CHAPTER VI

GROUP-THEORETIC STUDY OF
MODIFIED BESSEL FUNCTION
FROM DIFFERENT POINTS OF VIEW

(A) FROM THE VIEW-POINT OF KAUFMAN
(B) FROM THE VIEW-POINT OF LIE'S CANONICAL VARIABLES
(C) FROM THE VIEW-POINT OF WEISNER
6.0 INTRODUCTION

Here we have studied modified Bessel function from three different points of view, viz.

(A) from Kaufman's viewpoint, where group theoretic study starts with the two fundamental differential difference relations suitable for constructing the raising and lowering operators by the interpretation of the index of the function.

(B) from the viewpoint of Lie's canonical variables, where the raising and lowering operators are brought into canonical (or normal) forms, so that the action of operators upon any function can be easily visualized.

(C) from Weisner's viewpoint, where the raising and lowering operators, which commute with the partial differential operator obtained by suitable interpretation of the index from the ordinary linear differential equation satisfied by the function under consideration, are directly obtained.

Although the modified Bessel function $I_n(r)$ is related to $J_n(r)$ by means of the relation

$$I_n(r) = i^{-n} J_n(ir),$$

yet many physical problems lead to the study of Bessel function of purely imaginary argument and therefore we like to make a direct group-theoretic study of the said function to obtain the generating of functions.
6.1 MODIFIED BESSEL FUNCTION FROM THE VIEW-POINT OF KAUFMAN

In this section [1] we like to show that group-theoretic study of modified Bessel function can be made from the viewpoint of B. Kaufman [2].

The group-theoretic study of Bessel function of the first kind was made by Kaufman by means of differential recursion relations

\[(\frac{n-1}{r} - \frac{d}{dr}) J_n(r) = J_{n+1}(r)\]

\[(\frac{n}{r} + \frac{d}{dr}) J_n(r) = J_{n-1}(r)\]

From which the Bessel differential equation follows:

\[\left[ -\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} + 1 \right] J_n(r) = 0\]

In [3] we have shown that the differential recursion relations for modified Bessel functions are

\[(-\frac{d}{dk} - \frac{n}{k}) I_n(k) = I_{n+1}(k)\]

(6.1.1)

\[(-\frac{d}{dk} + \frac{n}{k}) I_n(k) = I_{n-1}(k)\]

From which the differential equation follows

(6.1.2) \[\left( -\frac{d^2}{dk^2} + \frac{1}{k} \frac{d}{dk} - \frac{n^2}{k^2} - 1 \right) I_n(k) = 0\]
This differential equation will not play any role in the following consideration and is presented only as an aid in identifying the function.

In order to obtain $R$ and $L$ in a form which does not refer to index of the modified Bessel function, we interpret $n$ as a result of operating with $-i \frac{\partial}{\partial \Phi}$ upon the function $e^{in\phi}$. So that we have

$$e^{i\phi} \left( -\frac{\partial}{\partial k} + \frac{i}{k} \frac{\partial}{\partial \Phi} \right) \left[ e^{in\phi} I_n(k) \right]$$

$$= \left[ e^{i(n+1)\phi} I_{n+1}(k) \right]$$

$$e^{-i\phi} \left( -\frac{\partial}{\partial k} - \frac{1}{k} \frac{\partial}{\partial \Phi} \right) \left[ e^{in\phi} I_n(k) \right]$$

$$= \left[ e^{i(n+1)\phi} I_{n-1}(k) \right].$$

The raising and lowering operators are thus (expressed in polar co-ordinates).

$$R = e^{i\phi} \left( -\frac{\partial}{\partial k} + \frac{i}{k} \frac{\partial}{\partial \Phi} \right)$$

(6.1.4)

$$L = e^{-i\phi} \left( -\frac{\partial}{\partial k} - \frac{i}{k} \frac{\partial}{\partial \Phi} \right)$$

Expressed in cartesian co-ordinates they become

$$R = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$$

(6.1.5)

$$L = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}.$$
Here \([\mathcal{R}, \mathcal{L}] = 0\), hence the composition law within Lie group is additive:

\[
\exp c\mathcal{R} \cdot \exp \beta \mathcal{L} = \exp (c\mathcal{R} + \beta \mathcal{L}).
\]

We now proceed to consider some properties for the function:

\[
F_n \equiv \left[ e^{in\phi} I_n(k) \right]
\]

which we consider alternately as functions of polar or of cartesian co-ordinates. For dealing with polar coordinates it proves convenient to introduce the variables

\[
u = x + iy = ke^{i\phi}
\]

\[
v = x - iy = ke^{-i\phi}
\]

in terms of which

\[
\mathcal{R} = 2 \frac{\partial}{\partial v} ; \quad \mathcal{L} = 2 \frac{\partial}{\partial u}.
\]

In the space of \(u\) and \(v\), the finite operators \(\exp(c\mathcal{R})\) and \(\exp(\beta \mathcal{L})\) are translation operators. \(\exp(c\mathcal{R})\) sends \(v\) into \((v+2a)\) leaving \(u\) invariant, while \(\exp(\beta \mathcal{L})\) send \(u\) into \((u+2\beta)\) leaving \(v\) invariant.

