a finite subcover $\{C\ \Delta(a_i) : i = 1, \ldots, r\}$. Thus if $I \in \mathcal{A}$, then $I \in C\ \Delta\ (a_i)$ for some i which implies that $a_i \notin I$. Hence $a, \ldots, a_r$ are the required finite number of elements.

Coversely, let $a_1, \ldots, a_r$ be elements of S such that for

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$I \in \mathcal{A}$, there exists a_i satisfying $a_i \notin I$. Then the basic open sets $\{C\ \Delta(a_i) : i = 1, \ldots, r\}$ form a finite open cover for $\mathcal{A}$. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

Corollary 1.10. If S is finitely generated, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

Proof. Let $S = \langle a_1, a_2, \ldots, a_n \rangle$. Then for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$, since I is a proper ideal. Hence by theorem 1.9, $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

Theorem 1.11. $(\mathcal{A}, \tau_{\mathcal{A}})$ is disconnected if and only if there exist an ideal I of S and a collection of points $\{a_\alpha\}_{\alpha \in \Lambda}$ of S not belonging to I such that if $I' \in \mathcal{A}$ and $a_\alpha \in I'$, $\forall \alpha \in \Lambda$ then $I \setminus I' \neq \phi$.

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be not connected. Then there exists a non-trivial open and closed subset of $\mathcal{A}$. Let I be the k-ideal of S for which $\Delta(I)$ is closed as well as open. Then $\Delta(I) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha)$ where $\{a_\alpha\}_{\alpha \in \Lambda}$ is a collection of points of S. Now since $C\Delta(a_\alpha) \subseteq \Delta(I)$, $\forall \alpha \in \Lambda$ for any $I_\alpha \in C\Delta(a_\alpha)$ we have $I \subseteq I_\alpha$, therefore $a_\alpha \notin I$ as $a_\alpha \notin I_\alpha$, $\forall \alpha \in \Lambda$. Now for any $I' \in \mathcal{A}$ and $a_\alpha \in I'$, $\forall \alpha \in \Lambda$ we have $I' \notin \Delta(I)$, Consequently $I \nsubseteq I'$ i.e. $I \setminus I' \neq \phi$.

Conversely let the given condition hold. Then $\Delta(I) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha)$ is an open and closed non-trivial subset of $\mathcal{A}$ and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is disconnected.

Example 1.12. Let $S = Z^+$ be the semiring of non-negative integers with respect to usual addition and multiplication. Then the completely prime k-ideals of S are $(p) = \{np : n \in Z^+\}$ where p is a prime number by using

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proposition 4.1 of Chapter- I. Let $\mathcal{A} = \{(p) : p\text{-prime}\}$ and $\tau_{\mathcal{A}}$ be the hull kernel topology defined in $\mathcal{A}$. Since any completely prime k-ideal of Z^+ is not contained in other completely prime k-ideal, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 - space by proposition 1.3. Now let (p_1) and (p_2) be two distinct elements of $\mathcal{A}$ and let n_1, n_2 be two elements of S such that $n_1 \notin (p_1)$ and $n_2 \notin (p_2)$, then we can always find an element (p) of $\mathcal{A}$ such that $n_1 \notin (p)$ and $n_2 \notin (p)$. In fact one can take $p > n_1, n_2, p_1, p_2$ and p is prime. Hence by Theorem 1.6, $(\mathcal{A}, \tau_{\mathcal{A}})$ is not a Housdorff space. $C\Delta(n)$ is infinite and its complements, which is closed in $\mathcal{A}$, is finite. So any two non-trivial open sets intersect. Consequently, $(\mathcal{A}, \tau_{\mathcal{A}})$ is neither a T_2 - space nor a regular space. For this reason $(\mathcal{A}, \tau_{\mathcal{A}})$ is a connected space. Now we prove that $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact. Let p_1, p_2 be two distinct prime numbers. Then we can easily verify that for any $(p) \in \mathcal{A}$, either $p_1 \notin (p)$ or $p_2 \notin (p)$. Hence by Theorem 1.9 $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

2. Semialgebra.

Definition 2.1 If H and T are hemirings, then an (H,T) bi-semimodule M is both a left H-semimodule and a right T-semimodule such that $(bx)a = b(xa)$ for all $x \in M$, $b \in H$ and $a \in T$.

