"A definition of growth can only be of an "operational" nature, and its value is exactly equal to what it may offer for a specific problem of research.

Growth is an abstraction; only in this sense can one discuss foundation, theory and laws of growth".

L. V. Bertalanffy (1960)
0.1 Preliminary Background: Objectives and Motivation

Evolution and growth are the most impressive phenomena in the world. Every year when the nature wakes up a new, we observe the growth of plants and animals and the awakening of a large variety of species. We always see life and death against the background of a rich manifold of evolutionary processes. There are tremendous number of natural processes around us. The world we see around is not stable, it is continuously growing, the growth being slowly [Peshel and Mende, 1983].

The growth is a significant phenomenon of life process and the analysis of growth may lead to basic problems and insight in biology and beyond. [Bertalanffy, 1960]. The branch of science dealing with the theoretical study of growth and development of living system is the subject ontogeny and falls within the domain of molecular biology. There are no general theory of growth and development. In biology, where growth and development are most frequently encountered, there has been a decided bias towards explanations in terms of the smaller, or microscopic, realm. It is now a general trend in biology to explain every phenomena of life process from the point of view of molecular constituents of the system. It is true that molecular biology has made tremendous progress in recent times. But the utility and applicability of a model depends on the scale of the system and
the level of abstraction. The growth and development are, however, not confined to ontogeny. Ecosystems, economic communities, social structures, cultural movements and even meteorological and galactic forms are perceived to undergo what is commonly regarded as the growth and development [Ulanowicz, 1986]. Most meteorologists would consider it a waste of time to attempt to explain the growth or formation of hurricanes solely in terms of the properties of water molecules. Thus in spite of the importance of the molecular theory of growth and development, phenomenological theory of growth and development is of utmost necessity and utility. The phenomenological theory of growth of living systems based on thermodynamics of irreversible processes is first due to Prigogine and Weame [1940]. This theory was later developed by Zotin and Zotina [1967, 1983], Lurie and Wagenberg [1983], Bormann [1988], Chakrabarti [1989] and others. Modelling of ecosystem or population growth on the basis of phenomenological theory of non-equilibrium thermodynamics was also attempted by Ebeling [1990], Assimacopoulos [1987] and Chakrabarti [1991]. The non-equilibrium thermodynamic theory of biological growth (individual or population) has two advantages. Firstly, it leads to an unified approach to deal with both living and non-living systems together. Secondly, the modelling of a special biological or ecological system may lead to characteristic results of general importance. The process of growth and
development is a complex phenomena. Thermodynamics is considered to be a cornerstone of science and is the appropriate branch of science to deal with complex phenomena. Its methods and principles are of exceptional generality and universibility.

Again the deterministic methods of non-equilibrium thermodynamics are not sufficient or adequate to deal with system subjected to random noises, internal or external. A biological system whether an individual or a community is an open system and is always subjected to random disturbances of the surroundings. The dialectic of randomness has a great influence on growth and structure — building processes. So for approximate physico-mathematical modelling of the growth processes the non-equilibrium thermodynamic models must be supplemented by stochastic models to take account of non-equilibrium fluctuations due to random disturbance of the environment.

In view of the great importance and utility of non-equilibrium thermodynamic and stochastic modelling in the development of a rational physico-mathematical theory of biological processes [Lamprecht and Zotin, 1983], we, in the present thesis have investigated some basic problems of biological growths starting from the simplest and the primitive growth model of Malthus to relatively complicated problems of growth of biological communities with competition and co-operation. The methodologies
based on the general laws and techniques of non-equilibrium thermodynamic and stochastic differential equations has led to some important characteristic results of biological growth providing an extended and realistic picture of the phenomena of growth.
0.2 Relevant Thermodynamic and Stochastic Background

In this section we shall present some basic non-equilibrium thermodynamic and stochastic results relevant to our works.

0.21 Basic Concept of Non-equilibrium Thermodynamics

Non-equilibrium thermodynamics considers those systems which are not in thermodynamic equilibrium. These are non-homogeneous systems in which at least some of the intensive parameters are functions of time and position. The state of a substance forming such a system is specified by introducing the concept of local thermodynamic state.

A. The Postulate of local thermodynamic equilibrium

Postulate I: Although a thermodynamic system as a whole may not be in equilibrium, arbitrarily small elements of its volume are in local thermodynamic equilibrium and have state functions which depend on state parameters through the same relationship as in the case of equilibrium states in classical thermodynamics.

Postulate II: The generalized (thermodynamic) fluxes \( J_k \) are in first approximation depend on all the thermodynamic forces \( X_i \). The proportionality factors \( L_{ki} \) are called phenomenological
coefficients

\[ J_k = \sum_i L_{ki} \chi_i \]

where,

\[ L_{ki} = \left( \frac{\partial J_k}{\partial \chi_i} \right) \chi_i \quad (\chi_i = 0, 1 \neq 1) \]

Postulate III: With proper choice of thermodynamic forces and fluxes, the phenomenological co-efficients \( L_{ij} \) form a symmetric matrix, i.e.

\[ L_{ij} = L_{ji} \]

In qualitative terms if a force say "A" (corresponding, for example, to a temperature gradient) may influence a flux "B" (for example, a diffusion process), then force "B" (a concentration gradient) will also influence the flux "A" (the heat flow).

This relation follows strictly from the principle of microscopic reversibility. This symmetry relations hold for a system only for non-equilibrium state near the equilibrium. These are known as Onsager's reciprocal relation.

B. Entropy production and dissipation function

A change in the entropy of an elementary volume of substance,

\[ dS = d_{net} S + d_S \]

where \( d_S \) is the change of entropy due to the exchange with the
environment and $d_{\text{t}} S$ is the entropy production inside the elementary volume considered and by the second law of classical thermodynamics $d_{\text{t}} S \geq 0$ for irreversible processes and $d_{\text{t}} S = 0$ for reversible processes.

If entropy production is taken per unit time, the result is the entropy production rate

$$P = \frac{d_{\text{t}} S}{dt} \geq 0$$

If the system is characterized by many extensive variables (such as energy), the change of entropy with time is represented by the sum of the products of the generalized forces and generalized fluxes

$$\frac{dS}{dt} = \sum_{j} \chi_{j} J_{j}$$

According to the second law of thermodynamics

$$\frac{d_{\text{t}} S}{dt} = \int \sigma \, dV \geq 0$$

where, $\sigma = \sum_{j} \chi_{j}$ is called dissipation function, and a non-negative quantity. Here the forces and fluxes are referred to unit volume.
C. Steady state of thermodynamic (linear) system

In an open system there may be steady non-equilibrium state. In this state the entropy production inside the system is exactly compensated for by the flow of entropy to the outside, so that the total entropy of the system is not changed

\[ ds = d_s + d_s = 0 \]

Bertalanffly, one of the founders of the thermodynamics of open biological system, termed such a stationary state as the state of the flow equilibrium. It differs from true equilibrium by many specific features.

Thus the total entropy change per unit time can be decomposed into two parts

\[ \frac{ds}{dt} = \frac{d_s}{dt} + \frac{d_s}{dt} \]

where term \( P = \frac{d_s}{dt} \geq 0 \) is always positive resulting from the irreversible phenomena inside the system and is called the rate of entropy production.

For stationary state

\[ \frac{d_s}{dt} = - \frac{d_s}{dt} < 0 \]

The total entropy produced inside the system is compensated by entropy leaving the system across the boundary. For an
isolated system

\[
\frac{d s}{d t} = 0
\]

so in an isolated system non-equilibrium stationary state can not be attained.

Another characteristic difference of non-equilibrium stationary state from that of equilibrium stationary state lies in the fact that for equilibrium stationary state, we have

\[
P = \frac{d s}{d t} = 0
\]

so that,

\[
\frac{ds}{dt} = \frac{d s}{d t} = \frac{ds}{dt} = 0
\]

Whereas, for non-equilibrium stationary state, we have

\[
\frac{ds}{dt} = \frac{d s}{d t} + \frac{d s}{d t} = 0
\]

The entropy of an open system does not attain a maximum and the dissipation is different from zero.

D. Minimum entropy production theorem

As a single case let us consider an open system which exchange matter and energy with the environment but its mass remain unchanged. Then there is a flux of heat \( J \) in the system and the
flux of matter $J_M = 0$

Hence,

$$\sigma = J_T X_T + J_M X_M$$

and since,

$$J_T = \perp_{11} X_T + \perp_{12} X_M \neq 0$$

and

$$J_M = \perp_{21} X_T + \perp_{22} X_M = 0$$

and assuming force $X_T$ (say, a temperature difference) to be fixed, we have

$$\sigma = \perp_{11} X_T^2 + (\perp_{12} + \perp_{21}) X_T X_M + \perp_{22} X_M^2$$

Differentiating this expression with respect to $X_M$ at $X_T = \text{constant}$, we have

$$\frac{\partial \sigma}{\partial X_M} = 2 \perp_{22} X_T + (\perp_{12} + \perp_{21}) X_T$$

If the system is close to equilibrium and the Onsager relation is fulfilled, then $\perp_{12} = \perp_{21}$.

Therefore, $\frac{\partial \sigma}{\partial X_M} = 2 (\perp_{12} X_T + \perp_{22} X_M) = 2J_M = 0$

The second derivative

$$\frac{\partial^2 \sigma}{\partial X_M^2} = 2 \perp_{22} > 0$$
Thus, in the stationary state close to equilibrium the entropy production $\sigma$ is a minimum. This is Prigogine's minimum entropy production theorem.

Under the assumption of linear phenomenological relation and Onsager reciprocal relation the stationary state of an open system is characterized by the minimization of the entropy production $P(\sigma)$ provided the system is subject to constraints that fix the constant values of a certain set of generalized forces but the other generalized forces may vary (Vollensstein, 1983)

E. Stability Criteria

Examination of the stability of a system requires analysis of the relations describing the deviation of the state of the system from equilibrium. The rate of entropy production in an equilibrium state is zero, that is to say, is of second order magnitude relative to the deviation from equilibrium. The entropy $S$ of a non-equilibrium system is expressed by means of the entropy $S_{eq}$ of an equilibrium system up to second order terms,

$$S = S_{eq} + (\delta S) + \frac{1}{2} (\delta^2 S)_{eq}$$

Since the entropy $S_{eq}$ of an equilibrium system does not depend on time, differentiation of the above expression with respect to
time yields

\[ \frac{\partial s}{\partial t} = \frac{\partial (\delta s)_{eq}}{\partial t} + \frac{1}{2} \frac{\partial^2 (\delta s)_{eq}}{\partial t^2} \]

For an isolated system \( (\delta s)_{eq} = 0 \)

On the other hand, small equilibrium changes in the entropy of a diathermal system can be compensated by the entropy flow flowing across the system boundaries. A non-equilibrium process occurs if there is no such compensation. If the boundary conditions at the system boundaries are assumed to be constant and change in both the thermodynamic forces and fluxes at the system boundaries vanish then we get the thermodynamic criterion of evolution for states close to equilibrium states

\[ \frac{1}{2} (\delta^2 s)_{eq} = \int_0^t P \, dt = \int_0^t \, d s \leq 0 \]

This is the stability condition for thermodynamic equilibrium of a system. The stability of a system depends solely on the sign of the quantity \( (\delta^2 s)_{eq} \). It is not necessary however, to know the value of the entropy production due to deviations from equilibrium.

The stability condition for the system in a local form per unit volume can be written as

\( \delta^2 (\rho s) < 0 \)
or, per unit mass of substance

\[ \sigma^2_s = 0 \]

This condition can be written for the system as a whole by the second law of thermodynamics as

\[ \frac{\partial}{\partial t} (\sigma^2_s) \geq 0 \]

Whereas the stability conditions gives

\[ \sigma^2_s < 0 \]

This acts as Lyapunov function in the study of stability (Glansdorff and Prigogine, 1971).

### 0.22 Necessary Mathematical Tools

#### A. Langevin equation

The Langevin equation is a typical differential equation governing many physical phenomena subject to random forces. The simple-minded Langevin equation that turns up most often can be written in the form

\[ \frac{dx}{dt} = a(x, t) + b(x, t) \xi(t) \]

where \( x \) is the variable of interest, \( a(x, t) \) and \( b(x, t) \) are certain known functions and \( \xi(t) \) is the rapidly fluctuating random term. An idealised mathematical formulation of the concept of a "rapidly varying, highly irregular function" is that for \( t \neq t' \), \( \xi(t) \) and \( \xi(t') \) are statistically independent.
We also require \( \langle \xi(t) \rangle = 0 \), since any nonzero mean can be absorbed into the definition of the \( a(x, t) \), and thus require that
\[
\langle \xi(t) \xi(t') \rangle = \delta(t - t')
\]
which satisfies the requirement of no correlation at different times and furthermore, has the rather pathological result that \( \xi(t) \) has infinite variance. From a realistic point of view, no quantity can have such an infinite variance, but the concept of white noise as an idealisation of a realistic fluctuating signal does have some meaning. So \( \xi(t) \) is assumed here to be a Gaussian white noise.