Operating on the function of \(F_n(u,v)\) we have on the one hand,

\[
\exp c\mathcal{R} \cdot F_n(u,v) = \exp \left[ 2a \left( -\frac{\partial}{\partial v} \right) \right] F_n(u,v)
\]

\[
= F_n(u, v+2a)
\]
and on the other hand,

$$\exp aR \cdot F_n(u,v) = \sum_{m=0}^{\infty} \frac{a^m}{m!} F_{n+m}(u,v)$$

Combining these we get

$$(6.1.9) \quad F_n(u,v+2a) = \sum_{m=0}^{\infty} \frac{a^m}{m!} F_{n+m}(u,v).$$

We may now re-write (6.1.9) in the original variables $((k,\phi))$. To do this we note the relation

$$e^{i\phi} = \left(\frac{-u}{v}\right)^{1/2}; \quad k = (uv)^{1/2}$$

$$u = ke^{i\phi}$$

$$(6.1.10) \quad v = ke^{-i\phi}$$

Thus

$$(6.1.11) \quad F_n(u, v+2a) = \left\{ e^{in\phi'} \right\} [I_n(k)]$$

Putting $2au = h$ we have from (6.1.9), (6.1.10) and (6.1.11)

$$\frac{u^n}{(k^2 + h)^{n/2}} \frac{1}{m!} \left[ e^{i(n+m)\phi} I_{n+m}(k) \right].$$
In other words,

\[(6.1.12) \quad (k^2 + h)^{-n/2} I_n[(k^2 + h)^{1/2}] \]

\[= \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{h}{2}\right)^m k^{m+n} I_{n+m}(k)\]

Similarly using the operator \( L \) we obtain the companion equation

\[\exp(\beta L) F_n(u,v) = \exp \left[ 2\beta \left(-\frac{\partial}{\partial u}\right) \right] F_n(u,v) = F_n(u + 2\beta, v)\]

On the other hand,

\[\exp(\beta L) F_n(u,v) = \sum_{m=0}^{\infty} \frac{\beta^m}{m!} F_{n-m}(u,v)\]

\[\therefore F_n(u + 2\beta, v) = \sum_{m=0}^{\infty} \frac{\beta^m}{m!} F_{n-m}(u,v)\]

Thus

\[F_n(u + 2\beta, v) = \left[ e^{i\beta^2} \right] I_n(k') \]

\[= \left(-\frac{u+2\beta}{v}\right)^{n/2} I_n\left[ \left\{(u+2\beta)\right\}^{1/2} \right] \]

\[= \left(-\frac{uv+2\beta v}{\beta^2 v^2}\right)^{n/2} I_n\left[ \left\{(uv+2\beta v)\right\}^{1/2} \right] \]

Putting \( 2\beta v = h \), we get

\[(6.1.13) \quad (k^2 + h)^{n/2} I_n[(k^2 + h)^{1/2}] \]

\[= \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{h}{2}\right)^m k^{n-m} I_{n-m}(k)\]
Now we consider the combination
\[ \frac{1}{2} (R + L) = -\frac{\partial}{\partial x} \cdot \]

This is a translation operator in the plane of \((x, y)\).

\[ \exp \left( a\left( -\frac{\partial}{\partial x} \right) \right) x = x + a \]

\[ \exp \left( a\left( -\frac{\partial}{\partial x} \right) \right) y = y \cdot \]

By previous reasoning, we find

\[
\left[ \exp \left( \frac{1}{2} a (R + L) \right) \right] F_n(x, y)
\]

\[ = \exp \left( a\left( -\frac{\partial}{\partial x} \right) \right) F_n(x, y) = F_n(x + a, y) \]

But on the other hand

\[
\left[ \exp \left( \frac{1}{2} a (R + L) \right) \right] F_n(x, y)
\]

\[ = \exp \left( \frac{1}{2} a R \right) \exp(\frac{1}{2} a L) \cdot F_n(x, y) \]

\[ = \sum_{t=0}^{\infty} \frac{1}{t!} (-\frac{a}{2})^t (-\frac{a}{2})^s \sum_{s=0}^{\infty} (-\frac{a}{2})^s F_{n+t-s}(x, y) \]

\[ = \sum_{t-s-m=-\infty}^{\infty} F_{n+m}(x, y) \left[ \sum_{t=0}^{\infty} \frac{1}{t!} (-\frac{a}{2})^{2t-m} \right] \]

Equating these results and returning to the polar coordinates we arrive at

\[ (6.1.14) \quad e^{i\phi} I_n(k') = \sum_{m=-\infty}^{\infty} e^{i(n+m)\phi} I_{n+m}(k) \]

\[ \times \sum_{t=0}^{\infty} \frac{1}{(t-m)! t!} (-\frac{a}{2})^{2t-m} \]
It contains the well-known series expansion of \( I_n(k) \) as a special case. If we specialise (6.1.14) for the case \( k = \phi = 0 \) which implies \((k = a, \ \phi' = 0)\), we obtain

\[
I_n(a) = \sum_{m=-\infty}^{\infty} I_{n+m}(0) \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} \left(-\frac{a}{2}\right)^{2t-m}
\]

But from the recursion relation we see that \( I_n(o) = 0 \) for every \( r \) and except \( r = 0 \). Hence if \( n \) is an integer [normalising \( I_0(o) = 1 \),

\[
I_n(a) = \sum_{t=0}^{\infty} \frac{1}{(t+n)! t!} \left(-\frac{a}{2}\right)^{2t+n}
\]

Introducing \( I_n(a) \) in (6.1.14) instead of series expansion, we find analogous graf's addition theorem

\[
e^{in(\phi'-\phi)} I_n(k') = \sum_{m=-\infty}^{\infty} e^{im\phi} I_{n+m}(k) I_{-m}(a)
\]

\[
e^{im\phi} I_{n+m}(k) I_{m}(a)
\]

\[
I_{-m}(a) = I_m(a) \text{ for integer } m
\]

To find further expansion, we make use of the most general operator available to us \((aL + \beta R)\).

