CHAPTER - THREE

FOURIER EXPANSION METHOD FOR DOUBLY-CONNECTED REGIONS BOUND BY DISSIMILAR BOUNDARIES.
INTRODUCTION

Solution of the two dimensional Laplace's equation with non-homogeneous boundary conditions for multiply connected regions have been attempted by many authors. Lin (1947) found the torsion function of a circular tube with longitudinal circular symmetrically spaced holes. He assumed three series of harmonic functions which are invariant with respect to rotation about the axis of the tube and finally expressed the solution in terms of an infinite set of equations. Hockney (1964) considered the problem of fluid flow through the central circular hollow of a square brick. He, too, assumed a particular series of harmonic functions which is also invariant with respect to rotation about the axis of the tube. But his method depends on fitting the solution on particular points of the external boundary and in an actual case he has fitted the solution to eight symmetrical points viz. the four angular points and the four middle points of the sides of the square boundary. The accuracy of his solution increases as the number of points fitted in the outer boundary increases. But the equations corresponding to any point on the external boundary excepting the eight mentioned above are much more complicated.
Solutions in terms of infinite set of equations have gained some importance as several authors, Abramian and Babloian (1960), Deutsch (1962), Ghosh (1964) have solved torsion problems of practical interests in terms of infinite set of equations. The author, therefore, chooses the corresponding problem of Hockney in Elasticity viz. the problem of finding the torsion function of a square beam with a central circular hollow and deduces the harmonic function which is invariant with respect to rotation according to symmetry of the configuration and finally expresses the result in terms of an infinite set of equations.
SECTION I

TORSION OF A SQUARE PEG WITH A CENTRAL CIRCULAR HOLLOW
Fig. 3.1
Statement of the problem

We consider a doubly connected region bounded internally by \( x^2 + y^2 = c^2 \) and externally by \((x^2 - a^2)(y^2 - a^2) = 0\). We are to find a function \( \psi \) which satisfies the equation

\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0
\]

(3.1)

and the boundary condition

\[
\psi = \frac{1}{2} (x^2 + y^2), \quad (x^2 - a^2)(y^2 - a^2) = 0
\]

(3.2)

and

\[
\rho = \frac{1}{2} c^2 + \psi_0, \quad x^2 + y^2 = c^2
\]

(3.3)

Preliminary Simplification

We take four different points in the central hollow as \( z = x + iy = l_m = l e^{i \pi / 2} \) and consider them as poles in four different polar coordinate system. We then define the following polar coordinate system:
where \( m = 0, 1, 2, 3 \) and \( \rho, \phi \) are the polar coordinates referred to the centre of the cross-section as origin and \( r_m, \Theta_m \) are the polar coordinates referred to \( z = b_m \) as origin. With \( z = b_m \) as pole, the solution of (3.1) may be written as

\[
\psi = A_m \log r_m + \sum_{n=0}^{\infty} \left( B_{mn} r_m^n + \frac{C_{mn}}{r_m^n} \right) \cos n \Theta_m \\
+ \sum_{n=0}^{\infty} \left( D_{mn} r_m^n + \frac{E_{mn}}{r_m^n} \right) \sin n \Theta_m
\]

Since the stress distribution is symmetrical about the diameter through \( z = b_m \), we take \( \psi \) as

\[
\psi = A_m \log r_m + \sum_{n=0}^{\infty} \left( B_{mn} r_m^n + \frac{C_{mn}}{r_m^n} \right) \cos n \Theta_m
\]
where \( m = (0, 1, 2, 3) \). Due to symmetry of the configuration, the function \( \psi \) expressed in four different co-ordinate systems referred to the symmetrical points \( \Omega = \Omega_m \) as poles must be similar in form. Therefore we have

\[
\begin{align*}
A_0 &= A_1 = A_2 = A_3 = A \\
\beta_{0n} &= \beta_{1n} = \beta_{2n} = \beta_{3n} = B_n \\
c_{0n} &= c_{1n} = c_{2n} = c_{3n} = C_n
\end{align*}
\]

and we take \( \psi \) as

\[
\psi \equiv A \sum_{m=0}^{3} \log \kappa_m + \sum_{m=0}^{3} \sum_{n=0}^{\infty} \left( \beta_n \kappa_m^n + \frac{c_n}{\kappa_m^n} \right) \cos n \theta_m \cos n \theta_m
\]

