CHAPTER V

TWO PARAMETER LIE GROUPS AND CANONICAL VARIABLES ASSOCIATED WITH SOME SPECIAL FUNCTIONS
5.0 INTRODUCTION

We have considered the following two parameters

\[ R = \xi_1(x,y) - \frac{\partial}{\partial x} + \eta_1(x,y) \frac{\partial}{\partial y} + \lambda_1(x,y) \]

and

\[ L = \xi_2(x,y) - \frac{\partial}{\partial x} + \eta_2(x,y) \frac{\partial}{\partial y} + \lambda_2(x,y) \]

Such that \( [R,L] = RL - LR \neq 0 \) and \( L \neq P(x,y)R \), in order to find their canonical forms in Lie's sense by suitable transformation of variables. Two special cases viz.

(i) \( \lambda_2(x,y) \equiv 0 \)

and

(ii) \( \lambda_1(x,y) \equiv 0 \)

\( \lambda_2(x,y) \equiv 0 \) are also considered for the application of Lie's canonical variables to the Bessel function of the first kind and to the Bessel polynomials due to H.L. Krall and O. Frink owing to the fact that a two parameter Lie group generated by \( R \) and \( L \) such that \( [R,L] = 0 \) and \( L \neq P(x,y)R \), can be constructed for each of special functions under consideration.
Group theoretic study of Bessel function of the first kind was made by B. Kaufman [1] by means of the relations.

\begin{align*}
(5.1.1) & \quad \mathcal{R}(\exp(i\phi) J_n(r)) = \exp(i(n+1)\phi) J_{n+1}(r) \\
(5.1.2) & \quad \mathcal{L}(\exp(i\phi) J_n(r)) = \exp(i(n-1)\phi) J_{n-1}(r)
\end{align*}

where

\begin{align*}
\mathcal{R} &= \exp(i\phi) \left( - \frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \\
\mathcal{L} &= \exp(-i\phi) \left( - \frac{i}{r} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r} \right)
\end{align*}

and \([\mathcal{R}, \mathcal{L}] = \mathcal{R}\mathcal{L} - \mathcal{L}\mathcal{R} = 0\), \(\mathcal{L} \neq \mathcal{P}(r, \phi) \mathcal{R}\).

In connection with simple Bessel polynomials S.K. Chatterjea [4] considered a two parameter Lie group by means of the relations

\begin{align*}
(5.1.3) & \quad \mathcal{R}(\exp(i(n + \frac{1}{2})\phi) Z_n(r)) = \exp(i \frac{2n+3}{2} \phi) Z_{n+1}(r) \\
(5.1.4) & \quad \mathcal{L}(\exp(i(n + \frac{1}{2})\phi) Z_n(r)) = \exp(i \frac{2n-1}{2} \phi) Z_{n-1}(r)
\end{align*}

where

\begin{align*}
\mathcal{R} &= \exp(i\phi) \left( - \frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \\
\mathcal{L} &= \exp(-i\phi) \left( - \frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right)
\end{align*}
and \( Z_n(r) = r^{-1/2} e^r y_n \left( \frac{1}{r} \right) \).

\[ [R, L] = 0 \quad , \quad L \neq P(r, \phi) R \]

In this chapter we wish to point out that study of the above two parameter Lie groups can be much more simplified if one uses Lie's canonical form of such groups. For this purpose, let us consider, in general, the following two operators:

\[ R = \xi^1(x, y) \frac{\partial}{\partial x} + \eta^1(x, y) \frac{\partial}{\partial y} + \lambda^1(x, y) \]
\[ L = \xi^2(x, y) \frac{\partial}{\partial x} + \eta^2(x, y) \frac{\partial}{\partial y} + \lambda^2(x, y) \]

such that \([R, L] = 0 \) and \( L \neq P(x, y) R \).

According to Lie, a set of variables \( \bar{x}, \bar{y} \) can be determined so that \( R \) takes the form \( R = \frac{\partial}{\partial \bar{x}} \).

If then the resulting form of \( L \) is

\[ L = \bar{\xi}^2 \frac{\partial}{\partial \bar{x}} + \bar{\eta}^2 \frac{\partial}{\partial \bar{y}} + \bar{\lambda}^2 (\bar{x}, \bar{y}) \]

then

\[ [R, L] f = \frac{\partial \bar{\xi}^2}{\partial \bar{x}} \frac{\partial f}{\partial \bar{x}} + \frac{\partial \bar{\eta}^2}{\partial \bar{x}} \frac{\partial f}{\partial \bar{y}} + \frac{\partial \bar{\lambda}^2}{\partial \bar{x}} f. \]

But \([R, L] \equiv 0 \).

Thus we have

\[ \frac{\partial \bar{\xi}^2}{\partial \bar{x}} = 0 ; \quad \frac{\partial \bar{\eta}^2}{\partial \bar{x}} = 0 ; \quad \frac{\partial \bar{\lambda}^2}{\partial \bar{x}} = 0. \]
so that \( \bar{\xi}_2, \bar{\eta}_2 \) and \( \bar{\lambda}_2 \) are at most functions of \( \bar{y} \).