Introducing

\[
\frac{1}{2} z = (a\phi)^{1/2} \text{ and } \gamma = \left(-\frac{a}{\beta}\right)^{1/2}
\]
we have 

\[ (6.1.18) \quad \exp(aL + \beta R) = \exp \left[ -\frac{z}{2} (YL + \frac{R}{Y}) \right] = \sum_{s,t=0}^{\infty} \frac{1}{s! t!} \left( -\frac{z}{2} \right)^{s+t} Y^s L^s \gamma^t R^t \]

\[ = \sum_{s,t=0}^{\infty} \frac{1}{s! t!} \left( -\frac{z}{2} \right)^{s-t} Y^{s-t} L^s R^t \]

We can now make use of the fact that the operators of \( R \) and \( L \) on the function 

\[ F_m(u,v) = [e^{\im \phi} I_m(k)] \]

cancel each other out:

\[ R \cdot L \cdot F_m = F_m = L \cdot R \cdot F_m \]

Hence operator \( R \) may be replaced by \( L^{-1} \) out of which

\[ \exp(aL + \beta R) = \sum_{p=-\infty}^{\infty} \sum_{t=0}^{\infty} \frac{1}{(p+t)!} \left( -\frac{z}{2} \right)^{p+2t} \gamma^p L^p \]

or by equation (6.1.16)

\[ (6.1.19) \quad \exp(aL + \beta R) = \sum_{p=-\infty}^{\infty} \gamma^p I_p(z) L^p \]

Using this operator equation we obtain the addition theorem

\[ (6.1.20) \quad F_m(u', v') = \exp(aL + \beta R) F_m(u,v) \]

\[ = \sum_{p=-\infty}^{\infty} \gamma^p I_p(z) F_{m-p}(u,v) \]
where \( u' = u + 2a \); \( v' = v + 2\beta \).

Translating back from the co-ordinates \((u,v)\) to the polar co-ordinates, equation (6.1.20) takes the form

\[
(6.1.21) \quad \left(\frac{ke^{i\phi} + 2a}{ke^{-i\phi} + 2\beta}\right)^{m/2} \text{Im}\left\{\left[(ke^{i\phi} + 2a)(ke^{-i\phi} + 2\beta)\right]^{1/2}\right\}
\]

\[
= \sum_{p=-\infty}^{\infty} \left(\frac{a}{\beta}\right)^{p/2} I_p\left[2(a\beta)^{1/2}\right] e^{i(m-p)\phi} I_{m-p}(k)
\]

We may now make various specialisations of our general addition theorem.

Choosing \( \alpha = \beta = z/2 \), equation (6.1.21) becomes

\[
(6.1.22) \quad \left(\frac{ke^{i\phi} + z}{ke^{-i\phi} + z}\right)^{m/2} \text{Im}(Z) = \sum_{p=-\infty}^{\infty} I_p(z) \times e^{i(m-p)\phi} I_{m-p}(k)
\]

where

\[
Z = \left[(ke^{i\phi} + 2a)(ke^{-i\phi} + 2\beta)\right]^{1/2}
\]

\[
= (k^2 + z^2 + 2kz \cos \phi)^{1/2} \quad \text{and} \quad 2(a\beta)^{1/2} = 2a = z .
\]

In particular for \( m = 0 \),

\[
(6.1.23) \quad I_0(Z) = \sum_{p=-\infty}^{\infty} e^{-ip\phi} I_p(z) I_p(k)
\]

Choosing \( \alpha = \frac{1}{2} ke^{i\phi} t^2 \) and \( \beta = \frac{1}{2} k e^{-i\phi} t^2 \), we get from (6.1.21)
\[
(e^{i\Phi})^m I_m [r(1+t^2)] = \sum_{p=-\infty}^{\infty} e^{i\phi p} I_p(r^2) e^{i(m-p)\phi} I_{m-p}(k)
\]

or

\[
(6.1.24) \quad I_m [r(1+t^2)] = \sum_{p=-\infty}^{\infty} I_p(r^2) I_{m-p}(k)
\]

It may be of interest to note that the operator equation (6.1.19) may be applied to any function on which \( R \) and \( L \) cancel each other. Thus

\[
\frac{1}{2} L \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right] = a \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right]
\]

where

\[
\frac{1}{2} R \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right] = \frac{1}{a} \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right]
\]

so that \( R/2 \) acts as a reciprocal of \( L/2 \) on the function \( \exp(\alpha v + \frac{-u}{\alpha}) \).

We may use the operator equation (6.1.19) to obtain

\[
(6.1.25) \quad \left[ \exp \frac{1}{2} a (L + R) \right] \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right]
\]

\[
= \sum_{p=-\infty}^{\infty} I_p(2a) (L/2)^p \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right]
\]

\[
= \sum_{p=-\infty}^{\infty} I_p(2a) a^p \left[ \exp(\alpha v + \frac{-u}{\alpha}) \right]
\]

on the other hand,
\[(6.1.26) \quad [\exp \frac{1}{2} a (L + R)] [\exp (av + \frac{u}{a})]\]

\[= \exp \left[ a(v + a) + \frac{u + c}{a} \right] \]

\[= \exp \left[ (av + \frac{u - c}{a}) + a (a + \frac{1}{a}) \right] \]

\[= [\exp a (a + \frac{1}{a})] [\exp (av + \frac{u}{a})] \]

Equating these two, we get

\[\exp \left[ a (a + \frac{1}{a}) \right] = \sum_{p=-\infty}^{\infty} a^p I_p (2a) .\]
6.2 MODIFIED BESSEL FUNCTION FROM THE VIEW-POINT OF LIE'S CANONICAL VARIABLES

In this section [4] we have shown that group theoretic study for modified Bessel function can be made from the viewpoint of canonical variables.

B. Kaufman [2] considered group theoretic study for Bessel function of first kind by means of relation

\[ \mathcal{R} \left[ \exp(i\phi) J_n(r) \right] = \exp \left[ i(n+1)\phi \right] J_{n+1}(r) \]

\[ \mathcal{L} \left[ \exp(i\phi) J_n(r) \right] = \exp \left[ i(n-1)\phi \right] J_{n-1}(r) \]

where

\[ \mathcal{R} = \exp(i\phi) \left( -\frac{1}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]

and

\[ \mathcal{L} = \exp(-i\phi) \left( -\frac{1}{r} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r} \right) \]

and

\[ [\mathcal{R},\mathcal{L}] = \mathcal{R}\mathcal{L} - \mathcal{L}\mathcal{R} \equiv 0 \]

where \[ L \neq \mathcal{P}(r,\phi)\mathcal{R}. \]

In connection with simple Bessel polynomial S.K. Chatterjea [5] considered a two parameter Lie group which is shown by the relation

\[ \mathcal{R} \left[ \exp \left( i(n+\frac{1}{2}) \right) Z_n(r) \right] = \exp \left[ i\left( -\frac{2n+3}{2} \right) \phi \right] Z_{n+1}(r) \]
\[ L \left[ \exp(i(n+\frac{1}{2})Z_n(r)) \right] = \exp\left[ i\left(-\frac{2n-1}{2}\right)\phi \right] Z_{n-1}(r) \]

where
\[ R = \exp(i\phi) \left( \frac{-i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]
\[ L = \exp(-i\phi) \left( \frac{i}{r} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \]

and \( Z_n(r) = r^{-1/2} e^{-r} y_n\left(\frac{1}{r}\right) \)

\[ [R;L] = 0 \quad , \quad L \neq P(r,\phi) R \]

Here we like to point out that study of two parameter Lie group can be simplified by using Lie's canonical form with illustration to modified Bessel function.