(3.5)

To transform in terms of the polar co-ordinates referred to the centre as origin we notice

\[
\kappa_m^n \cos n \theta_m = \text{Re} \left( \xi_m \right)^n
\]

\[
= \text{Re} \left[ \frac{z - \Omega_m}{\ell_m} \right]^n
\]

or
\[ r_n \cos n \theta_m = \Re \left\{ \left( \frac{z}{\ell_m} \right)^n (1 - \frac{\alpha_m}{z})^n \right\} , \]

\[ |z| > |\ell_m| \]

which is equivalent to

\[ \Re \sum_{\delta > 0}^{n} (-1)^\delta \left( \begin{array}{c} n \\ \delta \end{array} \right) \left( \frac{z}{\ell_m} \right)^{n-\delta} \]

Therefore

\[ \sum_{m=0}^{3} \sum_{n=0}^{\infty} B_n r_n \cos n \theta_m \]

\[ = \sum_{m=0}^{3} \sum_{n=0}^{\infty} \sum_{\delta=0}^{n} B_n (-1)^\delta \left( \begin{array}{c} n \\ \delta \end{array} \right) x \]

\[ \times \left( \frac{p}{\beta} \right)^{n-\delta} \cos \left\{ (n-\delta) \left( \frac{m \pi}{a} - \phi \right) \right\} \]

which is equal to
where \( p = \frac{b}{a} \). It is evident that in this series terms in which \((n-8)\) is a multiple of four are non-zero and others vanish. We take then \( n-8 = 4q \).

Then

\[
\sum_{m=0}^{3} \sum_{n=0}^{\infty} B_n \xi_m^n \cos \theta_m = 4 \sum_{2+4q=0}^{\infty} \sum_{2=0}^{2+4q} B_{2q+4q} (-1)^q \times \\
x \left( \frac{p}{r} \right)^{4q/8} \cos 4q/\varphi.
\]

Also

\[
\frac{\cos n \theta_m}{\xi_m^n} = \text{Re} \left( \frac{1}{\xi_m} \right)^n = \text{Re} \left[ \frac{b_m}{z - \xi_m} \right]^n,
\]

\(|z| > |b_m|\).
or

\[
\frac{\cos n\theta_m}{\tau_m^n} = \Re\left\{ \sum_{s=0}^{\infty} \left( n+s-1 \right) \left( \frac{\ell_m}{s} \right)^n \right\}
\]

Therefore

\[
\sum_{m=0}^{3} \sum_{n=0}^{\infty} c_n \frac{\cos n\theta_m}{\tau_m^n}
\]

\[
= 4 \sum_{4q-8=0}^{\infty} \sum_{s=0}^{\infty} C_{4q-8s} \left( \frac{b}{p} \right)^{4q} \phi
\]

as before, where we have taken \( n + s = 4q \).

Again

\[
A \sum_{m=0}^{3} \log \tau_m = A \Re \sum_{n=0}^{3} \log \xi_m = A \Re \sum_{n=0}^{3} \log \frac{2-\ell_m}{\xi_m}
\]

or

\[
A \sum_{m=0}^{3} \log \tau_m
\]

\[
= 4A \log \frac{p}{b} - A \sum_{q=1}^{\infty} \frac{1}{b} \left( \frac{p}{b} \right)^{4q} \cos 4q \phi,
\]

\(|z| > b\).
Finally \( \psi \) is given by

\[
\psi = 4A \log \left( \frac{f}{p} \right) - \sum_{q=1}^{\infty} \frac{1}{q^4} \left( \frac{f}{p} \right)^4 \cos 4q \phi.
\]

\[
+ 4 \sum_{q=1}^{\infty} \sum_{s=0}^{8+4q} B_{2s+4q} \left( \frac{f}{p} \right)^{4q} \cos 4q \phi
\]

\[
+ 4 \sum_{q=1}^{\infty} \sum_{s=0}^{8+4q} C_{4q-8} \left( \frac{4q-1}{p} \right)^{4q} \cos 4q \phi
\]

This shows that \( \psi \) is invariant with respect to rotation about the axis of the beam.