Definition 2.2. Let R be an arbitary hemiring (we do not require that R be commutative nor R should possess an identity element). Let S be also a hemiring. R is said to be an S-semialgebra iff R is a bi-semimodule over S such that $(ax)b = a(xb)$ for all $a, b \in R$ and $x \in S$.

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Definition 2.3. Let S be a hemiring with 1 and R an S - semialgebra. Then R is said to be unitary iff $1x = x$ for all $x \in R$.

Definition 2.4. An h-ideal A of the S-semialgebra R is an h-ideal of the hemiring R such that $SA \subseteq A$ and $AS \subseteq A$.

Definition 2.5. An h-ideal A of the S-semialgebra R is said to be a left modular h-ideal iff there exists an element $e \in R$ such that

(i) each $x \in R$ satisfies: $ex + a + z = x + b + z$ for some $z \in R$ and some $a, b \in A$;

(ii) each $s \in S$ satisfies : $se + c + h = es + d + h$ for some $h \in R$ and some $c, d \in A$.

In this case e is called a left unit modulo A.

We have similar definition for a right modular h-ideal.

Definition 2.6. If A is both a left and a right modular h-ideal of the S - semialgebra R, then A is called a bimodular h-ideal.

Definition 2.7. : A left (right) h-ideal A of the S-semialgebra R is said to be a maximal left (right) modular h-ideal iff $A \neq R$ and A is not contained properly in any left (right) modular h-ideal of R other than R.

By using Zorn's Lemma we can prove the following :

Lemma 2.8. Every proper left (right) modular h-ideal of an S-semialgebra R is contained in a maximal left(right) modular h-ideal of the S-semialgebra R.

Lemma 2.9 : If an h-ideal A of the S-semialgebra R is bimodular , then there exists $e \in R$ such that

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(1) each x R satisfies:

$ex + a + h = x + b + h$, $xe + c + u = x + d + u$

for some $a, b, c, d \in A$ and $h, u \in R$ (depending on x) ;

and (2) each $s \in S$ satisfies:

$se + c + r = es + d + r$ for some $c, d \in A$ and $r \in S$ (depending on s).

Proof. Since A is a left modular h-ideal of the S-semialgebra R, there exists $e_1 \in R$ such that

(i) each $x \in R$ satisfies :

$e_1x + a_1 + h_1 = x + b_1 + h_1$ for some $a_1, b_1 \in A$ and $h_1 \in R$;

and (ii) each $s \in S$ satisfies:

$se_1 + c_1 + r_1 = e_1s + d_1 + r_1$ for some $c_1, d_1 \in A$ and $r_1 \in S$.

Similarly, since A is also a right modular h-ideal, there exists $e_2 \in R$ such that

(iii) each $x \in R$ satisfies:

$xe_2 + a_2 + h_2 = x + b_2 + h_2$ for some $a_2, b_2 \in A$ and $h_2 \in R$.

and (iv) each $s \in S$ satisfies :

$e_2s + c_2 + r_2 = se_2 + d_2 + r_2$ for some $e_2, d_2 \in A$ and $r_2 \in S$.

Then $e_1\sigma_A$ and $e_2\sigma_A$ are respectively a left identity and a right identity in the Iizuka factor hemiring R/σ_A. It follows that $e_1\sigma_A = e_2\sigma_A$ and hence $e_1 + a_3 + h_3 = e_2 + b_3 + h_3$ for some $a_3, b_3 \in A$ and $h_3 \in R$.

Let us take $e_2 + b_3 = e \in R$. Then

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$e_1x + a_3x + h_3x = e_2x + b_3x + h_3x$ yields

$e_1x + a_1 + h_1 + a_3x + h_3x = ex + a_1 + h_1 + h_3x$. This shows that

$x + b_1 + h_1 + a_3x + h_3x = ex + a_1+h_1 + h_3x$ by (i).