For modified Bessel function we notice that

\[(6.2.1) \quad R \left[ \exp(in\phi) I_n(k) \right] = \exp(i(n+1)\phi) I_{n+1}(k) \]

\[(6.2.2) \quad L \left[ \exp(in\phi) I_n(k) \right] = \exp(i(n-1)\phi) I_{n-1}(k) \]

where
\[ R = e^{i\phi} \left( -\frac{\partial}{\partial k} + \frac{i}{k} - \frac{\partial}{\partial \phi} \right) \]
\[ (6.2.3) \quad L = e^{-i\phi} \left( \frac{\partial}{\partial k} - \frac{i}{k} - \frac{\partial}{\partial \phi} \right) \]

where
\[ [R;L] = 0 \quad , \quad L \neq P(k,\phi) R \]

Thus using the formula (6.2.1) and (6.2.2) we obtain the following canonical variables
\[
\Phi = \frac{k}{2} e^{-i\phi} \\
R = \frac{k}{2} e^{i\phi}
\]

(6.2.4)

So that

\[
(6.2.5) \quad R = \frac{1}{2} \frac{\partial}{\partial \phi} ; \quad L = \frac{1}{2} \frac{\partial}{\partial R}
\]

We note that

\[
k = 2(\Phi R)^{1/2}
\]

\[
e^{i\Phi} = \left(\frac{R}{\Phi}\right)^{1/2}
\]

If we put

\[
F_n(\Phi, R) = (-\frac{R}{\Phi})^{n/2} I_n \left(2\sqrt{\Phi R}\right),
\]

we have on using canonical forms (6.2.5)

(6.2.6) \quad e^{cR} F_n(\Phi, R) = F_n(\Phi + a, R)

on the other hand

(6.2.7) \quad e^{cR} F_n(\Phi, R) = \sum_{m=0}^{\infty} \frac{a^m}{m!} F_{n+m}(\Phi, R)

\[
\therefore F_n(\Phi + a, R) = \sum_{m=0}^{\infty} \frac{a^m}{m!} F_{n+m}(\Phi, R)
\]

Now, translating back to the original variables, we get

(6.2.8) \quad (1 + \frac{2e^{i\phi}}{k} e^{i\phi})^{-n/2} I_n \left(\sqrt{k^2 + 2ake^{i\phi}}\right)

\[
= \sum_{m=0}^{\infty} \frac{(a e^{i\phi})^m}{m!} I_{n+m}(k)
\]
which can be written in elegant form

\[(6.2.9) \quad (1 + \frac{2\rho}{k})^{-n/2} I_n \left( \sqrt{k^2 + 2\rho^2} \right) \]

\[= \sum_{m=0}^{\infty} \frac{(ae^{i\phi})^m}{m!} \, I_{n+m}(k) \]

\[= \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \, I_{n+m}(k) \]

where \( \rho = ae^{i\phi} \), which may be easily compared with the result (6.1.12).

Similarly applying \( e^{aL} \) on \( F_n(\Phi, R) \) we get

\[(6.2.10) \quad e^{aL} F_n(\Phi, R) = F_n(\Phi, R + a) \]

on the other hand,

\[(6.2.11) \quad e^{aL} F_n(\Phi, R) = \sum_{m=0}^{\infty} \frac{a^m}{m!} \, F_{n+m}(\Phi, R) \]

\[(6.2.12) \quad F_n(\Phi, R + a) = \sum_{m=0}^{\infty} \frac{a^m}{m!} \, F_{n-m}(\Phi, R) \]

Translating to the original variables we get,

\[(6.2.13) \quad (1 + \frac{2a}{k} e^{-i\phi})^{n/2} I_n \left( \sqrt{k^2 + 2a e^{-i\phi}} \right) \]

\[= \sum_{m=0}^{\infty} \frac{(ae^{-i\phi})^m}{m!} \, I_{n-m}(k) \]

which can be put in elegant form
(6.2.14) \[ (1 + \frac{-2P'}{k^2})^{n/2} I_n (\sqrt{k^2 + 2kP'}) \]
where \( P' = ae^{-i\phi} \), which may be easily compared with the result (6.1.13).

Operating the function of \( (\Phi, R) \) we have on the one hand,

(6.2.15) \[ \exp(-\frac{\alpha^2}{2} L) \exp(-\frac{\alpha^2}{2} R) F_n (\Phi, R) = \exp(-\frac{\alpha^2}{2} L) F_n (\Phi + \frac{\alpha^2}{2}, R) = F_n (\Phi + \frac{\alpha^2}{2}, R + \frac{\alpha^2}{2}) \]
on the other hand,

(6.2.16) \[ \exp(-\frac{\alpha^2}{2} L) \exp(-\frac{\alpha^2}{2} R) F_n (\Phi, R) = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{t!s!} (-\frac{\alpha^2}{2})^{t+s} F_{n+t-s} (\Phi, R) \]
\[ = \sum_{t-s=m=-\infty}^{\infty} F_{n+m} (\Phi, R) \left[ \sum_{t=0}^{\infty} \frac{1}{t!(t-m)!} \left( \frac{\alpha^2}{2} \right)^{2t-m} \right] \]