Method of solution

We take then \( \psi \) as

\[
\psi = D_0 + E \log p + \sum_{m=1}^{\infty} F_{4m} p^{4m} \phi + \sum_{m=1}^{\infty} G_{4m} \frac{\cos 4m \phi}{p^{4m}}
\]

By the boundary condition (3.3) at \( \rho = \frac{c}{a} = 1, \psi = \frac{1}{2} c^2 + \psi_0 \).

Therefore
\[ \frac{1}{2} c^2 + \psi_0 = D_0 + E \log d \]
\[ + \sum_{m=1}^{\infty} F_{4m} \cos 4m \phi + \]
\[ + \sum_{m=1}^{\infty} G_{4m} \frac{\cos 4m \phi}{d^{4m}} \]

We have therefore,
\[ D_0 + E \log d = \frac{1}{2} c^2 + \psi_0 \]  
(3.7)

and
\[ F_{4m} \phi + \frac{G_{4m}}{d^{4m}} = 0 \]  
(3.8)

By the boundary condition (3.2) we have at \( \rho = \sec \phi \), \( \psi = \frac{1}{2} a^2 \sec^2 \phi \) and at \( \rho = \cos \sec \phi \), \( \psi = \frac{1}{2} a^2 \cos \sec^2 \phi \). Then we get the conditions
\[ \frac{1}{2} a^2 \sec^2 \phi = D_0 + E \log |\sec \phi| \]
\[ + \sum_{m=1}^{\infty} F_{4m} \sec \phi \cos 4m \phi \]
\[ + \sum_{m=1}^{\infty} G_{4m} \frac{\cos \phi \cos 4m \phi}{d^{4m}} \]
for the regions $-\frac{3\pi}{4} \leq \phi \leq \frac{3\pi}{4}$, $\frac{3\pi}{2} \leq \phi \leq \frac{3\pi}{4}$.

\[ \frac{1}{2} a^2 \csc^2 \phi = D_0 + E \log \csc \phi \]

\[ + \sum_{m=1}^{\infty} F_{4m} \csc \phi \cot 4m\phi \]

\[ + \sum_{m=1}^{\infty} G_{4m} 8m \phi \cot 4m\phi \]

for the regions $\frac{3\pi}{4} \leq \phi \leq \frac{3\pi}{2}$, $\frac{3\pi}{4} \leq \phi \leq \frac{7\pi}{4}$.

We assume

\[ f_1(\phi) = D_0 + E \log \csc \phi \]

\[ + \sum_{m=1}^{\infty} F_{4m} \csc \phi \cot 4m\phi \]

\[ + \sum_{m=1}^{\infty} G_{4m} 8m \phi \cot 4m\phi - \frac{1}{2} a^2 \csc^2 \phi \]

and define a function $f(\phi)$ such that

\[ f_2(\phi) = D_0 + E \log \csc \phi \]

\[ + \sum_{m=1}^{\infty} F_{4m} \csc \phi \cot 4m\phi \]

\[ + \sum_{m=1}^{\infty} G_{4m} 8m \phi \cot 4m\phi - \frac{1}{2} a^2 \csc^2 \phi , \]
\[ f(\varphi) = f_1(\varphi), \quad 0 \leq \varphi \leq \frac{\pi}{4}, \quad \frac{3\pi}{4} \leq \varphi \leq \pi \]

\[ = f_2(\varphi), \quad \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4} \]

together with \( f(\varphi) = f(-\varphi) \). Then the boundary condition (3.2) reduces to

\[ f(\varphi) = 0 \]

Expanding \( f(\varphi) \) in a Fourier series we get

\[ \sum_{n=0}^{\infty} H_n \cos n\varphi = 0 \]