Langevin's equation was the first example of the stochastic differential equation with a random term. Therefore, its solution is, in some sense, a random function. Each solution of Langevin's equation represents a different random trajectory and, using only rather simple properties of fluctuating term, measurable results can be derived (Gardiner, 1983).

The correlation of \( \xi(t) \) obey the conditions:

All correlations of \( \xi(t) \) of odd order vanish
\[
\langle \left[ \xi(t_1) \xi(t_2) \ldots \xi(t_{2n-1}) \right] \rangle = 0
\]

All correlation of \( \xi(t) \) of even order can be written as sums of products of second order correlation.
In particular let us consider the first order Langevin equation of the form

\[
\frac{dy}{dt} + a(t) y = x(t)
\]

where \(a(t)\) and \(x(t)\) are random variables satisfy the Gaussian white noise conditions.

Without loss of generality, let us assume that the mean value of \(x(t)\) and \(a(t)\) are zero. Then \(y\) can be written in the form

\[
y(t) = y_0 e^{-\beta(t)} + \int_0^t x(t') e^{-\beta(t)} + \beta(t') \, dt',
\]

\[
\beta(t) = \int_0^t a(t') \, dt'
\]

where \(y(t)\) takes the value \(y_0\) at \(t = 0\).

The mean value of \(y(t + \Delta) - y(t)\), subject to the condition \(y(t) = y, \) is given by

\[
\left[ y(t + \Delta) - y(t) / y \right]
\]
\[ \exp \left( \int_t^{t + \Delta} a(t') \, dt' \right) - 1 \right), \\
= y \, D_{11} \, \Delta + O(\Delta) \\
\]
where \( D_{11} \) is connected to the second-order correlation of \( a(t) \) by

\[ \begin{bmatrix} a(t_1) & a(t_2) \end{bmatrix} \cdot 2 \, D_{11} \, \delta(t_2 - t_1) \]

The conditional mean square value of \( y(t + \Delta) - y(t) \) is given by

\[ \left( \frac{y(t + \Delta) - y(t)}{y} \right)^2 \]

\[ = (D_{11} \, y^2 - 2 \, D_{10} \, y + D_{00} \, \Delta + O(\Delta)) \]

where \( D_{10} \) and \( D_{00} \) are defined by

\[ \begin{bmatrix} a(t_1) \times (t_2) \end{bmatrix} \cdot 2 \, D_{10} \, \delta(t_2 - t_1) \]

\[ \begin{bmatrix} x(t_1) \times (t_2) \end{bmatrix} \cdot 2 \, D_{00} \, \delta(t_2 - t_1) \]

If \( x(t) \) and \( a(t) \) are statistically independent, then \( D_{10} = 0 \). All higher order moments of \( y(t + \Delta) - y(t) \) vanish faster than \( \Delta \) (Srinivasan and Vasudevan, 1971).
B. Fokker–Planck equation

We now obtain the Fokker–Planck equation for the probability frequency function $\pi(y / y_o, t)$, starting from the Langevin equation. If we assume that the process $y(t)$ is markovian and homogeneous with respect to $t$, we have the Chapman–Kolmogorov equation

$$\pi(y / y_o; t + \Delta) = \int dz \pi(z / y_o; t) \cdot \pi(y / z; \Delta).$$

Next we consider the integral

$$\int dy P(y) \frac{\partial}{\partial t} \pi(y / y_o; t)$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} P(y) \left[ \pi(y / y_o; t + \Delta) - \pi(y / y_o; t) \right] dy$$

where $P(y)$ is an arbitrary smooth function of $y$ going to zero as $u \to \pm \infty$. Substituting the Kolmogorov equation in the right-hand side of the above equation and expanding $\pi(z / y_o; t)$ about $y$, we obtain

$$\frac{\partial}{\partial t} \pi(y / y_o; t) = - \frac{\partial}{\partial y} (\pi(y / y_o; t) \beta(y))$$

$$+ D \frac{\partial^2}{\partial y^2} \pi(y / y_o; t)$$

an equation generally known as the Fokker–Planck equation (Srinivasan and Vasudevan, 1971)
0.3 Relevant Previous Works

In this section we shall describe some previous works which are relevant to our present investigation.

0.31 Biological Growth and Development

The first attempt to the thermodynamic theory of biological growth was made by Prigogine and Weame (1946). The Prigogine - Weame's theory was later developed by Zotin and Zotina (1967, 1978, 1983-85). The present section is a short review of their works on the thermodynamics of biological growth developed by Zotin, Zotina (1967), Lurie - Wagenberg (1979).

There are two basic problems in connection with the applicability of the non-equilibrium thermodynamics of irreversible processes to the problems of biological growth and development. The first is the validity of the linear phenomenological relation between thermodynamic fluxes and forces and the second is the applicability of Prigogine's principle of minimum entropy production for the steady non-equilibrium state in the context of biological growth and development.

A. Validity of linear phenomenological relations

We know that the thermodynamics of irreversible processes is concerned with phenomena occurring close to the steady state. The
linearity of the relations between thermodynamic fluxes and forces in the process under observation is the basic limitation for phenomena which can be embraced by the thermodynamics of irreversible processes (Prigogine, 1947; de Groot, 1955) i.e. the following relation must hold

$$I_j = \sum_k L_{kj} \chi_k$$  \hspace{1cm} (0.1)

Where $L_{kj}$ represents the phenomenological coefficients, $I_j$ thermodynamic fluxes and $\chi_k$ thermodynamic forces. If the flux depends only on one force, then

$$I_j = L_{j1} \chi_1$$ \hspace{1cm} (0.2)

Many different formulae and equations have been proposed to describe animal growth. We, in particular, take the following equation of growth (Zotin and Zotina, 1967).

$$\frac{1}{P} \frac{dp}{dt} = \gamma(T_m - t)$$ \hspace{1cm} (0.3)

Where $P$ represents the animal's weight, $t$, animal's age, $T_m$, maximum age, $\gamma$, growth constant.

It is clear from this equation that the specific current weight (flux) of an animal ($I/P$) ($dp/dt$) is linearly connected to the force, which in this case is proportionate to the difference ($T_m - t$). Equation (0.3) is better confirmed by experimental data than other equations of growth (Zotin and Zotina, 1967).
But even if one takes Robertson's equation to be more correct (Bordy, 1945).

$$\frac{1}{P} \frac{dp}{dt} = k(P_m - P) \quad \ldots \ldots \ldots (0.4)$$

or any other equation of growth based on the rate of growth being proportionate to the gradient of forces causing growth, then it can be concluded that the linear relations between the current weight (flux) and forces remains same. Thus, for animal growth a condition of linear relations between fluxes and forces is fulfilled, i.e. the basic requirement of processes - described by the thermodynamics of irreversible processes is fulfilled.

**B. Validity of Prigogine's minimum entropy production principle**

According to Prigogine - Weame theory during the steady state the rate of entropy production in an open system is constant in time and minimal in quantity. The concept of steady state is associated with the concept of basal metabolic rate. The basal metabolic rate, which is measured from the rate of heat production or respiration per unit of animal's weight, is determined as the animal's minimum metabolic rate during a state of rest. Consequently, the basal metabolic rate is characterized by the animal's minimum rate of heat production or minimum rate of entropy production. The estimate of the rate of specific heat production in the embryos of the trout according to the data of
Smith (1947) reveals that this assertion is true in relation to the embryonic development of fish. Similarly, we can find that a reduction in the rate of specific heat production in embryos takes place during the embryonic development of birds from the earliest stages (Romijn and Lochhorst, Needham, 1931). Similar conclusions can be reached if one examines data on the respiration intensity in the embryos of mammals (Bordy, 1945). Thus the Prigogine - Weame theory is applicable to the embryonic development of animals.

On the growth and aging of animals and men, the conclusion of this theory can be justified. The data on the change in the basal metabolic rate during the growth of calves, rabbits and in during the growth of man, a continuous reduction in the rate of specific heat production can be observed (Tomme and Hm, 1938).

With some animals, for example fish, growth continues throughout life it is therefore difficult to differentiate the processes of growth and ageing. In this respect mammals and birds are more interesting since their growth continues only into the first third or half of life, and then ceases. Data from biological literature seems to indicate that the processes of aging in animals and men are accompanied by a decrease in the rate of specific heat production. In man growth ceases at about 20 to 25 years of age but the basal metabolic rate in man—beginning with the age of 25—decreases every ten years by 3.0 to 7.5% (Berkovitch, 1964). A significant reduction in the
basal metabolic rate continues even in the late stages of aging (Lewis, 1938; Shock, 1955; Kleiber, 1961; Berkovitch, 1964; and others). All these observations show that the Prigogine - Weame theory is correct even in connection with the processes of aging in organisms.

Zotin, however, (1972) observed the increase in the specific heat production during the early stages of embryo genesis in a typical biological (specific) thermogram. This feature has been the chief source of difficulty in the way of the Prigogine - Weame theory. Various suggestions have been made as to how this difficulty might be circumvented. Zotin has argued that the assumption of fixed external constraints upon which Prigogine's minimum entropy production theorem is based, must be relaxed during the embryogenetic stage. A more drastic suggestion discussed by Hiernaux and Babloyantz (1976) is that the assumption of linear relations between thermodynamic fluxes and forces must be abandoned during the early stage of development. In this view, the embryogenetic stage is characterized by non-linear thermodynamics of systems far from equilibrium (Glansdorff and Prigogine, 1971). Such non-linear systems can evolve through a sequence of instability thresholds thereby undergoing transitions to states of increasing entropy production. Lurie and Wagenberg (1979) stated that neither linearity nor fixed constraints need abandoned to account for the observed increase in dissipation during embryogenesis. The
thermogram data of Zotin is seen to be perfectly compatible with the validity of the minimum entropy production theorem once it is realized that the identification (to within a sign) of the heat dissipation with the entropy production may become a poor approximation precisely during embryogenesis.

There is a complicated question: whether animals reach the steady state in the thermodynamic sense of the word during the process of aging. During the process of aging a change occurs in the relationship between the active cellular mass and the more passive parts such as the skeleton, adipose tissue, etc. (Kleiber, 1947). Consequently, to determine the change in the basal metabolic rate during the process of aging in animals, we must calculate the heat production per unit of metabolic body size. If we use Kleiber's method, the calculation turns out that the basal metabolic rate in women decreases as old age advances (Kleiber, 1961). On the basis of these calculations one can accept that animals and man do not reach the steady state until the latest stages of aging - the steady state being characterized by constancy in the rate of entropy production. This conclusion indicates that for living organisms attainment of the steady state means death.

We shall now examine the question of the rate of heat production during the processes of regeneration and healing of wounds. The quantity of heat production during regeneration is determined not only by the oxidative processes but also by the
level of aerobic glycolysis, the intensity of which charges significantly during the process of regeneration. During regeneration glycogen at first accumulates in the tissues and then quickly disappears because of the intensification of aerobic glycolysis. During the process of regeneration, respiration and aerobic glycolysis change significantly at first they increase, attaining maximum and then gradually slacken off as the extremities grow. Therefore, a gradual reduction in respiration intensity and aerobic glycolysis and hence in the rate of heat production takes place during the second half of regeneration in accordance to the Prigogine - Weame theory.

C. Dependence of the rate of heat production on the animal's weight during growth

As a result of numerous investigation (Brody, 1945; Kleiber, 1961) it has been ascertained that a relationship between the rate of respiration and body weight in animals of various species exist.

\[ \bar{\sigma}_{oz} = a \cdot P^k \] ........(0.5)

Where \( \bar{\sigma}_{oz} \) represents respiration rate; \( P \), animal's weight and \( a \) and \( k \) are constants. For the specific respiration rate we have

\[ \bar{q}_{oz} = \frac{\bar{\sigma}_{oz}}{P} = a \cdot P^{k-1} \] ........(0.6)
Since the constant $k$ is less than unity, then $b = k - 1 < 0$ and we can write (0.6) in the form

$$\log q = \log a - b \log p \quad \ldots \ldots (0.7)$$

The equations (0.5), (0.6) and (0.7) are valid not only for animals of various species but also for growing animals of the same species.

According to the Prigogine-Weame theory, the relation

$$\frac{ds}{dt} = \frac{1}{\overline{U}} \sum_k I_k \chi_k$$

is applicable to growing organization. Let us assume that the rate of heat production $\overline{q}$ is proportionate to the rate of entropy production $\overline{U} = T \left( \frac{ds}{dt} \right)$ and represent the rate of specific heat production by $\overline{q} = \overline{q}/\overline{U}$.