(6.2.17) \[ F_n (\Phi + \frac{\alpha^2}{2}, R + \frac{\alpha^2}{2}) = \sum_{m=-\infty}^{\infty} F_{n+m} (\Phi, R) \left[ \sum_{t=0}^{\infty} \frac{1}{t!(t-m)!} \left( \frac{\alpha^2}{2} \right)^{2t-m} \right] \]
Now lefthand side of (6.2.17) is

\[ F_n (\Phi + \frac{a}{2}, R + \frac{a}{2}) = \left( \frac{R + a/2}{\Phi + a/2} \right)^{n/2} I_n \left( 2 \sqrt{(\Phi + a/2)(R + a/2)} \right) \]

\[ = \left( \frac{ke^{i\Phi} + a}{ke^{-i\Phi} + a} \right)^{n/2} I_n \left( 2 \sqrt{kr \pm (e^{i\Phi} \pm e^{-i\Phi}) + a^2} \right) \]

and righthand side of (6.2.17) is

\[ \sum_{m=-\infty}^{\infty} F_{n+m} (\Phi, R) \left[ \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} \left( -\frac{a}{2} \right)^{2t-m} \right] \]

\[ = \sum_{m=-\infty}^{\infty} \left( \frac{R}{\Phi} \right)^{n+m} I_{n+m} \left( 2 \sqrt{\Phi R} \right) \left[ \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} \left( -\frac{a}{2} \right)^{2t-m} \right] \]

\[ = \sum_{m=-\infty}^{\infty} e^{i(n+m)\Phi} I_{n+m}(k) \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} \left( -\frac{a}{2} \right)^{2t-m} \]

Thus we get

\[ (6.2.18) \quad \left( \frac{ke^{i\Phi} + a}{ke^{-i\Phi} + a} \right)^{n/2} I_n \left( \sqrt{k^2 + a^2 + ak (e^{i\Phi} + e^{-i\Phi})} \right) \]

\[ = \sum_{m=-\infty}^{\infty} e^{i(n+m)\Phi} I_{n+m}(k) \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} \left( -\frac{a}{2} \right)^{2t-m} \]
which may be compared with the result (6.1.14) with a slight computation. In fact, from the following figure, we notice that

\[ x = k \cos \phi \]
\[ y = k \sin \phi \]

so that in (6.1.14), we have

\[ (k')^2 = y^2 + (x + a)^2 \]

\[ k' = \sqrt{(k \sin \phi)^2 + (k \cos \phi + a)^2} \]
\[ = \sqrt{k^2 + a^2 + 2ka \cos \phi} \]

Again in (6.1.14) we have

\[ e^{i \phi'} = (e^{2i \phi'})^{n/2} \]

Now

\[ e^{2i \phi'} = \cos 2\phi' + i \sin 2\phi' \]
\[ = \frac{1 - \tan^2 \phi'}{1 + \tan^2 \phi'} + i \frac{2 \tan \phi'}{1 + \tan^2 \phi'} \]

But

\[ \tan \phi' = \frac{y}{x + a} \]

\[ e^{2i \phi'} = \frac{(k \cos \phi + a)^2 - (k \sin \phi)^2 + 12k \sin \phi(k \cos \phi + a)}{(k')^2} \]
\[
\begin{align*}
\frac{(k^2 \cos^2 \phi + a^2 - k^2 \sin^2 \phi + 2ka \cos \phi) + i(k^2 \sin 2\phi + 2ka \sin \phi)}{k^2 + a^2 + 2ka \cos \phi} & = \frac{(k^2 \cos 2\phi + a^2 + 2ak \cos \phi) + i(k^2 \sin 2\phi + 2ka \sin \phi)}{k^2 + a^2 + 2ka \cos \phi} \\
& = \frac{k(\cos 2\phi + i \sin 2\phi) + a(\cos \phi + i \sin \phi)}{k + a (\cos \phi + i \sin \phi)} \\
& = \frac{ke^{2i\phi} + ae^{i\phi}}{ke^{-i\phi} + a} = \frac{ke^{i\phi} + a}{ke^{-i\phi} + a}.
\end{align*}
\]

\[
\sin' = (e^{2i\phi'})^{n/2} = \left(\frac{ke^{i\phi} + a}{ke^{-i\phi} + a}\right)^{n/2}.
\]
In this section [6] we have shown that group-theoretic study for modified Bessel function can be made from the viewpoint of Weisner [7].

The recurrence relations for modified Bessel functions are

\[(6.3.1) \quad \frac{d}{dx} I_n(x) = -\frac{n}{x} I_n(x) + I_{n-1}(x)\]

and

\[(6.3.2) \quad \frac{d}{dx} I_n(x) = -\frac{n}{x} I_n(x) + I_{n+1}(x)\]

These two independent differential recurrence relations determine the linear ordinary differential equations

\[(6.3.3) \quad \left(-\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{n^2}{x^2} - 1\right) I_n(x) = 0\]

Replacing \(-\frac{d}{dx}\) by \(\frac{\partial}{\partial x}\), \(n\) by \(y \frac{\partial}{\partial y}\), and \(I_n(x)\) by \(u(x,y)\) we get

\[(6.3.4) \quad \left[x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - y^2 \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial y} - x^2 \right] u(x,y) = 0\]

For simplicity of notation we shall write

\[L' = x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - y^2 \frac{\partial^2}{\partial y^2} - y \frac{\partial}{\partial y} - x^2\]
We now seek linear differential operators which will commute with \( I_n \).

Every linear differential operator of the first order generates one parameter Lie group. If we know the linear differential operators, we shall be able to determine the extended form of group generated by the operators.

Let us consider two linear differential operators \( B \) and \( C \) such that

\[
(6.3.6) \quad B \left[ I_n(x) y^n \right] = b_n I_{n-1}(x) y^{n-1}
\]

and

\[
(6.3.6) \quad C \left[ I_n(x) y^n \right] = c_n I_{n+1}(x) y^{n+1}
\]

where \( b_n \) and \( c_n \) are independent of \( x \) and \( y \).