Let us put

\[
L = \bar{\xi}_2 (\bar{y}) - \frac{\partial}{\partial \bar{x}} \bar{\eta}_2 (\bar{y}) - \frac{\partial}{\partial \bar{y}} + \bar{\lambda}_2 (\bar{y}).
\]

Now the operator \( R = -\frac{\partial}{\partial \bar{x}} \) remains unchanged due to change of variables of the type

\[
X = \bar{x} - \phi (\bar{y})
\]

\[
Y = \psi (\bar{y}),
\]

where \( \phi (\bar{y}) \) and \( \psi (\bar{y}) \) are at our disposal.

This change of variables causes \( Lf \) to take the form

\[
\left[ \bar{\xi}_2 (\bar{y}) - \bar{\eta}_2 (\bar{y}) \phi'(\bar{y}) \right] \frac{\partial f}{\partial \bar{x}} + \bar{\eta}_2 (\bar{y}) \psi'(\bar{y}) \frac{\partial f}{\partial \bar{y}} + \bar{\lambda}_2 (\bar{y}) f.
\]

If we choose \( \phi = \int \frac{\bar{\xi}_2 (\bar{y})}{\bar{\eta}_2 (\bar{y})} \, d\bar{y} \), and \( \psi = \int \frac{d\bar{y}}{\bar{\eta}_2 (\bar{y})} \),

then \( L \) assumes the form \( -\frac{\partial}{\partial \bar{y}} + \bar{\lambda}_2 (\bar{y}) \).

In particular, if \( \lambda_2 (x,y) = 0 \), then \( L \) assumes the form \( -\frac{\partial}{\partial \bar{y}} \). On the other hand, the operators (5.1.5) and (5.1.6) can be reduced to the form

\[
R = -\frac{\partial}{\partial \bar{x}}, \quad L = -\frac{\partial}{\partial \bar{y}}
\]

provided \( X \) satisfies
\[ \xi_1 \frac{\partial x}{\partial x} + \eta_1 \frac{\partial x}{\partial y} + \lambda_1 x = 1 \]  
(5.1.7) 

\[ \xi_2 \frac{\partial x}{\partial x} + \eta_2 \frac{\partial x}{\partial y} + \lambda_2 x = 0 \]

and \( y \) satisfies

\[ \xi_1 \frac{\partial y}{\partial x} + \eta_1 \frac{\partial y}{\partial y} + \lambda_1 y = 0 \]  
(5.1.8) 

\[ \xi_2 \frac{\partial y}{\partial x} + \eta_2 \frac{\partial y}{\partial y} + \lambda_2 y = 1 \]  

In particular, if \( \lambda_1 (x, y) = 0 \) and \( \lambda_2 (x, y) \equiv 0 \) then (5.1.7) and (5.1.8) reduce to the forms

\[ \xi_1 \frac{\partial x}{\partial x} + \eta_1 \frac{\partial x}{\partial y} = 1 \]  
(5.1.9) 

\[ \xi_2 \frac{\partial x}{\partial x} + \eta_2 \frac{\partial x}{\partial y} = 0 \]

\[ \xi_1 \frac{\partial y}{\partial x} + \eta_1 \frac{\partial y}{\partial y} = 0 \]  
(5.1.10) 

\[ \xi_2 \frac{\partial y}{\partial x} + \eta_2 \frac{\partial y}{\partial y} = 1 \]
5.2 BESSEL FUNCTION OF THE FIRST KIND FROM THE VIEW-POINT OF LIE’S CANONICAL VARIABLES

In [3] we notice that

\[ R(e^{i\phi} J_n(r)) = e^{i(n+1)\phi} J_{n+1}(r) \]

and

\[ L(e^{i\phi} J_n(r)) = e^{i(n-1)\phi} J_{n-1}(r) \]

where

\[ R = e^{i\phi} \left( -\frac{1}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]

\[ L = e^{-i\phi} \left( -\frac{1}{r} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r} \right) \]

Evidently \([R,L] = 0\) and \(L \neq P(r,\phi)R\). Thus using the formula (5.1.1) to (5.1.4) we obtain the following canonical variables (which is not considered in [3])

\[ \Phi = -\frac{r}{2} e^{-i\phi}; \quad R = \frac{r}{2} e^{i\phi}; \]

so that

\[ R = \frac{\partial}{\partial \Phi}, \quad L = \frac{\partial}{\partial R}. \]