The total rate of specific heat production is composed of specific heat production connected with growth ($\overline{q}_g$) and heat production unconnected with growth ($\overline{q}_o$).

$$\overline{q} = \overline{q}_g + \overline{q}_o \quad \ldots \ldots (0.8)$$
Therefore, from the relation \( \frac{ds}{dt} = \frac{1}{T} \sum_{k} I_k \dot{x}_k \) we obtain

\[
\overline{q}_g = \frac{1}{T} \sum_{k} I_k \dot{x}_k
\] ........ (0.9)

Substituting the value of (0.9) into (0.8) we obtain

\[
\overline{q} = \frac{1}{T} \sum_{k} I_k \dot{x}_k + \overline{q}_0
\]

\[
= \frac{1}{T} \sum_{k} I_k \dot{x}_k + \overline{q}_0
\] ........ (0.10)

where \( T = \frac{1}{k_4} \) = constant

Proceeding from the equation (0.3) we accept that the current weight (flux) of a growing animal is equal to

\[
I_j = \frac{1}{P} \frac{dP}{dt}
\]

and the force proportionate to the time \( x_j = T_m - t \), then from (0.3) and (0.10)

\[
\overline{q} = k_2 (T_m - t)^2 + \overline{q}_0
\] ........ (0.11)

we now find the solution of the equation (0.3)

\[
\frac{1}{P} \frac{dP}{dt} = k(T_m - t)
\]

or \( \frac{dP}{dt} = k(T_m - t) \)dt
Integrating, we get

$$\log p = - k \frac{(T - t)^2}{2} + C$$

where $C$ is the constant of integration.

When $t = 0$, $T = 0$ and $P = P_m$ (say)

"". log $P_m = C$

"". log $P - log P_m = - k \frac{(T_m - t)^2}{2}$

or $log \frac{P}{P_m} = - k \frac{(T_m - t)^2}{2}$

"". $P = P_m \exp \left( - \frac{1}{2} \frac{(T_m - t)^2}{2} \right)$

from (0.12) we get

$$\frac{(T_m - t)^2}{2} = \frac{2}{k} \left( log \frac{P}{P_m} - log P \right)$$

Substituting this in (0.11) we obtain

$$\bar{q} = \frac{2k}{l} \left( log \frac{P}{P_m} - log P \right) + \bar{q}_o$$

$$= \frac{2k}{l} log \frac{P_m}{P} + \bar{q}_o - \frac{2k}{l} log P$$
\[ q = \log a' - b' \log P \]

where

\[ \log a' = \frac{2k}{k} P + q_0 \]

and \( b' = \frac{2k}{z} \)

Thus proceeding from the growth equation (0.3) and Prigogine - Weame theory, we can conclude that, for a growing organism the rate of specific heat production is linearly related to the logarithm of weight.

If the heat production is determined by caloriemetric methods then equation (0.15) is valid, but if the heat production is determined by the indirect method of measuring the respiration of animals the equation (0.7) is correct.

The Prigogine - Weame thermodynamic theory not only points to the more correct one, but also enables us to see ways of uniting the different equations in a single theory of animal growth. And, in fact, according to equation \( \frac{ds}{dt} = \frac{1}{k} \sum k_k \), the rate of entropy production is determined by the product of fluxes and forces. It is obvious that, in particular, the process of differentiation is closely linked to the processes of growth. Proceeding from (0.1) we can express the interaction of these two processes during growth in the form of two phenomenological equations.
Where \( I_1 = \text{Current weight (flux)}, \)

- \( X_1 = \text{Thermodynamic force determining growth} \)
- \( I_2 = \text{Differentiation flux} \)
- \( X_2 = \text{Thermodynamic force determining differentiation} \)

and \( L_{11}, L_{12}, L_{21}, L_{22} \) are phenomenological coefficients.

It may be supposed that the growth equation most widely used corresponds to the phenomenological law of growth (Vinberg, 1966), but equation (0.3) corresponds to the phenomenological law of differentiation. In this case the thermodynamic force which determines growth will be proportionate to the difference between the assimilation and dissimilation in the growth organism and the thermodynamic force of differentiation proportionate to the gradient of biological time.

0.32 Stochastic models of Logistic and Gompertzian growth

In this section we shall review two of the most primitive but very important types of population growths due to Verhulst and Gompertz respectively. The discussion is based on the paper of Goel et al (1971).
Let us consider the Verhulst equation for the population growth of a single species with saturation level $\theta$.

\[
\frac{dN}{dt} = kN \left( \theta - N \right) / \theta
\]

.........(0.16a)

The solution of this equation is

\[
f(t) = f(0) / \left[ f(0) + (1 - f(0)) \exp(-kt) \right]
\]

with $f(t) = N(t) / \theta$

.........(0.16b)

.........(0.16c)

It is well known that the time variation of the population of many countries can be fitted quite well by equation (0.16) (Pearl, 1924 and Montroll, 1968). An excellent guide to the literature of population growth and its theory has been given by Glass (1967). The Malthusian exponentiation of population growth is just the $\theta \rightarrow \infty$ limit of (0.16a)

\[
\frac{dN}{dt} = kN \quad \text{or} \quad N(t) = N(0) \exp(kt)
\]

.........(0.16d)

There are differential equations beside (0.16a) which lead to population saturation. One is the equation of Gompertz (1825)

\[
\frac{dN}{dt} = -kN \log \left( \frac{N}{\theta} \right), \quad 0 < N \leq \theta
\]

.........(0.17a)

which was invented for the investigation of mortality rates rather than population growth. It was, however, used for fitting growth statistics.
Equation (0.17a) implies that
\[ \log \log \left[ \frac{N(t)}{\Theta} \right] - \log \log \left[ \frac{N(0)}{\Theta} \right] = -kT \]
and, therefore, that
\[ N(t) = \Theta \exp \left[ \exp \left(-kT \left( \log \frac{N(0)}{\Theta} \right) \right) \right] \]
which approaches the saturation level \( \Theta \) as \( t \to \infty \). The Gompertz form apparently became unfashionable because of computational difficulties, in precomputer days, in using the method of least squares to fit observational data (Davis, 1941).

Generally one might consider arbitrary saturation including functions \( G(N/\Theta) \) with the property
\[ G(x) \to 0 \text{ as } x \to 1 \]
and which lead to the differential equation
\[ \frac{dN}{dt} = kNG(N/\Theta) \]
Often \( G(x) \to 1 \text{ as } x \to 0 \) as in the Verhulst case in which \( G(x) = 1 - x \).

Now apart from other specific species of the set of \( n \) species if the species of interest is influenced by species, say bacteria, (if the \( n \) species are larger animal) and other parasites, plant life which varies in intensity from season to season, and unspecified migrating species, etc. which affect the
population of \( n \) species in a random way then the basic equations for population growth might be considered to be of the form

\[
\frac{dN_j}{dt} = k_j N_j G \left( \frac{N_j}{\Theta_j} \right) + N_j \left[ U_j(t) + \beta_j \sum_{i=1}^{N} a_{ij} N_i \right] \quad \ldots \quad (0.19)
\]

where \( G(x) \) is a saturation including term and \( U_j(t) \) represents random unspecified influences. When the number of specified related species, \( n \), is large and each species interacts with a fairly large number of others, \( a_{jk} \)'s can have both signs to appear in (0.19) for most \( j \). Since the population of each of the species \( N_1, N_2, \ldots \) varies with the time when each is influenced by random unspecified species, the sum in (0.19) might also be considered to be a random function of time. The combination of \( U_j(t) \) and the sum might then be considered as a random function of time, \( F_j(t) \). This consideration would lead to the species being coupled in only a random way. Since only terms with index \( j \) will appear in the resulting equation, \( j \) is suppressed and the equation (0.19) becomes

\[
\frac{dN_j}{dt} = k_j N_j G \left( \frac{N_j}{\Theta_j} \right) + N_j F(t) \quad \ldots \quad (0.20)
\]

where \( F(t) \) is considered to be a random function. It is also assumed that the average value of \( F(t) \) vanishes i.e.

\[
\langle F(t) \rangle = 0 \quad \ldots \quad (0.21)
\]
The Fokker-Planck equation for this process follows from the standard hypothesis made in the theory of Brownian motion and random processes that

\[
\langle F(t_1) F(t_2) \rangle = \sigma \delta(t_1 - t_2) \quad \text{........(0.22)}
\]

Equation (0.20) also represents the growth of a species (in the absence of other species) in a random environment or equivalently the growth when growth coefficient is \( \Gamma + F(t) \) where \( \Gamma \) is the average growth coefficient.

Instead of investigating the Fokker-Planck equation for the variable \( N \), let us investigate the variable \( V \) defined by

\[
V = \log \left( \frac{N}{\Theta} \right) \quad \text{........(0.23)}
\]

Thus the basic equation for deriving the Fokker-Planck equation is

\[
\frac{dN}{dt} = kG(\exp(V)) + F(t) \quad \text{........(0.24)}
\]

In a short time \( \Delta t \), the variation in \( V \) is

\[
\Delta V = kG(\exp(V)) \Delta t + \int_{t}^{t+\Delta t} F(t_1) \ dt_1 + O(\Delta t^2) \quad \text{........(0.25)}
\]

Then if (0.21) is valid, we find

\[
A = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = kG(\exp(V)) \quad \text{........(0.26)}
\]
while

\[
\langle (\Delta V)^2 \rangle = k^2 \left[ G \left( \exp \left( \frac{v}{2} \right) \right]^2 (\Delta t)^2 + \int_t^{t+\Delta t} \int_t^{t+\Delta t} F(t_1) F(t_2) \right. \\
\left. \frac{dt_1}{t_2} + \Delta t \right] + U (\Delta t)^2 \quad \ldots \ldots (0.27)
\]

If the classical Brownian motion postulate (0.22) is made, then

\[
B = \frac{\Delta t}{\Delta t \to 0} \left. \langle \Delta V^2 \rangle \right|_{\Delta t} = \sigma^2 \quad \ldots \ldots (0.28)
\]

If one assumes that \( F(t) \) is generated by a Gaussian random process, the standard form for the Fokker-Plank equation for the probability that \( V(t) \) has a value \( V \) at time \( t \) is (Wang and Uhlenbeck, 1930)

\[
\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial V} (AP) + \frac{1}{2} \frac{\partial^2}{\partial V^2} (BP) \quad \ldots \ldots (0.29)
\]

where \( A \) is given by (0.26), this becomes

\[
\frac{\partial \rho}{\partial t} = -k \frac{\partial}{\partial V} (AP) [PG(\exp (v))] + \frac{1}{2} \sigma^2 \frac{\partial^2 \rho}{\partial V^2} \quad \ldots \ldots (0.30)
\]

It may be noted that the Fokker-Planck equation for the variable \( N \) derived by using (0.20) when transformed into the variable \( V \) is different from (0.30). The steady state distribution function for this case is obtained by setting

\[
\frac{\partial \rho}{\partial t} = 0 \quad \text{and letting} \]

\[
\rho = p
\]
\[ U = k \int_{0}^{v} G \left( \exp (v) \right) dv \quad \text{with} \quad \frac{\partial U}{\partial v} = kG \left( \exp (v) \right) \] 

Then (15) becomes:

\[ \left[ \frac{\partial}{\partial v} \left( -P \left( \frac{\partial U}{\partial v} \right) + \frac{1}{2} \sigma^2 \left( \frac{\partial P}{\partial v} \right) \right) \right] = 0 \quad \ldots \ldots .(0.32) \]

from which see that

\[ \frac{\partial}{\partial v} \left[ \exp \left( \frac{2v}{\sigma^2} \right) \frac{\partial}{\partial v} \left[ P \exp \left( -\frac{2v}{\sigma^2} \right) \right] \right] = 0 \quad \ldots \ldots .(0.33) \]

The solution of the equation is, with \( P_0 \) being a normalizing constant,

\[ P = P_0 \exp \left( \frac{2U}{\sigma^2} \right) = P_0 \exp \left[ 2\sigma^{-2} \int_{0}^{v} G \left( \exp (v) \right) dv \right] \ldots .(0.34) \]

In the Gompertz and Verhulst cases, this distribution becomes, respectively

\[ \text{Gompertz} : P = P_0 \exp \left( -k\sigma^2 / \sigma^2 \right) \quad \ldots \ldots .(0.35a) \]

\[ \text{Verhulst} : P = P_0 \exp \left[ 2k \left( v - \exp (v) / \sigma^2 \right) \right] \ldots \ldots .(0.35b) \]

The distribution functions in terms of the population variable are
Gompertz: \( P(N, \alpha) = \left(\frac{k}{2\pi \sigma^2}\right)^{1/2} N^{-1} \)

\[
\exp \left[ -k \left( \log \left( \frac{N}{\theta} \right) \right)^2 / 2\sigma^2 \right]
\]

...........(0.36)

Verhulst: \( P(N, \alpha) = \left( \frac{2\kappa N}{\theta \sigma^2} \right)^{-1} \left( 2 \kappa N / \theta \sigma^2 \right)^{2k/\sigma^2} \)

\[
\exp \left( \frac{-2Nk}{\theta \sigma^2} \right)
\]

.............(0.37)

Where \( \Gamma(x) \) is the classical gamma function

Equation (0.35b) was first derived by Leigh (1969).
In this section we shall discuss the work of Mauersberger, Feistel and Ebeling (1981). Thermodynamical methods seem to be fruitful for the investigation of the extremely complicated process in ecosystems. The present investigation is based on the simple "model of the ideal solution" which has been developed independently by Mauersberger, Feistel and Ebeling. The application of the model to evolutionary systems is based on the following assumptions:

1. The evolutionary species are embedded in a surrounding (solute) which is in thermodynamical equilibrium at the temperature $T$.