Let \( B = B_1(x,y) \frac{\partial}{\partial x} + B_2(x,y) \frac{\partial}{\partial y} + B_0(x,y) \)

where each \( B_i \) \((i = 1,2,0)\) is a function of \( x \) and \( y \) but independent of \( n \). With the aid of (6.3.1) we have

\[
(6.3.6) \quad B\left[ I_n(x) y^n \right] = B_1 y^n \left[ - \frac{n}{x} I_n(x) + I_{n-1}(x) \right] + B_2 I_n(x) \left[ n y^{n-1} \right] + B_0 I_n(x) y^n
\]

To make the coefficient of \( I_{n-1}(x) y^{n-1} \) free from \( x \) and \( y \) we put \( B_1 = y^{-1} \). Then (6.3.6) becomes
(6.3.7) \[ B \left[ I_n(x) \, y^n \right] = - y^{n-1} \frac{n}{x} I_n(x) + I_{n-1}(x) \, y^{n-1} \]
+ \[B_2 I_n(x) \, n \, y^{n-1} + B_0 I_n(x) \, y^n\]

\[ = I_{n-1}(x) \, y^{n-1} + I_n(x) \, y^n \left[ y^{-1} \left( n \, B_2 - \frac{n}{x} \right) + B_0 \right] \]

In (6.3.7) we can make the coefficient of \( I_n(x) \, y^n \) equal to zero by choosing \( B_2 = \frac{1}{x} \) and \( B_0 = 0 \). Therefore,

(6.3.8) \[ B = y^{-1} \frac{\partial}{\partial x} + x^{-1} \frac{\partial}{\partial y} \]

and

(6.3.9) \[ B \left[ I_n(x) \, y^n \right] = I_{n-1}(x) \, y^{n-1} \]

Similarly, let the operator \( C \) is given by

(6.3.10) \[ C = C_1(x,y) - \frac{\partial}{\partial x} + C_2(x,y) \frac{\partial}{\partial y} + C_0(x,y) \]

with the aid of (6.3.2) we have

(6.3.11) \[ C \left[ I_n(x) \, y^n \right] = C_1 y^n \left[ - \frac{n}{x} I_n(x) + I_{n+1}(x) \right] \]
+ \[C_2 I_n(x) \left( n \, y^{n-1} \right) + C_0 I_n(x) \, y^n \]

To make the coefficient of \( I_{n+1}(x) \, y^{n+1} \) a function of \( n \) only, we choose \( C_1 = y \). Then (6.3.11) becomes

(6.3.12) \[ C \left[ I_n(x) \, y^n \right] = I_{n+1}(x) \, y^{n+1} + I_n(x) \, y^n \]
\[ \times \left[ y - \frac{n}{x} + C_2 \, n \, y^{-1} + C_0 \right] . \]
Since we require \( C_2 \) and \( C_0 \) be functions of \( x \) and \( y \) which are independent of \( n \) we choose \( C_2 = -y^2/x \) and \( C_0 = 0 \) in order to make the coefficient of \( I_n(x) y^n \) equal to zero. Then

\[
(6.3.13) \quad C = y \frac{\partial}{\partial x} - y^2 x^{-1} \frac{\partial}{\partial y}
\]

and

\[
(6.3.14) \quad C \left[ \frac{I_n(x)}{y^n} \right] = I_{n+1}(x) y^{n+1}.
\]

Let \( A = y \frac{\partial}{\partial y} \). We will use the commutator notations \( [A,B] \) with

\[
[A,B] u = (AB-BA)u
\]

we find that

\[
ABu = -y^{-1} \frac{\partial u}{\partial x} + x^{-1} y \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial x}
\]

and

\[
BAu = \frac{\partial^2 u}{\partial x \partial y} + x^{-1} \frac{\partial u}{\partial y} + x^{-1} y \frac{\partial^2 u}{\partial y^2}.
\]

Then for \( \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 u}{\partial x \partial y} \) we have

\[
[A,B] u = -(x^{-1} \frac{\partial u}{\partial y} + y^{-1} \frac{\partial u}{\partial x}) = -Bu
\]

\[
[A,B] = -B.
\]

In a similar manner we find that

\[
[A,C] = C \quad \text{and} \quad [B,C] = 0.
\]
Therefore, these operators generate a three parameter Lie group.

From the computation of \([B,C]\) we find that

\[
BC - 1 = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial}{\partial x} - y x^2 \frac{\partial}{\partial y} - y^2 x^2 \frac{\partial^2}{\partial y^2} - 1
\]

\[
= \frac{1}{x^2} \left[ x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - y^2 \frac{\partial^2}{\partial y^2} - x^2 \right]
\]

\[
= x^{-2} L'
\]

\[
\therefore BC - 1 = x^{-2} L'.
\]

By using the commutator relations we prove that

\[
\begin{align*}
[A (x^{-2}L') - (x^{-2}L') A] u &= [A (BC-1) - (BC-1)A] u \\
&= [ABC - BCA] u \\
&= [(BA-B)C - B(AC-C)] u \\
&= [0] u
\end{align*}
\]

Thus A commute with \(x^{-2}L'.\)

Similarly

\[
\begin{align*}
[B (x^{-2}L') - (x^{-2}L')B] u &= [B(BC-1) - (BC-1)B] u \\
&= [B(BC - CB)] u \\
&= [0] u
\end{align*}
\]

\[
\therefore B commute with x^{-2}L'.
\]
In a same manner for the operator $C$, we have

$$\left[ C(x^{-2} L) - (x^{-2} L) C \right] u = \left[ C(BC-I) - (BC-I) C \right] u$$

$$= \left[ (CB - BC) C \right] u$$

$$= \left[ 0 \right] u .$$

Thus using the commutator relations it is proved that operators $A, B, C$ commute with $x^{-2} L$.

We now express the extended form of the group generated by the operators $C$ and $B$.

First we observe that the differential operator

$$C = y \frac{\partial}{\partial x} - y^2 x^{-1} \frac{\partial}{\partial y}$$

is the same as $E$.