We note that

\[ r = 2(-\Phi R)^{1/2} \]

\[ e^{i\phi} = (-R/\Phi)^{1/2}. \]
If we put
\[ F_n(\Phi, R) = (-\frac{R}{\Phi})^{n/2} J_n(2\sqrt{-\Phi R}) , \]
we have on using canonical form (5.2.2)
\[ e^{aR} F_n(\Phi, R) = F_n(\Phi + a, R) , \]
on the other hand
\[ e^{aR} F_n(\Phi, R) = \sum_{m=0}^{\infty} \frac{a^m}{m!} F_{n+m}(\Phi, R) \]
Equating these two results and translating back to the original variables we get
\[ (5.2.4) \quad (1 - \frac{2a}{r} e^{i\phi})^{-n/2} J_n(\sqrt{r^2 - 2\alpha e^{i\phi} r}) \]
\[ = \sum_{m=0}^{\infty} \frac{(a e^{i\phi})^m}{m!} J_{n+m}(r) \]
which can be written in the elegant form
\[ (5.2.5) \quad (1 - \frac{2\rho}{r})^{-n/2} J_n(\sqrt{r^2 - 2\rho r}) \]
\[ = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} J_{n+m}(r) \]
Similarly applying the operator \( e^{aL} \) on \( F_n(\Phi, R) \), we obtain
\[ (5.2.6) \quad (1 + \frac{2\rho}{r})^{n/2} J_n(\sqrt{r^2 + 2\rho r}) \]
\[ = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} J_{r-m}(r) \]
5.3 BESSSEL POLYNOMIALS FROM THE VIEW-POINT OF LIE'S CANONICAL VARIABLES

In [4] we notice that

\[ R(e^{i(n+\frac{1}{2})\phi} Z_n(r)) = e^{i(-\frac{n+3}{2})\phi} Z_{n+1}(r) \]

and

\[ L(e^{i(n+\frac{1}{2})\phi} Z_n(r)) = e^{i(-\frac{n-1}{2})\phi} Z_{n-1}(r) \]

where

\[ R = e^{i\phi} \left( -\frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]

\[ L = e^{-i\phi} \left( \frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]

and

\[ Z_n(r) = r^{-\frac{1}{2}} e^{-r} y_n \left( \frac{1}{r} \right) \]

and \( y_n(x) \) is the Bessel polynomial satisfying the differential equation

\[ x^2 \frac{d^2 u}{dx^2} + 2(x+1) \frac{du}{dx} = n(n+1)u \]

Evidently \( [R,L] = 0 \) and \( L \neq P(r,\phi) R \).

So using the formula (5.1.1) to (5.1.4) we obtain the following canonical variables

\[ (5.3.1) \quad \phi = -\frac{r}{2} e^{-i\phi} \quad ; \quad R = -\frac{r}{2} e^{i\phi} \]
so that

\[(5.3.2) \quad R = \frac{\partial}{\partial \Phi} , \quad L = \frac{\partial}{\partial R} . \]

We note that

\[(5.3.3) \quad r = 2 \left( \frac{\Phi R}{2} \right)^{1/2} , \quad e^{i\phi} = \left( \frac{R}{\Phi} \right)^{1/2} \]

If we put

\[F_n(\phi, R) = (4\Phi R)^{-1/4} \left( \frac{R}{\Phi} \right)^{(2n+1)/4} \]

\[\times \exp(-2\sqrt{\Phi R}) \quad y_n \left( \frac{1}{2} \sqrt{\frac{1}{\Phi R}} \right) \]

we have after applying the operator \( e^{aR} \) upon \( F_n(\phi, R) \) and translating back to the original variables

\[(5.3.4) \quad (1 - \frac{2a}{r} \quad e^{i\phi} \quad -(n+1)/2 \quad \exp \left[ r - \sqrt{r^2 - 2ar e^{i\phi}} \right] \]

\[\times y_n \left( \frac{-1}{\sqrt{r^2 - 2ar e^{i\phi}}} \right) = \sum_{m=0}^{\infty} \frac{a^m}{m!} \quad e^{im\phi} \quad y_{n+m} \left( \frac{-1}{r} \right) \]

which can be written in elegant form

\[(5.3.5) \quad (1 - \frac{2a}{r} \quad -(n+1)/2 \quad \exp \left[ r - \sqrt{r^2 - 2ar} \right] \quad y_n \left( \frac{-1}{\sqrt{r^2 - 2ar}} \right) \]

\[= \sum_{m=0}^{\infty} \frac{a^m}{m!} \quad y_{n+m} \left( \frac{-1}{r} \right) \]

Similarly, applying the operator \( e^{aL} \) on \( F_n(\phi, R) \), we obtain
(5.3.6) \( (1 - \frac{2\rho}{r})^{n/2} \exp \left[ r - \sqrt{r^2 - 2\rho r} \right] \)

\[ x \, y_n \left( \frac{r}{\sqrt{r^2 - 2\rho r}} \right) = \sum_{m=0}^{\infty} \frac{r^m}{m!} \, y_{n-m} \left( \frac{1}{r} \right) \]

where it should be remembered that definition of Bessel polynomials can be extended to negative subscripts by defining \( y_{-n}(x) \) to \( y_{n-1}(x) \). It may be pointed out that the formula (5.3.5) and (5.3.6) are analogous to (4.2.5) and (4.2.6) respectively.
REFERENCES