2. The solution is able to penetrate freely through the surface of the ecosystem. The pressure in the interior is $p + \gamma$ and outside $p$, i.e. $\gamma$ is a kind of "osmotic pressure".

3. The time behaviour of the number variable $N_i$ of the species $i$ is characterized by dynamical equations

$$\frac{d}{dt} N_i(t) = f(N_1, N_2, \ldots, N_n), \quad i = 1, 2, \ldots, n$$

which presumably are of a Lotka - Volterra form.
(4) The internal energy is additive with respect to the population numbers of the ecological species.

We define now the model of ideal solutions.

Let us consider an arbitrary extensive thermodynamic function

\[ Z(p, T, N_1, N_2, \ldots, N_n) \]

depending on pressure, temperature and the number variables \( N_1, \ldots, N_n \).

We define now the evolutionary part \( \tilde{Z} \) by

\[ \tilde{Z} = Z(p, T, N_1, N_2, \ldots, N_n) - Z(p, T, 0, \ldots, 0) \]

\[ \ldots \ldots (0.41) \]

The Gibbs equation is assumed to be

\[ d\tilde{U} = T d\tilde{S} - p dV + \sum \mu_l dN_l \]

\[ \ldots \ldots (0.42) \]

where \( \tilde{U} \) denotes the evolutionary part of the internal energy and \( \tilde{S} \) denotes the entropy.

Extending Planck's assumption to ecosystems, let us assume a linear relation between \( \tilde{U} \) and the \( N_l \)

\[ \tilde{U} = \sum N_l u_l (p, T) \]

\[ \ldots \ldots (0.43) \]

and the entropy to be an ideal mixture entropy

\[ \tilde{S} = \sum N_l \left[ s_l (p, T) - \psi \ln \left( \frac{N_l}{V} \right) \right] \]

\[ \ldots \ldots (0.44) \]
The $s_i$ are specific quantities characterizing the different species thermodynamically. Both relations hold for ideal systems even in situations very far from equilibrium.

Ebeling and Feistel (1982) showed that all thermodynamic potentials may be derived, especially the free energy and the chemical potential form (0.42) and (0.43)

$$F = T \sum q_i s_i \quad \text{(0.45)}$$

$$\mu_i = \mu_i^{(0)} (p, T) + k_b T \ln (N_i / V)$$

By use of relative frequencies (mole fractions)

$$q_i = \frac{N_i}{N} \quad N = \sum N_i, \quad i = 1, 2, \ldots, n \quad \text{(0.46)}$$

we find the entropy per individual

$$\frac{\tilde{S}}{N} = \sum q_i s_i - k_b \sum q_i \ln q_i - k_b \ln (N / V) \quad \text{(0.47)}$$

which is a more general expression than in common ecology (Stugren, 1978). It has a maximum value at (Mauersberger, 1981)
The maximum itself increases logarithmically with the ratio \( n / N \) of the numbers of species and individuals. Note that this maximum is reached at an "evolutionary equilibrium" which is thermodynamically a situation very far from equilibrium. Without input of free energy (free enthalpy) the system will tend necessarily to equilibrium for example, to a canonical or grand canonical distribution i.e. the mean occupation numbers are given by

\[
q_{t} = \left[ \sum_{j=1}^{n} \exp \left( \frac{s_{j} - s}{k_{B}} \right) \right]^{-1} \approx n^{-1} + 0(s_{t} - s) 
\]

\[\text{...........(0.48)}\]

The last inequality expresses the fact that in thermodynamic equilibrium none of the species has a real chance to appear in one exemplar due to the practical impossibility of its creation by thermodynamic fluctuations in a molecular system. In other words the occurrence of living beings in equilibrium is an extremely improbable event. If \( \mu_{t}^{oq} \) denotes the chemical constitution of the species we may write

\[
N_{t} \sim \exp \left( - \frac{\mu_{t}}{k_{B}} \right); \quad N_{t}^{oq} = 1 
\]

\[\text{...........(0.49)}\]

\[
\mu_{t} = \mu_{t}^{oq} + k_{B} T \ln \left( \frac{N_{t}}{N_{t}^{oq}} \right) 
\]

\[\text{...........(0.50)}\]
With (0.49) we get the inequalities

\[ \mu \leq T \quad \text{or} \quad \tilde{F} > \tilde{F}_{eq} \]  

\[ \ldots (0.51) \]

for any realistic ecosystem with \( N \gg 1 \).

From the point of view of thermodynamics, evolutionary systems are extremely far from equilibrium and are held in this state by a permanent influx of free energy.

Let us discuss now the second law and the role and character of irreversible processes. Evolutionary systems are open systems, therefore, changes of the thermodynamical functions consist of internal and external parts

\[ \begin{align*}
\text{d}U &= \text{d}U + \text{d}U_i, \quad \text{d}s = \text{d}s + \text{d}s_i \\
\text{d}F &= \text{d}F + \text{d}F_i
\end{align*} \]  

\[ \ldots (0.52) \]

The basic law of thermodynamics read for irreversible process

\[ \begin{align*}
\text{d}U &= 0 ; \quad \text{d}s > 0 \\
\ldots (0.53)
\end{align*} \]

For isothermal processes which we consider here it follows

\[ \text{d}F \neq 0 \]  

\[ \ldots (0.54) \]
For a functioning ecosystem the entropy cannot increase i.e.
\[ dS \leq 0 ; ( - dS ) \geq dS > 0 \] \hspace{1cm} (0.55)

or for isothermal systems
\[ dF \geq T \, dS > 0 \] \hspace{1cm} (0.56)

This is the second law for a living ecosystem. In other words a living ecosystem exists under the condition only that the import of free energy exceeds the entropy production multiplied with the temperature. At a steady state internal and external contributions cancel so that

\[ \begin{align*}
    dU &= dU = dU = 0 \\
    dF &= -dF = T \, dS = -T \, dS > 0
\end{align*} \] \hspace{1cm} (0.57)

In our model energy is stored exclusively within the organism, hence every change or transfer for energy is determined by the ecological evolution equations (0.40). However, entropy is produced even if there are no changes in the population numbers.

\[ P = \frac{dS}{dt} = \sum_j N_j p_j + \tilde{p} \] \hspace{1cm} (0.58)
where $p_j$ is the entropy production per individual of the species $j$ and where $\tilde{P}$ is the ecological entropy production which is connected with changes of the $N_j$:

\[
\tilde{P} = \frac{dS}{dt} = - \sum_j \frac{\mu_j}{T} \frac{dN_j}{dt} = \sum_j \chi_j I_j .
\]

\[
\chi_j = - \frac{\mu_j}{T} , \quad I_j = \frac{dN_j}{dt}.
\]

\[\ldots\ldots(0.59)\]

Linear relations for the vicinity of the equilibrium state are of no value for ecosystems. Instead of those let us consider the neighbourhood of a stationary state $N_1^0, N_2^0, \ldots$. Then the entropy is given by

\[
S = S_0 + \delta S + \frac{1}{2} \delta^2 S + \ldots .
\]

\[
\delta S = - \sum l \delta N_l \left( \frac{\mu_l^0}{T} \right) .
\]

\[\ldots\ldots(0.60)\]

\[
\delta^2 S = - k_a \sum \left( \delta N_l \right)^2 / N_l^0 .
\]

$S_0$ contains the ecological and nonecological entropy at the steady state. Evidently the partial change $\delta_x P$ due to variation of the forces and $\delta^2 S$ satisfy the Glansdorff - Prigogine relations.
Let us apply now these concepts to Lotka–Volterra systems. A rather general model of ecosystems is given by the dynamic equations

\[
\frac{d}{dt} \delta P = \delta I, \quad \delta I = \frac{d}{dt} \left( \frac{1}{2} \delta^2 S \right) + \mathcal{O}(\delta^3) \quad \cdots \cdots (0.61)
\]

Here, \( \delta^2 \to 0 \) is a very small term standing for the random creation of individuals by mutations etc. \( \lambda \to 0 \) is the reciprocal natural living time, \( p \geq 0 \) a pumping rate (e.g., autotrophic multiplication by photosynthesis) and \( \gamma_{ij} \) the community matrix of interaction between species. The balance of internal energy per species

\[
U_i = N_i u_i \quad \cdots \cdots (0.63)
\]

is governed by

\[
\frac{d}{dt} U_i = \delta_i u_i + U_i \left( p_i - \lambda_i - \sum_j \gamma_{ij} N_i N_j \right) \quad \cdots \cdots (0.64)
\]

The matrices are defined by

\[
\begin{align*}
\delta_i &= \frac{\gamma_{ij}}{u_j}, \\
\delta_i^S &= \frac{1}{\varepsilon} \left( \delta_i + b_{ij} \right), \\
\delta_i^A &= \frac{1}{\varepsilon} \left( \delta_i - b_{ij} \right)
\end{align*}
\]
The terms $\delta u$ and $p U$ describe the gain of internal energy by import. The next two terms correspond to loss of internal energy by export or degeneration (entropy production). The last term is the exchange of energy between the species that cancels in the balance of the whole system. Obviously, the symmetric part $b_{ij}^S$ has to be a non-negative matrix to describe dissipation. Moreover we shall assume that $b_{ij}^S$ is positive semidefinite. Every real antisymmetric matrix can be interpreted as a so-called connection matrix to a corresponding digraph. This graph to $b_{ij}^A$ gives a picture of the energy flow through the ecosystems; it is called "trophic system". For all species at the first level the condition $p_l \cdot \lambda_l$ is necessary for their existence. Finally, a positive semidefinite matrix $b$ ensures the boundedness of the whole population. We may express the distance to the steady state by a relative affinity:

$$y_l = -\delta x_l = \beta(\mu_l - \mu_l^C) = \ln \left(\frac{N_l}{N_l^0}\right) \quad \ldots \ldots \quad (0.66)$$

![Trophic system](image)

**Fig.** Trophic system: Nodes correspond to species and arrows to energetic flow. $b_{ij}^A > 0.$
where the index zero stands again for steady state values of the variables. From the evolution equation (0.62) follow flux – force relations of exponential form

\[
\dot{\gamma}_i = - \sum_j \Omega_{ij} N_j^C \left[ \exp (\gamma_j) - 1 \right]
\]

\[
\Omega_{ij} = \gamma_{ij} + \delta_i \delta_{ij} \exp (- \gamma_i) (N_j^C)^2
\]

which suggest to introduce a kinetic potential \( L \). An identical reformulation of equation (0.67) yields

\[
\dot{\gamma}_i = - \sum_j \Omega_{ij} \frac{\partial L}{\partial \gamma_j} \quad L = \sum_i N_i^C \left[ e^{\gamma_i} - 1 - \gamma_i \right]
\]

We see that \( L \) is a Lyapunov function

\[
\frac{dL}{dt} = - \sum_j \Omega_{ij} \frac{\partial L}{\partial \gamma_j} \frac{\partial L}{\partial \gamma_j} \leq 0 \quad \ldots \ldots \ldots (0.68)
\]

because of the positive definiteness of \( \Omega_{ij} \). We note that \( \gamma_{ij} \) is semidefinite if \( b_{ij} \) is for \( u_i > 0 \). Thus \( \Omega_{ij} \) is positive definite if \( \gamma_{ij} \) is positive semidefinite and all \( \delta_i > 0 \). We see that \( L \) always decreases with time and approaches zero. It is interesting to note that for small distances from the steady state the kinetic potential corresponds to the excess entropy
Therefore the evolution criterion (0.68) reduces in the vicinity of the steady state to the famous Glansdorff – Prigogine criterion

\[ \frac{d}{dt} \delta^2 S \leq 0 \]

Let us now summarize these results. Ecosystems can be treated as ideal solutions within a certain approximation. Especially internal energy and entropy are quantities which are defined also in situations very far from equilibrium. All other thermodynamic potentials can then be deduced formally. The main difference from common solutions is the wide distance of ecosystems from equilibrium, expressed in inequalities as \( \mu_l \gg kT \). First and second law of thermodynamics and Glansdorff – Prigogine relations can be stated. They lead to restrictions of the physically possible forms of evolution equations, as demonstrated for Lotka-Volterra systems. For such systems a dynamic potential is given, which is in the vicinity of the steady states identical with the second order excess entropy. In far from equilibrium situations this potential will always monotonically decrease. Written in affinities, the non-linear thermodynamic forces are of exponential shape for Lotka-Volterra systems.
A population balance model for fish population dynamics

In this section we have done the review of the eminent work of R.W. Thompson and D.A. Cauley (1979) which is the study of a population balance model of fish population dynamics for batch systems. Solutions of the model provide prediction of fish size distribution characteristics as average size, standard deviation and coefficient of variation.