Consequently, $ex + a + h = x + b + h$, where

$h = h_1 + h_3x \in R$, $a = a_1$ and $b = b_1 + a_3 x \in A$.

Similarly, $xe + c + u = x + d + u$ for each $x \in R$ and some $c, d \in A$ and $u \in R$,

This completes the proof (1).

Also, by using (iv) we have,

$sb_3 + b_3s + se_2 + d_2 + r_2 = e_2s + c_2 + r_2 + sb_3 + b_3s$.

Then $se + d_2 + r_2 + b_3s$

$= es + c_2 + r_2 + sb_3$.

This shows that

$se + c + r = es + d + r$ for all $s \in R$ and suitable $c, d \in A$ and $r \in R$ (depending on s), where $c = d_2 + b_3 s \in A$, $d = c_2 + sb_3 \in A$ and $r = r_2$

This proves (2).

The following lemma is essentially due to [45] :

Lemma 2.10. The k-ideals of a halfring H are precisely the intersection with H of the ideals of its ring of differences $\widetilde{H}$.

A halfring R is said to be h-simple iff R has only trivial h-ideals:

Lemma 2.11. Let R be an h-simple halfring. If e is a left identity of R, then e is also a right identity of R.

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Proof. Let e be a left identity. Then $ex = x$ for all $x \in R$. Suppose $\widetilde{R}$ is a ring of difference of R.

In $\widetilde{R}$, define

$$I = \{ (a, b)(e, o) - (a, b) : a, b \in R \}$$
$$= \{ (ae, be) + (b, a) : a, b \in R. \}$$
$$= \{ (ae + b, be + a) : a, b \in R \}.$$

Then I is an ideal of $\widetilde{R}$. Hence $I \cap R$ is a k-ideal of R by lemma 2.10. Then $I \cap R = \{o\}$ or $I \cap R = R$ as in this case h-ideals and k-ideals of R are the same. But $I \cap R \neq R$.

Hence $I \cap R = \{o\}$.

Let $a \in R$ Then $(o,a)(e,o) + (a, o) \in I \cap R$.

Hence $(o, ae) + (a,o) = (o,o)$

i.e. $(a \; ae) = (o, o)$. Consequently, it follows from the construction of $\widetilde{R}$ that $ae = a$ for all $a \in R$.

Lemma 2.12. Let A be a maximal left (right) modular h-ideal of the S-semialgebra R. Then A is bimodular.

Proof. Let $e \in R$ as in definition 2.5. Since A is a maximal left modular h-ideal of R, the Iizuka factor hemiring R/A is additively cancellative having only trivial h-ideals with $e\sigma_A$ a left identity. Hence from lemma 2.11, $e_1\sigma_A$ is also a right identity in $R/\sigma_A = R/A$. Let $x \in R$. Then $x\sigma_A \; e\sigma_A = x\sigma_A$ and hence $(xe)\,\sigma_A = x\sigma_A$. This shows that $xe + a + h = x + b + h$ for some $a, b \in A$ and $h \in R$.

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The following theorem shows that the set of maximal modular h-ideals of the S-semialgebra R depends only on the hemiring structure of R.

Theorem 2.13. Let R be an S - semialgebra. Then A is a maximal modular h-ideal of the S-semialgebra R iff A is a maximal modular h-ideal of the hemiring R.

Proof. Suppose A is a maximal modular h-ideal of the hemiring R and e a unit modulo A. Then for each $x \in R$, we have

$ex + a + h = x + b + h$ and $xe + a_1 + h_1 = x + b_1 + h_1$

for some $a, b, a_1, b_1 \in A$ and some $h, h_1 \in R$.

For $s \in S$ and $x \in A$, we have

$e(sx) + a_2 + h_2 = sx + b_2 + h_2$ and $(es)x \in A$

for suitable $a_2, b_2 \in A$ and $h_2 \in R$. As A is an h-ideal of the hemiring R, it follows that $sx \in A$. Similarly for $s \in S$ and $x \in A$, $xs \in A$.