This gives the conditions

\[ H_0 = 0 \]  \hspace{1cm} (3.9) \]

\[ H_n = 0, \quad n = 1, 2, \ldots \infty \]  \hspace{1cm} (3.10) \]

and \( H_n \) are given by

\[ H_n = \frac{2}{\pi} \int_{0}^{\pi} f(\varphi) \cos n\varphi \, d\varphi \]

or
\[
H_n = \frac{2}{\pi} \left[ \int_0^{\pi/4} f_1(\varphi) \cos n \varphi \, d\varphi \\
+ \int_{\pi/4}^{3\pi/4} f_2(\varphi) \cos n \varphi \, d\varphi \\
+ \int_{\pi/4}^{3\pi/4} f_3(\varphi) \cos n \varphi \, d\varphi \right]
\]

which is equal to

\[
\frac{2}{\pi} \left[ \sum_{m=1}^{\infty} F_{4m} I_{4m,n} + \sum_{m=1}^{\infty} G_{4m} J_{4m,n} - \frac{E}{L} K \right]
\]

where

\[
K_n = 4 \cos \frac{n\pi}{2} \cos \frac{n\pi}{4} \int_0^{\pi/4} (\log \cos \varphi) \sin (\frac{n\pi}{4} - n \varphi) \, d\varphi
\]

\[
L_n = 4 \cos \frac{n\pi}{2} \cos \frac{n\pi}{4} \int_0^{\pi/4} \sec^2 \varphi \sin (\frac{n\pi}{4} - n \varphi) \, d\varphi
\]

\[
I_{4m,n} = 4 \cos \frac{m\pi}{2} \cos \frac{n\pi}{4} \int_0^{\pi/4} \sec^4 \varphi \cos 4m \varphi \cos (\frac{n\pi}{4} - n \varphi) \, d\varphi
\]

\[
J_{4m,n} = 4 \cos \frac{m\pi}{2} \cos \frac{n\pi}{4} \int_0^{\pi/4} \cos 4m \varphi \cos 4m \varphi \cos (\frac{n\pi}{4} - n \varphi) \, d\varphi
\]

(\( n = 0, 1, 2, \ldots \))
It is evident that if \( n \) is not a multiple of 4, relations (3.10) become identity. For the non-identity relations we take \( n = 4t \) then (3.11) gives

\[
H_{4t} = \frac{a}{2\pi} \left[ \sum_{m=1}^{\infty} F_{4m} I_{4m, 4t} + \sum_{m=1}^{\infty} G_{4m} J_{4m, 4t} - \frac{1}{2} \alpha^2 \right]
\]

with

\[
K_{4t} = 4 \int_{0}^{\pi/4} (\log \cos \phi) \cos 4t \phi \, d\phi
\]

\[
L_{4t} = 4 \int_{0}^{\pi/4} \sec^2 \phi \cos 4t \phi \, d\phi
\]

\[
I_{4m, 4t} = 4 \int_{0}^{\pi/4} \sec^4 \phi \cos 4m \phi \cos 4t \phi \, d\phi
\]

\[
J_{4m, 4t} = 4 \int_{0}^{\pi/4} \cos^4 \phi \cos 4m \phi \cos 4t \phi \, d\phi
\]

(3.13)
The equations (3.9) and (3.10) then reduce to

\[
\sum_{m=1}^{\infty} F_{4m} I_{4m,0} + \sum_{m=1}^{\infty} G_{4m} J_{4m,0} = E K_0 - \frac{1}{2} \alpha^2 L_0 + \pi \delta_0 = 0
\]

and

\[
\sum_{m=1}^{\infty} F_{4m} I_{4m,4t} + \sum_{m=1}^{\infty} G_{4m} J_{4m,4t} = E K_{4t} - \frac{1}{2} \alpha^2 L_{4t} = 0
\]

where \( t = 1, 2, 3, \ldots \). Making use of equations (3.8) these become

\[
\sum_{m=1}^{\infty} F_{4m} (I_{4m,0} - c_{4m} J_{4m,0}) = E K_0 - \frac{1}{2} \alpha^2 L_0 + \pi \delta_0 = 0
\]

and

\[
\sum_{m=1}^{\infty} F_{4m} (I_{4m,4t} - c_{4m} J_{4m,4t}) = E K_{4t} - \frac{1}{2} \alpha^2 L_{4t} = 0
\]