The ability to forecast the production rate of fish in a controlled environment would be a valuable tool for commercial fishery operations. Mathematical models for the dynamics of fish populations have appeared in the literature (Clark, 1976), but these models ignored size variations within the total population. A population balance is a method of accounting for changes in a population distributed in a specific variable (or variables). The variables of interest in the present work are the size and age of the fish. Solutions of the population balance result in determination of the fish size distribution of a particular population as a function of time. Randolph and White (1977) have incorporated the size dispersion effect, often observed in growing systems, into a population balance model. This effect can occur when members of a distribution grow at randomly different rates relative to the mean growth rate, and results in a broadening of the distribution with time.
The similarities between batchwise crystal growth and fish growth are sufficient to allow population balance methods to be applicable to fish systems. In both cases growth occurs in discrete particles. Particles are born and may disappear by 'death' mechanisms. The present work addresses the case of a fish population of uniform age distributed in size. The current model for fish growth can provide important information as size / age history, growth rates, and fish size distribution predictions. This information can then aid in formulating harvesting policies, in choosing operating conditions, or in evaluating other economic aspects of fish farming.

A. Theory

The population balance for the fish population can be expressed as:

\[
\frac{dN}{dt} + \frac{\partial}{\partial \theta} \left( NG \theta \right) + \frac{\partial}{\partial L} \left( NL \right) = B - D + D _q \frac{\partial^2 N}{\partial \theta^2}
\]

\[\ldots\ldots\ldots (0.70)\]

where \( N \) denotes the number density of fish, \( t \) is time, \( \theta \) is age, \( L \) represents fish size (some characteristic length), \( G_L \) is a growth rate function, \( dL/dt \), \( G_\theta \) represents the rate of change of age \( (d\theta/dt = 1) \), \( B \) is birth rate, \( D \) is death rate, and \( D _q \) is the growth rate diffusivity. \( D _q \) has been assumed independent of size.

For the simplification, it will be assumed that all fish in the
population are of the same age and that the birth and death rates are zero. That is, the present analysis will be restricted to populations which are only growing in a uniform environment. Also for the present, the growth diffusivity $D_\alpha$ will be ignored. Equation (0.70) is then reduced to:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial \alpha} (\alpha N L) = 0 \quad \text{.........(0.71)}$$

It should be mentioned that because time and age increase at the same rate ($\theta = t + \theta_0$) the terms may be used interchangeable here. This is true only for batch systems in which all particles are of uniform age, i.e. where birth does not occur in the system. Since it is customary to speak of living things in terms of their ages, the substitution $\frac{\partial N}{\partial \theta} = \frac{\partial N}{\partial t}$ will be made in equation (0.71):

$$\frac{\partial N}{\partial \alpha} + \frac{\partial}{\partial \theta} (\alpha N L) = 0 \quad \text{.........(0.72)}$$

One boundary condition and one initial condition are required to completely pose the problem. We assume that there are no fish having absolutely zero length ($N(L, \theta) = 0$ at $L = 0$). In addition, we assume that the initial size distribution of fingerlings is known ($N(L, \theta) = N(L, 0)$ at $\theta = 0$).

The characteristics of the growth rate function, $g_\alpha$, have been discussed in the literature (Brown, 1957) as they relate to
Medawar's (1945) "laws of growth". The salient features of Medawar's laws are:

(a) size is a monotonically increasing function, that is, a fish always grows larger,

(b) the process of growth is multiplicative and, in general, will be described by equations of logarithmic form,

(c) the specific acceleration of growth is always negative under conditions of growth, and

(d) the growth rate decays more slowly as the fish increases in age.

It is recognized that the above rules are not always applicable during the entire life of the fish. There may be periods where growth is affected by environmental or seasonal changes, or by physiological events such as achievement of sexual maturity. These special situations will be ignored here, and the framework will be laid for analysis of a population of fish which only grow to a terminal size in uniform surroundings. That is, here, it is assumed that the aquatic environment, characterized by temperature, feeding rate, sunlight, toxic level, water velocity, etc., will be independent in space and time.
An appropriate growth rate function should characterize the dependence of growth on size and age. This dependence is represented by the general expression:

\[
\frac{d \log x}{d \theta} = f(\theta) \quad \ldots \ldots \ldots (0.73)
\]

The time dependent function \( f(\theta) \) should be such that \( f(\theta) \) is always positive, \( f(\theta) \) decreases with age, and \( df(\theta)/d\theta \) increases with age towards zero. Various expressions for biological growth appearing in the literature (e.g. Brown (1957) for fish growth and Finn & Wilson (1954) for microorganisms) have chosen \( f(\theta) \) to be constant. Other growth expressions in the literature (Iwasa, 1978) utilized size dependence rather than age dependence:

\[
\frac{d \log L}{dt} = k \left( 1 - AL^3 \right) \quad \text{(logistic)} \quad \ldots \ldots \ldots (0.74)
\]

\[
\frac{d \log L}{dt} = k \left( \frac{A}{L} - 1 \right) \quad \text{(Von Bertalanfly)} \quad \ldots \ldots (0.75)
\]

\[
\frac{d \log L}{dt} = k \log \frac{A}{L} \quad \text{(Gompertz)} \quad \ldots \ldots (0.76)
\]

Here the form of the function be

\[ f(\theta) = \exp \left( -\alpha \theta \right) \quad \ldots \ldots (0.77) \]
to represent the age dependence of the growth rate, where \( \alpha \) is a positive non-zero constant characteristic of a particular fish species in a specific fixed environment. Equation (0.77) satisfies the criteria given by Medawar and Brown.

The growth law used in this work, equation (0.77) is unique in that it follows accelerating growth at early ages followed by decaying growth at later ages.

Having chosen the age dependence of the growth rate expression, it is a straightforward matter to develop the expression describing the size / age history for a single fish. Incorporating equation (0.77) into equation (0.73) and integrating the resulting expression gives:

\[
L = L_0 \exp \left\{ \frac{-k}{\alpha} \left( e^{-\alpha \Theta} - e^{-\alpha \Theta_0} \right) \right\} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (0.78)
\]

where \( L_0 \) is size of the fish at the initial age, \( \Theta_0 \). It is not necessary that the initial age be zero. Equation (0.78) can be applied over any region of interest.

The growth rate function, \( G_L \), required in the population balance can be found from equations (0.73) and (0.77)

\[
G_L = \frac{dL}{d\Theta} = k \, L \, e^{-\alpha \Theta} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (0.79)
\]
For a population of a given uniform age, equation (0.79) indicates that the larger fish in the system grow faster than the smaller fish.

Expression for the growth of a single fish are of limited partial critical value for populations having many members. More useful information is obtained regarding the growth of population with a distribution of sizes when equation (0.79) is combined with equation (0.72)

\[
\frac{\partial N}{\partial \Theta} + \frac{\partial}{\partial L} (N/L e^{-\alpha \Theta}) = 0
\]

.........(0.80)

The initial condition and boundary condition discussed earlier completely formulate the problem. Solution of equation (0.80) yields the fish size distribution at various times. An analytical solution is not apparent, but equation (0.80) can be solved using numerical techniques.

To proceed in the development the term ke^{-\alpha \Theta} should be removed from the partial derivative with respect to L. The partial differentiation with respect to L is carried out at constant time (or age). The proof of the independence of L and \( \Theta \) is given below. The question of the independence of age, \( \Theta \), and size, L, can be settled in the following way. We begin by writing the population balance for particles distributed in size,
for which the linear growth rate, $G$, may depend on $\theta$, $L$, or both:

$$\frac{\partial N}{\partial \theta} + \frac{\partial (NG)}{\partial L} = 0 \quad \text{..........(A1)}$$

In this discussion random fluctuations in the growth rate, $G$, will be ignored.

By multiplying equation (A1) by $L^1$ and integrating over all particle sizes the differential equation for the 1st moment may be developed:

$$\frac{dm_1}{d\theta} = 1 \int L^{1-1}(NG) \, dL \quad \text{..........(A2)}$$

The differential equation for the first moment is generated when $1 = 1$:

$$\frac{dm_1}{d\theta} = \int (NG) \, dL \quad \text{..........(A3)}$$

and is related to the change in the average size of the population, $\bar{L}$, by

$$m_o \frac{d \bar{L}}{d\theta} = \frac{dm_1}{d\theta} = \int (NG) \, dL \quad \text{..........(A4)}$$

Equation (A4) is strictly correct for any size and/or age dependent growth rate.
The definition of \( \Gamma \) gives,
\[
\Gamma = \frac{\int_0^\alpha x N_L \, dl}{\int_0^\alpha x N \, dl}
\]
.........(A5)

and evaluate the time (age) derivative of \( \Gamma \) from equation (A5). The denominator of the r.h.s. of equation (A5) is strictly independent of age when members of the population are subject to no birth, death, or harvest. Therefore, equation (A5) reduces to
\[
\Gamma = \frac{1}{m_0} \int_0^\alpha x N_L \, dl
\]
.........(A6)

The age derivative of \( \Gamma \) is, then given by.
\[
\frac{d}{d\theta} \Gamma = \frac{1}{m_0} \frac{d}{d\theta} \int_0^\alpha x N_L \, dl
\]
.........(A7)

or,
\[
\frac{d}{d\theta} \Gamma = \frac{1}{m_0} \int_0^\alpha \frac{\partial}{\partial \theta} (x N_L) \, dl
\]
.........(A8)

Assume that, for the moment, the size and age are uniquely related by
\[
L = L(\theta)
\]
.........(A9)

Now,
\[
\frac{d}{d\theta} \Gamma = \frac{1}{m_0} \int_0^\alpha \left( N \frac{\partial}{\partial \theta} L + L \frac{\partial}{\partial \theta} N \right) \, dl
\]
.........(A10)
By virtue of equation (A9),

\[
\frac{\partial}{\partial \theta} L = \frac{d}{d \theta} L = B \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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determine a fish's age by measuring its length.

Again, we come to the original problem by rewriting equation (0.80) in terms of the moment transformation of the distribution.

The $i$-th moment, $m_i$, is defined by:

$$m_i(\theta) = \int_{L}^{\infty} N(L, \theta) \, dL \quad \text{...............}(0.81)$$

By multiplying equation (0.80) by $L^i$ and integrating with respect to $L$, the moment transform is found:

$$\frac{d}{d\theta} m_i = 0 \quad \text{...............}(0.82)$$

Where it has been assumed that the number density $N(L, \theta)$, is zero at both zero size and at infinite size. The solution of equation (0.82) is

$$m_i = m_{i0} \exp \left\{ -\frac{ik}{\alpha} (e^{-\alpha\theta} - e^{\alpha\theta}) \right\} \quad \text{...............}(0.83)$$

Where $m_{i0}$ is the $i$th moment evaluation at the initial condition. By allowing $i$ to take on non-negative integer values, it follows that:

$$m_0 = m_{\infty} = N_T \quad \text{...............}(0.84)$$
\[ m_i = m_{i0} \exp \left\{ - \frac{k}{\alpha} (e^{-\alpha\theta} - e^{-\alpha\theta_0}) \right\} \] ............(0.85)

\[ m_z = m_{z0} \exp \left\{ - \frac{2k}{\alpha} (e^{-\alpha\theta} - e^{-\alpha\theta_0}) \right\} \] ............(0.86)

\[ m_3 = m_{30} \exp \left\{ - \frac{3k}{\alpha} (e^{-\alpha\theta} - e^{-\alpha\theta_0}) \right\} \] ............(0.87)

Where \( N \) is the total number of fish. The initial moment values must be found from analysis of the initial distribution.

Using the moment equation, it is possible to generate information about fish sizes as the fish population ages. The average length of a distribution, \( \bar{L} \), is equal to \( (m_1 / m_0) \) and is given by:

\[ \bar{L} = \bar{L}_0 \exp \left\{ - \frac{k}{\alpha} (e^{-\alpha\theta} - e^{-\alpha\theta_0}) \right\} \] ............(0.89)

The standard deviation of a number size distribution, \( \sigma \), is given by:

\[ \sigma^2 = \frac{m_2}{m_0} - \left( \frac{m_1}{m_0} \right)^2 \] ............(0.89)

\[ i.e. \sigma^2 = \sigma_0^2 \exp \left\{ - \frac{2k}{\alpha} (e^{-\alpha\theta} - e^{-\alpha\theta_0}) \right\} \] ............(0.90)
Where \( \sigma^2_o = \left( \frac{m^2_{o \text{var}}}{m^2_o} - \frac{m^2_{10}}{m^2_o} \right) \)

The coefficient of variation of a distribution, \( C_v \), is given by:

\[
C_v = \frac{\sigma}{\bar{\mu}} = \frac{\sigma_o}{\bar{\mu}_o} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (0.91)
\]

It is interesting to note that equation (0.91) predicts that the coefficient of variation should be constant as the population ages.