Hence A is an h-ideal of the S-semialgebra R. In order to show that A is a modular h-ideal of the S-semialgebra R it is sufficient to show that

$$se + a_3 + h_3 = es + b_3 + h_3 \text{ for any } s \in S \text{ and}$$

suitable $a_3, b_3 \in A$ and $h_3 \in R$.

Since $e\sigma_A$ is the unit element of the S-semialgebra $R/\sigma_A = R/A$, we have $(se)\sigma_A = (e\sigma_A)s$ for all $s \in S$. This shows that for any $s \in S$, $se + a_3 + h_3 = es + b_3 + h_3$ for suitable $a_3, b_3 \in A$ and $h_3 \in R$.

Hence A is a modular h-ideal of the h-ideal of the S-semialgebra R.

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Let us show that A is a maximal h-ideal of the S-semialgebra R. Suppose now A' is a modular h-ideal of the S-semialgebra such that $A \subseteq A'$. Then A' being a modular h-ideal of the hemiring R, $A' = A$ or $A' = R$. Hence A is a maximal modular h-ideal of the S-semialgebra R.

Conversely, suppose that A is a maximal modular h-ideal of the S-semialgebra R. Then A is a modular h-ideal of the hemiring R. By lemma 2.8 A is contained in a maximal modular h-ideal A' of the hemiring R. From the first part of the proof A' is a maximal modular h-ideal of S-semialgebra R. Hence $A = A'$.

Theorem 2.14. Let R be an S-semialgebra and $A \subset R$ an h-ideal. Then A is an S-semialgebra. Further

(i) for any maximal modular h-ideal M_1 of the S-semialgebra A, there exists a unique maximal modular h-ideal M of the S-semialgebra R such that $M_1 \subseteq M$ and $M \not\supseteq A$. If $e \in A$ is a unit element modulo M_1 in A, then

$$\begin{aligned} M &= \{x \in R : e\,x \in M_1\} \\ &= \{x \in R : x\,e \in M_1\} \\ &= \{x \in R : e\,x\,e \in M_1\} \end{aligned}$$

and $M_1 = M \cap A$;

and (ii) for any maximal modular h-ideal $M \subset R$ such that $A \not\subseteq M$, $M_1 = M \cap A$ is a maximal modular h-ideal of the S-semialgebra A..

Proof (i). Let M_1 be a maximal modular h-ideal of the S-semialgebra A and $e \in A$ a unit element modulo M_1. Then for any $x \in R$, $ex + a + z = x + b + z$ for some $a, b \in M_1$, $z \in R$. This shows that $e\,x \in M_1$ implies

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$x \in M_1$ and hence $x e \in M_1$. (since $e x \in M_1 \Longrightarrow x \in M_1 \Longrightarrow x e \in M_1$).
Again $xe + a_1 + z_1 = x + b_1 + z_1$ for $a, b \in M_1$, $z_1 \in R$ for any $x \in R$.
This shows that $x e \in M_1$ implies $x \in M_1$ and again this implies $ex \in M$.
Thus $ex \in M_1 \Leftrightarrow x e \in M_1 \Leftrightarrow e x e \in M_1$.
Define $M = \{ x \in R \ : \ e x \in M_1 \}$.
Then M is an ideal of the S-semialgebra R and if $x + a + h = b + h$ for $x, h \in R$, $a, b \in M$, then $ea, eb \in M_1$ and hence $ex + ea + eh = eb + eh$ implies that $ex \in M_1$ as M_1 is an h-ideal. Hence $x \in M$. Thus M is an h-ideal. Also for any $x \in M_1$, $ex \in M_1$ implies $x \in M$. Thus $M_1 \subseteq M$. Consequently, e is a unit element modulo M. We claim that $e \notin M$. Otherwise $ee \in M_1$ i.e. $e \in M_1$, yielding $M_1 = A$, a contradiction. Hence $e \notin M$ and consequently, $A \nsubseteq M$. Then M is a proper modular h-ideal of R. Consequently, there exists a maximal modular h-ideal M_o of the S-semialgebra R such that $M \subseteq M_o$. If $x \in M_o$, then $ex \in M_o$. Also $e x \in A$. Consequently, $ex \in M_o \cap A$. Now $M_1 = M \cap A \subseteq M_o \cap A = m_o$ (say). As M_o is a maximal h-ideal of R, m_o is a maximal h-ideal of A. M_1 is also a maximal h-ideal of A such that $M_1 \subseteq m_o$. Consequently, $M_1 = m_o$. As a result, $ex \in m_o = M_1$. This implies that $x \in M$. Consequently, $M_o \subseteq M$. Therefore $M = M_o$ is a maximal h-ideal of the S-semialgebra R with the property that $M \cap A = M_1$ and $A \nsubseteq M$. From the above discussion it also follows that M is the unique maximal modular h-ideal.