\(( t = 1, 2, \ldots )\)
These may be conveniently written as
\[ \sum_{m=1}^{\infty} \beta_{4m} F_{4m} - \int E K_0 - \frac{1}{2} \alpha^2 L_0 + \pi D_0 = 0 \]  \hspace{1cm} (3.14)

and
\[ \sum_{m=1}^{\infty} 4^t \beta_{4m} F_{4m} - \int E K_4t - \frac{1}{2} \alpha^2 L_4t = 0 \]  \hspace{1cm} (3.15)

\[ (t = 1, 2, \ldots, \infty) \]

and \[ 4^t \beta_{4m} \] is given by
\[ 4^t \beta_{4m} = I_{4m, 4t} - \alpha^8 m J_{4m, 4t} \]  \hspace{1cm} (3.15a)

\[ (t = 0, 1, 2, \ldots, \infty) \]

In order to determine \( \psi_0 \) of equation (3.3) we shall have to consider the circulation of shearing stress round the internal contour. The proper relation of circulation is given by
\[ \oint T_s \, ds = \varepsilon \frac{\mu}{\alpha} \]
where

\[ A = \text{Area included inside the contour}, \]

\[ \mu = \text{Co-efficient of rigidity}, \]

\[ \alpha = \text{Twist per unit length}, \]

and

\[ T_8 = T_{2\phi} \]
\[ = -\mu \alpha \left( \frac{\partial \psi}{\partial \rho} - \varphi \right) \]
\[ = -\mu \alpha \left[ \frac{E}{\rho} + \sum_{m=1}^{\infty} F_{4m} \cdot 4m \rho \cos 4m \varphi \right] \]
\[ - \sum_{m=1}^{\infty} G_{4m} \cdot \frac{4m \rho}{\rho^{4m+1}} \cos 4m \varphi - \varphi \]

So, we have

\[ E = \frac{c^2}{a^2} - \frac{c^2}{a^2} \]

(8.16)
Putting this value of $E$ in equation (3.7), (3.8), (3.14) and (3.15). We have finally

$$D_0 + \left(\frac{c^2}{a^2} - \frac{c'^2}{a'^2}\right) \log a = \frac{1}{2} c^2 + \psi_0 \quad (3.17)$$

$$F_{4m} \cdot a^{4m} + \frac{G_{4m}}{a^{4m}} = 0 \quad (3.18)$$

$$\sum_{m=1}^{\infty} \beta_{4m} F_{4m} \left(\frac{c^2}{a^2} - \frac{c'^2}{a'^2}\right) k_0 = -\frac{1}{8} a^2 L_0 + \pi D_0 = 0 \quad (3.19)$$

$$\sum_{m=1}^{\infty} 4^t B_{4m} F_{4m} \left(\frac{c^2}{a^2} - \frac{c'^2}{a'^2}\right) k_4^t = -\frac{1}{8} a^2 L_4 + \frac{3}{2} \pi \tilde{D}_0 \quad (3.20)$$

Solving the equations (3.20) we get the co-efficients $F_{4m}$. When the co-efficients $F_{4m}$ have been obtained, co-efficients $G_{4m}$, $D_0$ and $\psi_0$ are obtained from equations (3.18), (3.19) (3.17) respectively. The function $\psi$ is then completely known. The torsion function $\Phi$ which is complex conjugate of $\psi$ is then given by
\[ \Phi = E \varphi + \sum_{m=1}^{\infty} F_{4m} \frac{\varphi^{4m}}{\lambda_m^{4m}} \varphi + \sum_{m=1}^{\infty} G_{4m} \frac{\varphi^{4m}}{\rho_{4m}} \]  

(3.21)

For the case \( e = 0.5 \text{cm} \) and \( \alpha = 1 \text{cm} \), \( \Phi = \) constant lines which are shear stress lines are shown in Figure 3.2.

The series obtained is quickly converging near the outer boundary except near the corners. But it is slowly converging near the inner boundary in which case co-efficients are to be evaluated taking into consideration larger number of terms.