The total mass of the fish population, \( W_T \), can be determined by using the empirical weight length relationship presented by Brown (1957):

\[
W_T = a N_T \bar{L}_p \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (0.92)
\]

Where \( a \) is a constant and \( p \) is an exponent whose value typically lies 2.5 and 4.0 for fish. Both constants are found experimentally. When there is no change in shape during growth \( p = 3.0 \), in which case the total mass of the population is given by:

\[
W_T = a N_T \bar{L}_0^3 \exp \left( - \frac{3K}{\alpha} (e^{-\alpha \bar{L}} - e^{-\alpha \bar{L}_0}) \right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (0.93)
\]

from the result presented in equation (0.88). For \( P = 4.0 \), the mass is related to the fourth moment of the distribution.
The empirical relationship shown in equation (0.92) may be derived for a fish population using the relations presented thus far. If equation (0.92) were modified to apply to the size-mass relationship for a member of the species of interest to be:

\[ W(L) = a' L^p \] ............(0.94)

Then the total mass of the population, having the size distribution, \( N(L, \theta) \), is given by:

\[ W(\theta) = \int W(L) \cdot N(L, \theta) \, dL = a' \int L^p \cdot N(L, \theta) \, dL \] ............(0.95)

or

\[ W(\theta) = a'm_\theta \] ............(0.96)

The \( p \)th moment at age \( \theta \) for the population can be found from equation (0.83) by letting \( t \) take on the value of \( p \). In addition, the \( p \)th moment of the distribution can be related to the first moment through the properties of the exponential in equation (0.83). Specifically, the \( p \)th moment is given by:

\[ m_p(\theta) = m_\theta \left( \frac{m_1(\theta)}{m_1(0)} \right)^p \] ............(0.97)
and the total mass of the fish population is given by:

\[ W(\theta) = a'm_p \left( \frac{m_1(\theta)}{m_{10}} \right)^p \]  \hspace{1cm} \ldots \ldots (0.98) \]

\[ = \left[ \frac{a'm_p m_0^{p-1}}{m_{10}^p} \right] N_L^{-p} \]  \hspace{1cm} \ldots \ldots (0.99) \]

Equations (0.92) and (0.99) are reconciled by allowing:

\[ a = \frac{a'm_p m_0^{p-1}}{m_{10}^p} \]  \hspace{1cm} \ldots \ldots (0.100) \]

which is constant for a fish population experiencing no death. Thus, the total mass of the population at any time can be determined once \( a' \) and \( p \) are known. These results are equally valid for either integer or fractional values of \( p \).

B. Growth diffusivity

It is possible to include growth diffusivity effects in the model, which case the population balance becomes:

\[ \frac{\partial N}{\partial \theta} + \frac{\partial (N \theta)}{\partial L} = D \frac{\partial^2 N}{\partial L^2} \]  \hspace{1cm} \ldots \ldots (0.101) \]

Multiplying equation (0.101) by \( L^4 \) and integrating with respect to \( L \), the moment equations give:
\[ m_0 = m_{\infty} = N \] 
\[ m_1 = m_{\infty} \exp \left\{ - \frac{k}{\alpha} \left( e^{-\alpha t} - e^{-\alpha \theta} \right) \right\}. \]
\[ m_2 = m_{\infty} \exp \left\{ z - z_0 \right\} + \frac{2D_0 m_{\infty}}{\alpha} e^{-z} \left[ \log_e \left( \frac{z_0}{z} \right) + \frac{z^2 - z^2_0}{4} + \ldots + \frac{z^n - z^n_0}{n! n!} \right] \]
\[ m_3 = m_{\infty} \exp \left\{ \frac{3}{2} (z - z_0) \right\} + \frac{6D_0 m_{\infty}}{\alpha} e^{(z_0 - 3z)/2} \log_e \left[ \left( \frac{z_0}{z} \right) + (z - z_0) + \frac{z^2 - z^2_0}{4} + \frac{z^n - z^n_0}{n! n!} \right]. \]

where the variable \( Z \), defined by:
\[ Z = \frac{zK}{\alpha} e^{-\theta} \text{ with } z_0 = z \|_\theta = \theta_0 \]

has been introduced for convenience. The average size of a distribution, \( \Gamma \), is still expressed by equation (0.87) but the standard deviation when \( D_0 \neq 0 \) is:
\[
\sigma^2 = \sigma_0^2 e^{(z_0 - z) - z} + \frac{2D}{a} e^{-z} \left[ \log \left( \frac{z_0}{z} \right) + (z_0 - z) + \frac{z_0^2 - z^2}{4} + \frac{z_0^n - z^n}{n - n!} + \right], \text{ } (0.107)
\]

The second term on the right hand side of equation increases with time and causes \( \sigma \) to increase more rapidly relative to the case in which \( D = 0 \).

At large values of time, the standard deviation becomes very large. The coefficient of variation for the case of activated growth is given by:

\[
C_v = \left[ \frac{\sigma_0^2}{L_0} + \frac{2D}{a} e^{-z_0} \left[ \log \left( \frac{z_0}{z} \right) + (z_0 - z) + \frac{z_0^2 - z^2}{4} + \frac{z_0^n - z^n}{n - n!} + \right] \right]^{1/2}, \text{ } (0.108)
\]

The coefficient of variation given by equation (0.108) increases with time due to the second term on the right, whereas it was independent of time when \( D = 0 \).

Data from the literature (Brown, 1951) for brown trout fry indicate that the coefficient of variation of a distributed population increases with time. Equation (0.91) predicts \( C_v \) to be
constant, as a consequence of the simplification that $D_\alpha = 0$. These results would lead one to conclude that the choice of $D_\alpha = \nu$ is not adequate to represent real fish population size data.

C. Discussion

Data from the literature (Brown, 1951) for Brown trout fry indicate that the coefficient of variation of a distributed population increases with time. Equation (0.91) predicts $C_V$ to be constant, as a consequence of the simplification that $D_\alpha = 0$. These results would lead one to conclude that the choice of $D_\alpha = 0$ is not adequate to represent real fish population size data. Equation (0.108) is the expression for $C_V$ when the growth diffusivity takes on positive values and allows $C_V$ to increase with time.

A curious feature of the predicted result when $D_\alpha \neq 0$ is that the average size, $\bar{C}$, approaches a limiting value exponentially in time, while the standard deviation, $\sigma$, continues to increase with time without bound. Thus, some fish should continue to grow (although slowly) indefinitely. Also, to maintain the terminal value of $C$, it is necessary that some get smaller to compensate for some fish continuing to grow large. Since real living systems do not normally behave in this manner, it must be concluded that the case presented where $D_\alpha \neq 0$ loses its applicability at finite (or relatively large) ages.
The increase of $C_v$ with time observed by Brown (1951) must then be explained in some other manner, rather than in terms of growth diffusivity effects. It is suggested that the changes in $C_v$ may have resulted from unevenly distributed mortality. Brown's data indicate a death rate of typically 20% over the course of experiments (6 weeks). It is reasonable to suspect that death occurred predominantly among the smaller fish. Such as unbalanced mortality would account for the observed changes in $C_v$. It is likewise possible that the mortality experienced was evenly distributed over all sizes and the changes in $C_v$ are the result of a real effect (which the present model does not include).

A population balance model of fish population dynamics in batch system was developed. A growth rate expression for living systems was introduced and applied to the population balance. The model is capable of predicting length and growth rate dependence on age with the additional advantage of predicting fish size distribution characteristics such as average size, standard deviation and coefficient of variation. Given the constants $\alpha$ and $k$ for a species in a fixed environment and an initial size distribution it is possible to simulate the size / age history of the fish population. In addition it was shown that an empirical relationship describing the total mass of the
population could be derived from the population balance model developed.

The population balance model does not include the effects of life cycle changes, such as birth, mortality, possible changes in shape or attainment of sexual maturity. The explicit dependence of environmental or seasonal change on fish growth rates, for example water temperature and pH, food availability or population crowding, has been excluded in the current study. These effects can be included by evaluating the dependence of the growth parameters, \( \alpha \) and \( K \), on these environmental conditions. This would require an enormous amount of experimental work.

A parameter study was conducted to illustrate the nature of the growth law expression. It was concluded that the conditions conducive to high growth rates, notably large values of \( K \) and smaller values of \( \alpha \), also caused increased broadening of the fish size distribution.

0.35. Non-equilibrium Fluctuation in Volterra-Lotka System

The stochastic theory of non-equilibrium fluctuation plays a significant role in modern non-linear chemical physics and theoretical biology. The stochastic modelling of many physicochemical and biological processes involves non-linear
stochastic differential equations. The problem of non-equilibrium fluctuation was studied earlier by Kerner (1957, 1959), Goel et al. (1971), Nicolis and Prigogine (1977), Gardiner (1983) and others. Here we have discussed the work of Chakrabarti and Baishya (1987) about the problem of non-equilibrium fluctuation in Volterra-Lotka system. The procedure which is used in solution of non-linear stochastic differential equations is to replace it by a linear one in such a way that the ensemble average of the square of the error due to the replacement is minimum (Valsakumar et al. 1983).

The basic equation of a Volterra-Lotka system describing the interaction of two species are given by:

\[
\begin{align*}
\frac{dX}{dt} & = e_1 X - k_2 XY \\
\frac{dY}{dt} & = k_1 XY - e_3 Y
\end{align*}
\]

The main interest is on the chemical analog of the Volterra-Lotka model (0.109).

Let us set

\[
e = k_1 A, \quad e = k_2 D
\]

Equations (0.109) then represent the conservation of mass equations of the following set of irreversible autocatalytic chemical reactions (in the limit of ideal mixture):
The system of equations (0.109) admits a single non-zero steady state solution

\[
\begin{align*}
X &= \frac{K_3 D}{K_2}, & Y &= \frac{K_4 A}{K_2} \quad \text{.........(0.112)}
\end{align*}
\]

For the behaviour of the system near the steady state solution (0.112) we put

\[
X = X_0 + \beta_1 x, \quad Y = Y_0 + \beta_2 y \quad \text{.........(0.113)}
\]

We assume that

\[
k_1 = k_2 = k_3 = k_4 \quad \text{.........(0.114)}
\]

Substituting

\[
\tau = (r A B)^{1/2} t \quad \text{.........(0.115)}
\]

the system of the equations (0.109) reduces to the form:

\[
\frac{d X}{d \tau} = -\left(\frac{k D}{B}\right)^{1/2} y - \left(\frac{k}{k_3}\right)^{1/2} x y \quad \text{.........(0.116)}
\]
These are the basic deterministic equations governing the system's behavior near the steady state. To study the fluctuation around the steady state the equations (0.116) and (0.117) are extended to the stochastic differential equation (coupled bivariate Langevin equations):

\[
\frac{d x}{d \tau} = -\left( \frac{1D}{B} \right) \dot{x} - \left( \frac{1}{B} \right) \dot{x} y + \eta_1(t) \quad \ldots \ldots \ldots (0.118)
\]

\[
\frac{d y}{d \tau} = +\left( \frac{kD}{AB} \right) \dot{y} + \left( \frac{kD}{AB} \right) \dot{x} y + \eta_2(t) \quad \ldots \ldots \ldots (0.119)
\]

where \( \eta_1(t) \) and \( \eta_2(t) \) are assumed to be Gaussian white noises satisfying the conditions

\[ \langle \eta_i(t) \rangle = 0 \quad \text{and} \quad \langle \eta_i(t) \eta_j(t') \rangle = 2 \epsilon_{ij} \delta(t-t') \quad \ldots \ldots \ldots (0.120) \]

Equations (0.118) and (0.119) are the basic non-linear stochastic differential equations determining the statistical behaviour of the system near the steady state.