Let us take

$$(6.3.15) \quad E = y \frac{\partial}{\partial x} - y^2 x^{-1} \frac{\partial}{\partial y} .$$

Now we choose two variables $\xi$ and $\eta$ such that $E$ is transformed into the operator $\frac{\partial}{\partial \xi}$.

Thus it is sufficient to solve the following pair of partial differential equations

$$(6.3.16) \quad E\eta = 0 \quad \text{and} \quad E\xi = 1 .$$
For the general integral of $E\eta = 0$, we get

$$y \frac{\partial \eta}{\partial x} - y^2 x^{-1} \frac{\partial \eta}{\partial y} = 0$$

or,

$$x \frac{\partial \eta}{\partial x} - y \frac{\partial \eta}{\partial y} = 0.$$

The corresponding subsidiary equations are

(6.3.17) \quad \frac{dx}{x} = \frac{dy}{-y} \quad ; \quad d\eta = 0

The general integral of $E\eta = 0$ is

(6.3.18) \quad \eta = \sigma (xy), \quad \text{where} \ \sigma \ \text{is arbitrary.} \ \text{So we choose} \ \eta = xy.$$

Again for the general integral of $E\xi = 1$, we have

$$y \frac{\partial \xi}{\partial x} - y^2 x^{-1} \frac{\partial \xi}{\partial y} = 1$$

or,

$$x \frac{\partial \xi}{\partial x} - y \frac{\partial \xi}{\partial y} = xy^{-1}.$$

The corresponding subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{d\xi}{xy^{-1}}.$$

The general integral of $E\xi = 1$ is

(6.3.19) \quad \sigma (xy, xy^{-1} - 2\xi) = 0,
where $\sigma$ is arbitrary. So we choose $\zeta = x/2y$.

In other word we have,

\[(6.3.20) \quad x = \sqrt{2\eta \zeta} \quad \text{and} \quad y = \sqrt{\eta/2\zeta}.\]

So we can express the extended form of the group generated by the operator $C$ as follows:

For any arbitrary function $f(x,y)$ we get

$$e^C f(x,y) = e^{\mathcal{E}} f(x,y).$$

The substitution $x = \sqrt{2\eta \zeta}$ and $y = \sqrt{\eta/2\zeta}$ transforms $\mathcal{E}$ into $\mathcal{D}$ where $\mathcal{D} = \partial/\partial \zeta$. Making this substitution we have,

\[(6.3.21) \quad e^C f(x,y) = e^{\mathcal{D}} f(\sqrt{2\eta \zeta}, \sqrt{\eta/2\zeta}) = f(\sqrt{2\eta(\zeta+c)}, \sqrt{\eta/2(\zeta+c)}) = f(\sqrt{2xy(\frac{x}{2y} + c)}, \sqrt{\frac{xy}{2(x/2y+c)}}).\]

Now consider the case 1. $b = 0, c = 1$.

Since for any arbitrary function we have shown

\[(6.3.22) \quad e^C f(x,y) = f((x(x+2y))^{1/2}, y(\frac{x}{x+2y})^{1/2}),\]

we find
(6.3.23) \[ e^C \left[ y^n I_n(x) \right] = \left[ y^n \left( \frac{-x}{x+2y} \right)^{n/2} \right] I_n\left((x(x+2y))^{1/2}\right) \]

Also since
\[ C \left[ y^n I_n(x) \right] = y^{n+1} I_{n+1}(x) \]

we have

(6.3.24) \[ e^C \left[ y^n I_n(x) \right] = \sum_{k=0}^{\infty} \frac{-1}{k!} I_{n+k}(x) y^{n+k} \]

By equating the expressions for \( e^C \left[ y^n I_n(x) \right] \), we get

\[ \left[ y^n \left( \frac{-x}{x+2y} \right)^{n/2} \right] I_n\left((x(x+2y))^{1/2}\right) = \sum_{k=0}^{\infty} \frac{-1}{k!} I_{n+k}(x) y^{n+k} \]

or,
\[ \frac{-x}{x+2y}^{n/2} I_n\left((x(x+2y))^{1/2}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n+k}(x) y^k \]

or,
\[ \frac{x+2y}{x}^{-n/2} I_n\left(\sqrt{x^2+2xy}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n+k}(x) y^k \]

or,

(6.3.25) \[ (1 + \frac{2y}{x})^{-n/2} I_n\left(\sqrt{x^2+2xy}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n+k}(x) y^k \]

which may be easily compared with the result (6.1.12) or (6.2.9).
Next we consider the differential operator
\[ B = y^{-1} \frac{\partial}{\partial x} + x^{-1} \frac{\partial}{\partial y} \]
which is the same as
\[ (6.3.26) \quad E = y^{-1} \frac{\partial}{\partial x} + x^{-1} \frac{\partial}{\partial y} . \]

Now we choose the variables \( \xi \) and \( \eta \) such that \( E \) is transformed into the operator \( \partial / \partial \xi \). Thus it is sufficient to solve the following pair of partial differential equations
\[ E^\eta = 0 \quad \text{and} \quad E^\xi = 1 . \]

For the general integral \( E^\eta = 0 \), we get
\[ y^{-1} \frac{\partial \eta}{\partial x} + x^{-1} \frac{\partial \eta}{\partial y} = 0 \]
or,
\[ x \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} = 0 . \]

The corresponding subsidiary equations
\[ \frac{dx}{x} = -\frac{dy}{y} ; \quad d\eta = 0 . \]

the general integral of \( E^\eta = 0 \) is
\[ (6.3.27) \quad \eta = \sigma \left( \frac{x}{y} \right) \]
where \( \sigma \) is arbitrary. So we choose \( \eta = x/y \).

Again for the general integral for \( E^\xi = 1 \), we have
The corresponding subsidiary equations are,

\[
\frac{dx}{x} - \frac{dy}{y} = \frac{d\xi}{\xi}
\]

or,

\[
x \frac{d\xi}{dx} + y \frac{d\xi}{dy} = xy
\]

The corresponding subsidiary equations are,

\[
-x \frac{d\xi}{dx} + y \frac{d\xi}{dy} = \frac{d\xi}{\xi}
\]

\[\text{the general integral for } E_\xi = 1 \text{ is}
\]

\[
(6.3.28) \quad \sigma \left( \frac{x}{y} , \, xy - 2\xi \right) = 0
\]

where \(\sigma\) is arbitrary. So we choose \(\xi = xy/2\).