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(ii). Let M be a maximal modular h-ideal of the S semialgebra R such that $A \not\subseteq M$. Define $M_1 = M \cap A$. Then M is a proper h-ideal of S-semialgebra A. Hence there exists a maximal modular h-ideal M_2 of A containing M_1. Then by the first part of the proof there exists a maximal modular h-ideal M_0 of R such that $M_0 \supseteq M_2$ and $M_0 \not\supseteq A$ and $e \in A$ is a unit element modulo M_2. If $x \in M$, then as in the first part $ex \in M \cap A$ $= M_1 \subset M_2 \subseteq M_0$. But $ex + a + h = x + b + h$ for some $h \in R$ and $a, b \in M_0$. As M_0 is an h-ideal, and $e x \in M_0$, $x \in M$. Hence $M \subseteq M_0$. As M and M_0 are both maximal h-ideals of R such that $M \subseteq M_0$, it follows that $M = M_0$. But then we have $M_1 = M \cap A = M_0 \cap A$. This implies that M_1 is a maximal modular h-ideal of the S-semialgebra A.

3. Structure spaces of semialgebras

It appears from theorem 2.13 that the set $M(R)$ of all maximal modular h-ideals of the S-semialgebra R depends on the hemiring structure of R. A Galois connection may be defined between the partially ordered set of the subsets of R and the partially ordered set of the subsets of $M(R)$, which leads to the Stone-Jacobson-Zarisky topology on $M(R)$ (known also as the hull-kernel topology).

Define $h(E) = \{M \in M(R) : M \supseteq E\}$ for any subset $E \subset R$ and $k(\mathcal{A}) = \bigcap_{M \in \mathcal{A}} M$, for any subset $\mathcal{A}$ of $M(R)$. In particular, let $k(\mathcal{A}) = R$ if $\mathcal{A} \neq \phi$

We now show that

$h^*: \mathcal{A} \longrightarrow h(k(\mathcal{A}))$ is a closure operator on the set of subsets

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f M(R).

e first prove the following lemma.

emma. 3.1. Let $M \in M(R)$ and A, B two h-ideals of R such that $\cap B \subseteq M$. Then either $A \subseteq M$ or $B \subseteq M$.

roof. Suppose $B \not\subseteq M$. Define $M_1 = M \cap B$. Then M_1 is an h-ideal of R. herefore there exists a maximal modular h-ideal M_0 of R containing M uch that $e \in B$ is a unit element modulo M_1 by theorem 2.14 (i). en $ex \in B$ for all $x \in R$.

et $x \in A$. Then $ex \in A$. Thus $e\,x \in A \cap B \;\forall x \in A$.

ence $ex \in M$ for all $x \in A$ as $A \cap B \subseteq M$. Thus $ex + a + h = x + b + h$ for , b, $ex \in M$, $h \in R$ implies that $x \in M$. Hence $A \subseteq M$.

eorem 3.2.