A. **Statistical Linearization: Moment Equations**

The statistical linearization of the equations (0.118) and (0.119) consists of replacing the equations by the system of
linear equations

\[\frac{dx}{dt} = \alpha_1 x + \beta_1 y + c_1 + \eta_1 \quad \text{.........(0.121a)}\]

\[\frac{dy}{dt} = \alpha_2 x + \beta_2 y + c_2 + \eta_2 \quad \text{.........(0.121b)}\]

where the errors in the above linearization are given by

\[e_1 = -a_1 y - b_1 x - \alpha_1 x - \beta_1 y - c_1 \quad \text{.........(0.122a)}\]

\[e_2 = a_2 x + b_2 x y - \alpha_2 x - \beta_2 y - c_2 \quad \text{.........(0.122b)}\]

where

\[a_1 = \left[ \frac{\gamma D}{B} \right]^{\frac{1}{2}}, \quad b_1 = \left[ \frac{1}{B} \right]^{\frac{1}{2}}, \quad b_2 = \left[ \frac{\gamma D}{AB} \right]^{\frac{1}{2}} \quad \text{.........(0.123)}\]

Minimization of the average of the squares of the errors leads to

the equations of the motion of the first two moments (Valsalumar et al 1983)

\[\frac{d}{dt} \langle x \rangle = -\left[ \frac{\gamma D}{B} \right]^{\frac{1}{2}} \langle y \rangle - \left[ \frac{\gamma}{B} \right]^{\frac{1}{2}} \langle x \rangle \quad \text{.........(0.124)}\]

\[\frac{d}{dt} \langle y \rangle = +\left[ \frac{\gamma D}{B} \right]^{\frac{1}{2}} \langle x \rangle + \left[ \frac{\gamma D}{AB} \right]^{\frac{1}{2}} \langle y \rangle \quad \text{.........(0.125)}\]
\[
\frac{d}{dt} x'^2 = 2 \left[ -\left( \frac{kD}{B} \right)^{1/2} x'y + \left( \frac{k}{B} \right)^{1/2} y' + \epsilon_1 \right]
\]

\[
\frac{d}{dt} y'^2 = 2 \left[ \left( \frac{kD}{B} \right)^{1/2} x'y + \left( \frac{kD}{AB} \right)^{1/2} x'^2 - \left( \frac{y}{B} \right) \right]
\]

\[
+ \left( \frac{kD}{AB} \right)^{1/2} y' - \left( \frac{y}{B} \right)^{1/2} y'^2
\]

\[
\text{where the following relations are used:}
\]

\[\eta_1' = \epsilon_1; \quad \eta_2' = \epsilon_2;\]

\[x' \eta_2' = 0 = y' \eta_1'.\]

It is also assumed that the system size expansion is valid such that all correlation and variances are of the order of \(1/v\) compared to the averages (Nicolis and Prigogine, 1977; Vanampen, 1981; Gardiner, 1983)

\[x'y' \sim 0 \left( \frac{x'y'}{v} \right) \text{ or } 0 \left( \frac{y'}{v} \right)\]
where \( v \) is the volume of the system. We also assume that the correlations \( \epsilon_1 \) and \( \epsilon_2 \) given by the (0.129) decrease with the increase of the volume of the system and they are assumed to be of the order of the increase of the system volume \( v \) (Valsalummar et al., 1983):

\[
\epsilon_i \sim \frac{1}{v} \quad (i = 1, 2) \quad \text{.........(0.131)}
\]

Then using the expressions

\[
\langle x^2 y \rangle = 2 \cdot x \cdot \left[ \langle x y \rangle - x \cdot y \right] + x^2 \cdot y,
\]

\[
\langle x y^2 \rangle = 2 \cdot y \cdot \left[ \langle x y \rangle - x \cdot y \right] + x \cdot \langle y^2 \rangle \quad \text{.........(0.132)}
\]

and keeping the lowest order terms, equations (0.126) - (0.128) reduce to

\[
\frac{d}{dt} \langle x^2 \rangle = 2 \left[ - \left( \frac{kB}{g} \right)^{\frac{1}{2}} \langle x y \rangle - 2 \left( \frac{1}{g} \right)^{\frac{1}{2}} \langle x^2, y \rangle \right] \quad \text{.........(0.133)}
\]
Now replacing the averages $\langle x \rangle$ and $\langle y \rangle$ in equations (0.133) - (0.135) by their steady-state values given by

$$
\frac{d}{d\tau} \langle y^2 \rangle = 2 \left[ \left( \frac{KD}{B} \right)^{\frac{1}{2}} \langle x^2 \rangle - 2 \left( \frac{KA}{AB} \right)^{\frac{1}{2}} \langle x \rangle \langle y^2 \rangle \right]
$$

and

$$
\frac{d}{d\tau} \langle x \rangle = \left( \frac{KD}{B} \right)^{\frac{1}{2}} \left[ \langle x^2 \rangle - \langle y^2 \rangle \right]
$$

$$
+ 2 \left( \frac{V}{B} \right)^{\frac{1}{2}} \langle x \rangle \langle y^2 \rangle - 2 \left( \frac{KD}{AB} \right)^{\frac{1}{2}} \langle x^2 \rangle \langle y \rangle
$$

...........(0.134)

we have,

$$
\frac{d}{d\tau} \langle x^2 \rangle = 2 \left[ - \left( \frac{KD}{B} \right)^{\frac{1}{2}} \langle x \rangle \langle y \rangle + 2 \left( \frac{AK}{B} \right)^{\frac{1}{2}} D \right]
$$

...........(0.137)

Now replacing the averages $\langle x \rangle$ and $\langle y \rangle$ in equations (0.133) - (0.135) by their steady-state values given by

$$
\frac{d}{d\tau} \langle x \rangle = 0 \text{ and } \frac{d}{d\tau} \langle y \rangle = 0
$$

or $\langle x \rangle = - (D)^{\frac{1}{2}}$ and $\langle y \rangle = - (A)^{\frac{1}{2}}$

...........(0.136)

we have,

$$
\frac{d}{d\tau} \langle x^2 \rangle = 2 \left[ \left( \frac{KD}{B} \right)^{\frac{1}{2}} \langle x \rangle \langle y \rangle + 2 \left( \frac{AK}{B} \right)^{\frac{1}{2}} D \right]
$$

...........(0.137)

$$
\frac{d}{d\tau} \langle y^2 \rangle = 2 \left[ \left( \frac{KD}{B} \right)^{\frac{1}{2}} \langle x \rangle \langle y \rangle + 2 \left( \frac{AK}{B} \right)^{\frac{1}{2}} D \right]
$$

...........(0.138)
and
\[ \frac{d}{d\tau} \langle x, y \rangle = \left( \frac{4D}{B} \right)^{\frac{1}{2}} \left[ \frac{d}{d\tau} \langle x^2 \rangle - 2 \frac{d}{d\tau} \langle y^2 \rangle \right] + 2 \left( \frac{KD}{B} \right)^{\frac{1}{2}} (D - A). \]

Equations (0.137) - (0.139) are the required moment equations.

B. Non-Equilibrium Fluctuation

To solve the equations (0.137) - (0.139), differentiating (0.139) with respect to \( \tau \),

\[ \frac{d^2}{d\tau^2} \langle x, y \rangle = \left( \frac{4D}{B} \right)^{\frac{1}{2}} \left[ \frac{d}{d\tau} \langle x^2 \rangle - \frac{d}{d\tau} \langle y^2 \rangle \right] \]

using (0.137) and (0.138), we have

\[ \frac{d^2}{d\tau^2} \langle x, y \rangle = -4 \frac{KD}{B} \langle x, y \rangle \]

which is the equation of simple harmonic motion. The solution of (0.140) is given by

\[ \langle x, y \rangle = C_1 \cos 2 \left( \frac{KD}{B} \right)^{\frac{1}{2}} \tau + C_2 \sin 2 \left( \frac{KD}{B} \right)^{\frac{1}{2}} \tau \]

\[ \theta \]

\[ \theta \]

\[ \theta \]
Using the initial condition (Nicolis and Prigogine, 1977)

\[ \langle x \ y \rangle = 0, \ \langle x^2 \rangle = \langle y^2 \rangle = 0 \text{ at } \tau = 0 \] ........... (0.142)

and also from (30)

\[ \frac{d}{d\tau} \langle x \ y \rangle = 2 \left( \frac{KD}{B} \right)^{1/2} (D - A) \text{ at } \tau = 0 \]

........... (0.143)

we have

\[ \langle x \ y \rangle = (D - A) \sin 2 \left( \frac{KD}{B} \right)^{1/2} \tau \] ........... (0.144)

Putting this value of \( \langle x \ y \rangle \) in (0.137) and (0.138) and using the initial condition (0.142), we have

\[ \langle x^2 \rangle = (A - D) \left[ \cos \left( \frac{KD}{B} \right)^{1/2} \tau - 1 \right] + 2 \left( \frac{AK}{B} \right)^{1/2} \tau \] ........... (0.145a)

\[ \langle y^2 \rangle = (D - A) \left[ \cos \left( \frac{KD}{B} \right)^{1/2} \tau - 1 \right] + 2 \left( \frac{AK}{B} \right)^{1/2} \tau \] ........... (0.145b)
The expressions of fluctuations $\langle X^2 \rangle$ and $\langle Y^2 \rangle$ which increase linearly with time with a periodic background noise are in good agreement with those of Gardiner (1983).

This shows that the system exhibits abnormal critical fluctuation which from the biological or ecological context implies that the predator-prey system is never stable because there is no internal mechanism to reestablish the stability of prey distribution once it is perturbed by the predator.

The whole work aims to study the importance of the technique of statistical linearization in the statistical analysis of non-linear chemical reactions. It is mainly concerned with the problem of non-equilibrium fluctuation in non-linear systems satisfying Volterra-Lotka equations. In case of biological system of interacting species the volume $v$ should be replaced by the population size $N$ of the system (Nicolis and Prigogine, 1977). The stochastic extension of two variable Volterra-Lotka system are of the type of bivariate non-linear Langevin equations which are quite amenable to the technique of statistical linearization.
0.4 **Summary of the Present Works**

In this section we shall describe briefly the main results of our investigation and also specify the characteristic features of the methods and principles used in the investigation. The importance of the thermodynamic theory of biological processes and the role of the stochastic theory of fluctuation in the study of biological system has motivated us in studying some important non-equilibrium phenomena of biological growth from the consideration of both non-equilibrium thermodynamics and stochastic theory of irreversible processes. For the convenience of presentation and the coherence of the principles and methods used, we divided the works into four parts each consisting of two chapters.

0.41 In Chapter I we have presented some preliminary discussions on the primitive growth model of Malthus including its impact on the future development of biology and its drawback for the realistic picture of growth. We have then discussed the basic physical and mathematical characteristics of a growth law to build up the platform for the development of a generalized model equations of growth. As an example, we have considered logistic equation.
0.42 The Chapter II is on the stochastic modelling of Bertalanffy growth [Bertalanffy, 1962]. Bertalanffy equation is an important model equation based on the anabolism and catabolism of living system. This model have been successfully applied to a large number of growths of animals, particularly animals having tails [Bertalanffy, 1962]. There is, however, a drawback with the model equation, it takes infinite time to reach the terminal growth, which most animals fail to achieve. The equation have been modified to avoid the drawback. The stochastic extension of the modified growth equation leads to the result of decreasing law of randomness as the living system evolves towards the terminal growth.

0.43 The Chapter III deals with the non-equilibrium thermodynamic modelling of gompertzian growth of a population substantiated by a stochastic model of the system under random disturbance of the environment.

The study of growth phenomena is a dominant problem in ecology [Ulanowicz, 1986]. There are, however, no general physico-mathematical theory of population growth. The statistical mechanical approach to population biology initiated by Kerner, although very powerful, has, however, some limitations [Smith, 1978]. The utility of non-equilibrium thermodynamic models lies not only in finding out the usual results of dynamical ecology, but also in modelling of an ecosystem from the
more realistic physico-mathematical theory of energy-flow or transfer within and with the environment [Ulanowicz, 1986]

The object of the present chapter is to develop a non-equilibrium thermodynamic model of Gompertzian growth, particularly to study the role of entropy-production in the growth process. The non-equilibrium thermodynamic model has been substantiated by a stochastic model of the system under random disturbance of the environment. The comparison of the expression of entropy production obtained from the two models leads to the relation connecting temperature and growth rate,

\[ \alpha = \frac{D}{k_0 T} \]  \hspace{1cm} \text{(0.146)}

which expressed the growth rate \( \alpha \) in terms of \( k_0 \), temperature \( T \) and diffusion coefficient \( D \).

From (0.146) we see that the growth rate \( \alpha \) is proportional to the inverse temperature \( 1/T \). This relation, however, fails to account of the tolerance limit or reference temperature. As a result it fails to explain the general empirical observations that growth rate increases with the increase of temperature. As a matter of fact, The growth rate should not be proportional to the inverse temperature \( 1/T \), but to the inverse temperature difference \( (1/T_e - 1/T) \). Accordingly, the relation (0.146) should
be modified to the form

\[ \alpha = \frac{D}{K_0} (1/T - 1/T_c) \]  \hspace{1cm} \ldots \ldots \ldots \ldots (0.147)

This justifies the increase of growth rate with the increase of temperature within the temperature tolerance limit. When \( T \to T_c \), the growth rate \( \alpha \) tends to zero as it should be. When \( T \) tends to become large crossing the upper bound of tolerance limit, the organism dies down, which is a biological requisite and not a consequence of equation (0.147). The important results obtained here, is the growth temperature relation (0.146) and its modification (0.147). While the modified relation (0.147) is able to explain the empirical observational results of temperature growth relationship [Gross, 1980], the relation (0.146) is of great significance for radiation therapy of tumor growth which is also governed by gompertz growth. In radiation therapy, the temperature rises very high which destroy or sterilize tumor cells [Casey, 1962].

0.44 In the chapter - IV the stochastic behaviours of gompertzian growth of a single species population (both homogeneous and heterogeneous) with periodic carrying capacities have been investigated.

Gompertz equation is one of the most commonly used models for a wide class of growth - phenomena such as embryonic growth, tumor growth, growth of plants, animals and men. [Laird, 1956,
'62]. Various authors have investigated different deterministic and stochastic behaviours of Gompertzian growth [Smith and Tuckwell, 1976; Goel et al, 1977; Holgate, 1989]

The problems under consideration are the stochastic behaviours of population, both homogenous and heterogeneous under internal and external (environmental) noises having deterministic time dependent carrying capacities. The method is based on the extension of the deterministic equations to stochastic differential equations.