In otherword we have

\[
x = \sqrt{2\eta \xi} \quad \text{and} \quad y = \sqrt{2\xi/\eta}
\]

So we can express the extended form of group generated by the operator \(B\) as follows:

For the arbitrary function \(f(x,y)\), we have

\[
e^{Bf} f(x,y) = e^{Ef} f(x,y)
\]

The substitution \(x = \sqrt{2\eta \xi}\) and \(y = \sqrt{2\xi/\eta}\) transforms \(E\) into \(D\) where \(D = \partial/\partial \xi\). Making this substitution, we have

\[
(6.3.29) \quad e^{Bf} f(x,y) = e^{Df} f(\sqrt{2\eta \xi} , \sqrt{2\xi/\eta})
\]

\[= f(\sqrt{2\eta(\xi+b)} , \sqrt{2(\xi+b)/\eta})\]
\[ f(\sqrt{\frac{2x}{y}} (-\frac{xy}{2} + b), \sqrt{2(-\frac{xy}{2} + b)/ -\frac{x}{y}}) \]
\[ = f(\sqrt{xy^{-1} (xy+2b)}, \sqrt{yx^{-1} (xy+2b)}) \]

Now we consider the case 2. \( b = 1, c = 0 \).

Since for arbitrary function \( f(x,y) \)

\[ e^B f(x,y) = f(\sqrt{xy^{-1} (xy+2)}, \sqrt{yx^{-1} (xy+2)}) \]

we find

\[ e^B \left[ y^n I_n(x) \right] = (yx^{-1} (xy+2))^{n/2} I_n(\sqrt{yx^{-1} (xy+2)}) \]  

(6.3.30)

Also since

\[ B \left[ y^n I_n(x) \right] = y^{n-1} I_{n-1}(x) \]

we have

\[ e^B \left[ y^n I_n(x) \right] = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n-k}(x) y^{n-k} \]  

(6.3.31)

Equating these two expressions for \( e^B \left[ y^n I_n(x) \right] \), we get

\[ (yx^{-1} (xy+2))^{n/2} I_n(\sqrt{yx^{-1} (xy+2)}) \]

(6.3.32)

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n-k}(x) y^{n-k} \]

or

\[ (xy)^{-n/2} (xy+2)^{n/2} I_n(\sqrt{xy^{-1} (xy+2)}) \]

(6.3.33)

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} I_{n-k}(x) y^{-k} \]
or,

\[(6.3.33) \quad (1 + \frac{2}{xy})^{\nu/2} \quad I_n \left( \sqrt{\frac{xy-1}{xy}} (xy+2) \right) \]

\[= \sum_{k=0}^{\infty} \frac{1}{k!} \quad I_{n-k}(x) \quad y^{-k} \]

which may be easily compared with the result \((6.1.13)\) or \((6.2.14)\).

Now for any arbitrary function \(f(x,y)\) we have

\[e^{cC} f(x,y) = f(\sqrt{x(x+2cy)}, \quad y \sqrt{x(x+2cy)}) \]

and

\[e^{bB} f(x,y) = f(\sqrt{xy^{-1} (xy+2b)}, \quad \sqrt{yx^{-1} (xy+2b)}) \]

Then

\[(6.3.34) \quad e^{bB} e^{cC} f(x,y) \]

\[= e^{bB} f(\sqrt{x(x+2cy)}, \quad y \sqrt{x(x+2cy)}) \]

\[= f(\sqrt{y^{-1} (xy+2b)(x+2cy)}, \quad \sqrt{\frac{xy+2b}{x+2cy}}) \]

We now consider the case 3. \(bc \neq 0\), \(b = c = \frac{1}{2} w\).

Since for any arbitrary function \(f(x,y)\)

\[e^{\frac{1}{2} wB} e^{\frac{1}{2} wC} f(x,y) \]

\[= f(\sqrt{y^{-1} (xy+w)(x+wy)}, \quad \sqrt{\frac{xy+w}{x+wy}}) \]
we have

\[
(6.3.35) \quad \left( \frac{1}{e^2} w_B + \frac{1}{e^2} w_C \right) [ y^n I_n(x) ]
\]

\[
= \left[ y^{n/2} \left( \frac{xy+w}{x+wy} \right)^{n/2} \right] I_n \left( \sqrt{y}^{-1} (xy+w)(x+wy) \right)
\]

Again

\[
(6.3.36) \quad \left( \frac{1}{e^2} w_B + \frac{1}{e^2} w_C \right) [ y^n I_n(x) ]
\]

\[
= \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \frac{1}{t! s!} (-w)^t \left[ y^n I_n(x) \right] y^{n+t-s}
\]

\[
= \sum_{t-s=m=-\infty}^{\infty} \left[ y^n I_n(x) \right] y^{n+m} \left[ \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} (-w)^{2t-m} \right]
\]

From (6.3.35) and (6.3.36) it follows that

\[
(6.3.37) \quad \left[ y^{n/2} \left( \frac{xy+w}{x+wy} \right)^{n/2} \right] I_n \left( \sqrt{y}^{-1} (xy+w)(x+wy) \right)
\]

\[
= \sum_{m=-\infty}^{\infty} \left[ y^n I_n(x) \right] y^{n+m} \left[ \sum_{t=0}^{\infty} \frac{1}{t! (t-m)!} (-w)^{2t-m} \right]
\]

This result may be easily compared with the result (6.2.18) or (6.1.14).
## References

1. T.D. Banerjee: Group theoretic study of modified Bessel function from the viewpoint of Kaufman — Communicated.


4. T.D. Banerjee: Group theoretic study from the viewpoint of Lie's canonical variables — Communicated.


6. T.D. Banerjee: Group theoretic study of modified Bessel function from the viewpoint of Weisner — Communicated.