(i) $h^* \phi = \phi$

(ii) $\mathcal{A} \subseteq h^*(\mathcal{A})$

(iii) $h^{**}(\mathcal{A}) = h^*(\mathcal{A})$

(iv) $h^*(\mathcal{A} \cup \mathcal{B}) = h^*(\mathcal{A}) \cup h^*(\mathcal{B})$

r any subsets $\mathcal{A}$, $\mathcal{B}$ of M(R).

oof. We prove only (iv) and the proof of others are trivial.

$$(\mathcal{A} \cup \mathcal{B}) = \bigcap_{M \in \mathcal{A} \cup \mathcal{B}} M = \left\{\bigcap_{M \in \mathcal{A}} M\right\} \cap \left\{\bigcap_{M \in \mathcal{B}} M\right\}$$

$$= k(\mathcal{A}) \cap k(\mathcal{B}).$$

$$(\mathcal{A} \cup \mathcal{B}) = \{M \in M(R) : M \supseteq k(\mathcal{A} \cup \mathcal{B})\}$$

$$= k(\mathcal{A}) \cap k(\mathcal{B}) \supseteq h^*(\mathcal{A}) \cup h^*(\mathcal{B})$$

Cont... 141.

Let $M \in h^* (\mathcal{A} \cup \mathcal{B})$, Then $M \supseteq k \ (\mathcal{A} \cup \mathcal{B})$

$= k\ (\mathcal{A}) \cap\ k\ (\mathcal{B})$. Since $k\ (\mathcal{A})$ and $k\ (\mathcal{B})$ are h-ideals,

by lemma 3.1, either $k\ (\mathcal{A}) \subseteq M$ or $k\ (\mathcal{B}) \subseteq M$.

Consequently, $M \in h^*\ (\mathcal{A})$ or $M \in h^*\ (\mathcal{B})$. Hence

$h^*\ (\mathcal{A} \cup \mathcal{B}) \subseteq h^*\ (\mathcal{A}) \cup h^*\ (\mathcal{B})$. Consequently,

$h^*\ (\mathcal{A} \cup \mathcal{B}) = h^*\ (\mathcal{A}) \cup h^*\ (\mathcal{B})$

The closure operator $h^* : \mathcal{A} \longrightarrow h(k\ (\mathcal{A}))$ defines a topology on the set of the subsets of M(R), yielding the usual Stone - Jacobson - Zarisky topology.

M(R) endowed with this topology is called the structure space of the S-semialgebra R (of the hemiring R). Clearly, structure space M(R) is a T_1 - space.

Definition 3.3. Two elements $x, y \in R$ are said to be disjoint iff for any $M \in M(R)$ we have $x \in M$ or $y \in M$.

Theorem 2.4. M(R) is a Housdorff space iff for any two distinct maximal h-ideals $M_1, M_2 \in M\ (R)$, $M_1 \neq M_2$, there exist disjoint elements $e_i \in R$ such that each e_i is a unit element modulo M_i, $i = 1,2$.

Proof. Suppose that M(R) is a Housdorff space and let $M_1, M_2 \in M(R)$ be distinct. Then there exist open neighbourhoods U_i with $M_i \in U_i$

$i = 1, 2$ and $U_1 \cap U_2 = \phi$ ..... (1)

Since their complements $CU_i = F_i$ are closed sets, we have

$F_i = h(I_i)$, where $I_i = k\ (F_i)$, $i = 1, 2$. From (1) we have

$F_1 \cup F_2 = M\ (R)$

Cont... 142.

Since $M_i \not\in F_i = h(I_i)$. there exists $e_i \in I_i$ unit element modulo M_i, $i = 1, 2$. For any $M \in M(R)$, we have $M \in F_1$ or $M \in F_2$. Consequently $e_1 \in M$ or $e_2 \in M$, which shows that e_1 and e_2 are disjoint. Conversely, let M_1, M_2 be distinct elements of $M(R)$ and e_i, $i = 1, 2$ be disjoint unit elements moduls M_i. Then

$U_i = Ch(\{e_i\})$, are disjoint open sets and $M_i \in U_i$, $i = 1, 2$.

This completes the proof.