Gompertz equation for the time - evolution of a single species is given by

\[
\frac{dN}{dt} = \alpha N \ln (N/K) \tag{0.148}
\]

Where \(N(t)\) is the population size at any time \(t\), the parameters \(\alpha\) and \(K\) are the growth rate and carrying capacity respectively. Both \(\alpha\) and \(K\) may be constant, deterministic functions of time or randomly fluctuating variables. We have first considered the time dependent carrying capacity and have taken (Nisbet and Gurney, 1983)

\[
\ln K = K' = K_0 (1+b \cos wt) \tag{0.149}
\]

To consider the effect of external noise we also assume that the growth rate coefficient \(\alpha(t)\) is randomly fluctuating variable and
\[ \alpha(t) = \alpha_0 + \varepsilon(t) \quad \ldots \ldots \quad (0.150) \]

where \( \alpha_0 \) is the deterministic part and \( \varepsilon(t) \) is the random perturbation term which is assumed to be a white noise of unit intensity.

Investigating the model we have observed that after a sufficiently large time the average population exhibits oscillation of amplitude

\[ \frac{b \alpha_0^k}{(\alpha_0^2 + \omega^2)^{1/2}} \]

and with the same frequency \( \omega \) as that of the logarithm of the carrying capacity.

For \( \alpha_0 > \frac{1}{2} \), after a large time the second-order moment (as fluctuation) also exhibit oscillatory behaviour not of single amplitude and frequency but of a superposition of a number of oscillations of different amplitudes and frequencies.

However, for \( \alpha_0 < \frac{1}{2} \), the second-order moment becomes infinitely large. This implies that for \( \alpha_0 < \frac{1}{2} \) the system becomes unstable in the long run.

The case with internal noise is also similar to the case of external noise. The similarity is in the sense that both the expressions of second-order moment are superpositions of a number
of oscillations of different amplitudes and frequencies after a large time.

Next we have assumed the system to be heterogeneous and for simplicity we have considered only one-dimensional system. For finite value of $K$ (the carrying capacity), $\alpha_0$ (the deterministic part of the growth rate coefficient $\alpha$) and $D$ (the diffusion coefficient), the variance $\sigma^2$ has been found finite implying that the system is stable. However, if $D \rightarrow 0$ i.e. the case of infinitely small diffusion the variance $\sigma^2 \rightarrow \infty$ which implies the instability of the population.

0.45 The chapter V deals with the mathematical modelling and the study of the characteristic properties of size-structured distributions obeying Bertalanffy and Gompertz equations of growth.

In the mathematical modelling of population growth the difference among the individuals in each species plays a significant role. The difference among the individuals can be due to location in space, genetic difference, differences in age and size etc. Among them the size-difference leading to the size-distribution of a population is of significant interest not only for its theoretical importance of reflecting the dynamics of growth, but also for its commercial value, for example in fishery and forestry [Thomson and Cauley, 1979; Hara, 1984].
In this chapter we have studied the size-distribution of populations obeying both the Bertalanffy and Gompertz equations of growth. The mathematical model based on Kolmogorov forward equation or diffusion equation have helped to predict the characteristic properties of size-distribution and hence the pattern of growth of the corresponding population. The results obtained from the population growth models have been compared with observed results of Brown to test their validity at least for fish population. The first problem we have considered is the linear growth of animals and plants governed by Bertalanffy's growth equation:

$$g = \frac{dx}{dt} = E - \frac{x}{t}$$  \hspace{2cm} (0.151)

We have set $g = E - \frac{x}{t}$ in the continuity equation and derived the moment equations. In particular, the average length $\bar{x}$ and variance $\sigma^2(x)$ have been obtained. These expressions show that the average size $\bar{x}$ tends to the fixed value $E / (k + \mu)$ after a large time. The variance $\sigma^2$ also tends to the fixed value $E^2\mu / (k + \mu)^2 (2k + \mu)$ when the time increases infinitely. The results of the mathematical model is tested by comparing the results with the observational data available from the study of Brown [1951] on Brown trout fry. The increasing law of the coefficient of variation with time agrees with the observational results of Brown [1951]. We can thus conclude that Bertalanffy's...
growth model serves as a good model of size-structured fish population dynamics.

Next we have chosen the $g(x)$ as the Gompertzian growth function in the model equation and this has led to an interesting result for chemotherapy of tumor growth where some chemical reagents are used to stop the growth of effected cells. The expression of average size shows that it tends to zero as $t$ tends to infinity implying extinction of the malignant cells in the long run. Thus the tumor growth modelling by the size-structured Gompertzian growth equation provides a good model for chemotherapy.

0.46 The object of the chapter VI is to study the criterion of growth or extinction of some age-structured and physiological age-structured populations under constant and variable mortality rate.

We have considered a single-species population with age-distribution $f(a, t)$ such that $f(a, t) \, da$ denotes the number of individuals whose age $a$ time $t$ lies between $a$ and $a + da$. Let $\gamma(a)$ be the per capita death rate (or mortality) implying that $\gamma(a) \, dt$ is the average-fraction of individuals aged $a$ who die during the time-interval $dt$. Then the age-distribution $f(a, t)$ obeys the famous Foerster equation
Besides there is the age-specific birth rate (i.e., per capita birth rate) \( \beta(a) \), implying that \( \beta(a) \, da \) is the average number of off-springs produced per unit time by an individual aged between \( a \) and \( a + da \). In this case the boundary condition

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial a} + \gamma(a) \, f = 0 \tag{0.152}
\]

called the renewal condition, must be used in relation to the equation \( (0.152) \) to calculate the age-distribution at any time \( t \) subject to the given initial distribution at time \( t = 0 \). We also assume that the age-specific birth rate is given by

\[
\beta(a) = \beta_0 e^{-\alpha a} \tag{0.154}
\]

For constant mortality rate \( \gamma(a) = \gamma_0 \), we have shown that the population would decay or grow with times according as \( \lambda > 0 \) or \( \lambda < 0 \) i.e. according as \( \alpha + \gamma_0 > \beta_0 \) or \( \alpha + \gamma_0 < \beta_0 \). When the mortality rate is not a constant, we have taken \( \gamma = \gamma_0 \alpha \), then \( \lambda \) can be determined for arbitrary set of values of \( \alpha \) and \( \gamma_0 \) and henceforth the sign of \( \lambda \) will determine the criteria of decay or growth.
For physiological age-structured distribution we have taken a problem of tissue culture, a process for growing certain multicellular organism. We have obtained here the condition of growth or decay of the population of pellets. Next we have considered the Bertalanffy growth model in linear dimension with two separate cases of constant mortality rate and variable mortality rate. The physiological age is taken as the length of the individual. In the case of variable mortality rate the criteria for growth and decay can not be determined exactly, only approximate values have been obtained and numerical simulation for particular choices of the parameters have been provided.

The comparison of the criteria of growth (or decay for age-structured and physiological age-structured populations respectively) deserve special mention on the scaling of age. In the former case (age-structured population) the age is measured by the time $t$ (with an additive constant) and for the second case (physiological age-structured population) it is measured by the length '$x'$ of the organism.

0.47 The chapter VII deals with the non-equilibrium thermodynamic and stochastic analysis of an extended logistic model (due to May) of mutually interacting species. The most interesting and beneficial association between two-species is the act of co-operation (e.g. the clown fish and sea-anemones, the antacacia system and plant-pollinator system etc.).
In the absence of interspecific effects, the individual populations are assumed to be governed by logistic equations. May (1976) had proposed a model equation of two-species co-operation by modifying the carrying capacity of logistic equations to reflect dependence upon the density of the complementary population. The system of equations considered by May (1976) is:

\[
\begin{align*}
\frac{dN_1}{dt} &= r_1 N_1 \left[1 - \frac{N_1}{K_1 + \alpha N_2}\right] \\
\frac{dN_2}{dt} &= r_2 N_2 \left[1 - \frac{N_2}{r_2 + \beta N_1}\right]
\end{align*}
\] .......(0.155)

In this chapter we have studied the stability of the stationary state of the system from the thermodynamic and stochastic modelling of the system. This has strengthened the deterministic analysis of the study of criteria of stability.

At first we have presented the deterministic analysis of the stability of the stationary state of the system governed by the equations (0.155). The steady state sizes are given by

\[
N_1^* = \frac{K_1 + \alpha K_2}{1 - \alpha \beta}, \quad N_2^* = \frac{K_2 + \beta K_1}{1 - \alpha \beta}
\] .......(0.156)
It is evident from (0.156) that the steady state populations exist provided $\alpha \beta < 1$ and the mathematical calculations led to the result that for $\alpha \beta < 1$ the steady state population is locally stable.

The thermodynamic analysis has led to the following criteria of stability of the stationary state $(N_1^*, N_2^*)$

$$r > 0$$

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2 > 4 \quad \text{i.e.} \quad \alpha \beta < 1.$$ 

(that $r$, positive is assumed in the model).

We have thus obtained the same criteria of stability $\alpha \beta < 1$ as that obtained by the dynamical analysis.

In stochastic version of our model we have considered small fluctuations in the population size caused by the random disturbance of the environment. We add small fluctuation term $\phi(t)$ to both the equations of the linearised form of the basic model in order to consider the effect of variable environment. We have calculated the fluctuation intensity for the two populations and the corresponding auto correlation functions. The variances of the population sizes $N_1$ and $N_2$ remain finite provided $\alpha \beta < 1$. This is the criteria of stability of the
stationary state \((N^*_1, N^*_2)\) in the sense of second order moment.

Thus the deterministic, stochastic and non-equilibrium thermodynamic modelling of the system under co-operation lead to the same criteria of stability of the stationary state of the system under consideration.

0.48 The object of the chapter VIII is to study the stability of a simple two-species competing system from the consideration of non-equilibrium thermodynamic and stochastic modelling of the system to see whether these modellings would result in any new conditions of stability in addition to those determined by the usual dynamical analysis of the deterministic model equations.

When two or more species live in proximity and share the basic requirement, they usually compete for resources, habitat or territory and in some way inhibit each other growth. This is the famous principle of competitive exclusion in population biology [Edelstein-Keshat, 1987]. Some interesting phenomena have been found from the study of practical competitive model. There are much current arguments and literature about the role of competition in determination of structure of an ecological community [Hsu et al., 1979].
We are, however, not interested here in the generality of the competition model. Instead we shall consider a simple two species competing system and study the stability of the system from the entropic analysis of stability.

Firstly, we have considered the basic dynamical equations of two yeast populations in an alcohol medium:

\[
\frac{dN_1}{dt} = e_1 N - a_1 N (N + N_2) \quad \cdots \cdots \cdots (0.157a)
\]

\[
\frac{dN_2}{dt} = e_2 N - a_2 N (N + N_1) \quad \cdots \cdots \cdots (0.157b)
\]

where \(N_1(t)\) and \(N_2(t)\) are the population sizes of the two species at any time \(t\), \(e_1\) and \(e_2\) are the specific growth rates of the populations and are positive quantities.

There are three possible stationary points:

\((N_1^*, N_2^*) = (0, 0)\) (both species die out) \(\cdots \cdots \cdots (0.158a)\)

\((N_1^*, N_2^*) = (0, \frac{e_2}{a_2})\) (only the first species dies out) \(\cdots \cdots \cdots (0.158b)\)

\((N_1^*, N_2^*) = (\frac{e_1}{a_1}, 0)\) (only the second species dies out) \(\cdots \cdots \cdots (0.158c)\)
The stationary point (0,0) is unstable and uninteresting. We have considered only the stationary state (e_i/a_i,0). The non-equilibrium thermodynamic modelling of the system and the thermodynamic criteria of stability

$$\frac{\partial}{\partial t} (\delta^2 s) \bigg|_{st} > 0 \quad \ldots \ldots \quad (0.159)$$

has led to the following criteria of stability of the stationary state (e_i/a_i,0):

$$e_i < 0, a_{i1} - a_{i2} e_i < 0 \quad \ldots \ldots \quad (0.160)$$

The thermodynamic criteria of stability are the same as criteria of stability obtained directly from the dynamical analysis.

The stochastic analysis of the stability of the system depends on the stochastic entropy defined for the stochastic model of the system by adding random parturbation term to the r.h.s of the system of equations (0.157). The stochastic analysis for the stability of the system at p(\(\frac{e_i}{a_i}\),0) showed that the eigenvalues e_i and (e_i - a_{i1} \frac{e_i}{a_1}) in the deterministic analysis of stability and the noise intensity are the factors which determine the stability of the stochastic system.
The present chapter is an attempt, limited in its scope and field, of the entropic (both thermodynamic and stochastic) analysis of stability. Here we have seen that although the deterministic and non-equilibrium thermodynamic modelling has led to the same criteria of stability or instability, the stochastic model led to different criteria of stability. The result is in qualitative agreement with the stability analysis by May [1973].