CHAPTER THREE

ON SOME INTEGRAL TRANSFORM TECHNIQUES

AND CERTAIN APPLICATIONS IN RELIABILITY
Many of the phenomena of classical physics may be described by partial differential equations of simple type. For example, problems in potential theory, which embraces gravitational theory, electrostatics, magnetostatics, and the irrotational flows of a perfect fluid may be discussed with reference to either Laplace's equation or Poisson's equation, and similarly wave propagation and the diffusion phenomena may be described by wave equation and diffusion equation respectively. Transform techniques are commonly used in finding the solution of the boundary and initial value problems for ordinary and partial differential equations, integral equations and difference equations, e.g. momentum wave function in quantum mechanics is calculated by Dirac transformation theory. Thus the solution of a boundary or initial value problem for a partial differential equation reduces to the solution of an ordinary differential equation virtually or an algebraic equation. The use of Kontrovich-Lebedev transform has been common, which arise through the method of separation of variables for solution of boundary value problems in terms of a system of cylindrical coordinates, and similarly Mehler-Fock transform has been considerably used in axisymmetric potential and cracks problems of elasticity in terms of toroidal coordinates. Kontrovich-Lebedev transform utilizes the properties of modified Bessel function of the second kind $K_V(x)$ and MacDonald function $K_{i\tau}(x)$, whereas Mehler-Fock transform utilizes the ideas of Legendre function $P_V(x)$ and associated Legendre function $P_M(x)$ of the first kind of order $\nu$ and degree $M$. Lowndes [1964] developed integral transform as a study of boundary value problems of the wave and
diffusion equations involving wage and conically shaped boundaries in which the variable of integrals appeared as the order of Bessel and Legendre function, and a Parseval relation was also obtained for that type of transforms. Mandal obtained a transform associated with integration with respect to degree of Legendre's associated function from a boundary value problem for a complete interval and obtained a Parseval relation for the transform developed in this paper. In the first sub-section of section - I, a more general integral transform associated with respect to the degree of Legendre's associated function for a broken interval has been developed, and a Parseval relation for the transform has been obtained. Finally a particular integral using the transform developed has been evaluated. In the second sub-section of this section, we evaluate an integral involving Meijer-G-function and an associated Legendre function in the range 0 to $\sqrt{2}$, where a similar integral has been evaluated by Saxena [1961] in a different range.

As a part of interest in the report of the thesis towards reliability of digital machines, we have discussed in the first sub-section of section-II, different types of temporary and permanent faults which may occur in digital machines pertaining to different logic gates, and then discussed a standard technique for its identification using fault tables. In the second sub-section, we have developed an application of inverse Laplace transform in the problem of finding the probability factor of the failure rate of the output from a priority-Nand logic gate where the inputs arrive sequentially. A formula for the probability factor
has been obtained in this sub-section, and a comparison of time dependent exact and approximate solution has been given in Table - 3.7. Fussel et al [1976] derived a similar formula for the probability factor with priority-AND gate using inverse Laplace transform. A priority-NAND logic gate is logically equivalent to a NAND gate, where the input events occur sequentially with the order of occurrence as unique, and symbolised in Boolean logic as $\triangle$. Evidently, priority-NAND logic gate is more general than priority-AND gate as shown in fig - 3.1 (a) through fig - 3.4 (b) in the next page, and thus the formula for probability factor established in this section is more general and also the results obtained in table - 3.7 is more desirable than in Fussel et al [1976].
FIG: 3-1(a) (b)

FIG: 3-2(a) (b)

FIG: 3-3(a) (b)

FIG: 3-3(c) (d)

FIG: 3-4(a) (b)

EXCLUSIVE OR
A, B → A ⊕ B
SECTION - I

(i) NOTE ON AN INTEGRAL TRANSFORMS

(ii) A DEFINITE INTEGRAL INVOLVING ASSOCIATED LEGENDRE FUNCTION OF THE FIRST KIND
The LAPLACE's equation $\nabla^2 u = 0$ in the region $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < \alpha$, reduces to

$$s(s+1)u + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

--- (3.1)

where $u = \nu r^{-s-1}$, $Re s > -1$ taking $u \to 0$ as $r \to \infty$

The boundary conditions which $u(\theta, \phi)$ satisfies are,

$$u = f(\theta), \quad \phi = 0, \quad 0 \leq \theta \leq \alpha < \pi$$

$$u = 0, \quad \phi = \beta, \quad 0 \leq \theta \leq \alpha < \pi$$

Let

$$u_n = \int_0^\beta u \sin \lambda \varphi \, d\varphi$$

where

$$\lambda = \frac{2n\pi}{\beta}, \quad u = \frac{2}{\beta} \sum_{n=1}^\infty u_n(\theta) \sin \lambda \varphi$$

From (3.1), we see that $u_n(\theta)$ satisfies the equation,

$$\frac{d}{d\theta} \left( \sin \theta \frac{d u_n}{d\theta} \right) + \left[ s(s+1) \sin \theta - \frac{\lambda^2}{\sin \theta} \right] u_n = -\frac{\lambda f(\theta)}{\sin \theta}$$

--- (3.2)

The solution of (3.2) is

\[ \psi(\theta) = -\chi \int_{0}^{\chi} \frac{f(\theta')}{\sin \theta'} \ G(\theta, \theta') \ d\theta \]  

--- (3.3)

where \( G(\theta, \theta') \) is the Green's function corresponding to eqn.(3.2) and is given when \( G=0 \) on \( \Theta = 0 \) and \( \Theta = \alpha \):

\[
G(\theta, \theta') = -\frac{\pi}{2} \frac{\rho(s+1+\lambda)}{\rho(s+1-\lambda)} \frac{P_s^{\lambda}(\cos \theta)}{\sin(s+1-\lambda) \Pi}
\]

\[ \times \left[ P_s^{\lambda}(\cos \theta) \cdot P_s^{\lambda}(-\cos \alpha) - P_s^{\lambda}(-\cos \theta) \cdot P_s^{\lambda}(\cos \alpha) \right] \]

--- (3.4a)

To write Green's function for the region \( 0 < \theta' < \theta < \alpha \) and \( \theta \) and \( \theta' \) are to be interchanged.

Now \( G \) can be written as

\[
\hat{G}(\theta, \theta') = \frac{1}{2i} \int \frac{\rho(s+1+M)}{\rho(s+1-M)} \frac{P_s^{-M}(\cos \theta)}{\sin(s+1-M) \Pi}
\]

\[ \times P_s^{\lambda}(\cos \theta) \cdot P_s^{\lambda}(-\cos \alpha) - P_s^{\lambda}(-\cos \theta) \cdot P_s^{\lambda}(\cos \alpha) \]

\[ \times \frac{M \ dM}{M^{\chi - \lambda^2}} \]  

--- (3.4b)

where \( L \) is the straight line \( \text{Re} \ M = c \), \( -\lambda < s < c < \lambda \)  

(Taking \( |\text{Re} \ s| < \lambda \)). To show this, we observe that the integrand has singularities only at \( M = \pm \lambda \). When \( \text{Re} \ M \) is large, the integrand is

\[
\sim \frac{1}{2M} e^{-\frac{M}{2}} e^{-M^2} \]
where
\[
\gamma = \ln \left( \frac{\tan \theta/2}{\tan \theta'/2} \right)
\]
and
\[
\delta = \ln \left( \frac{\tan \theta/2 \cdot \tan \theta'/2}{\tan \alpha/2 \cdot \tan \alpha'/2} \right)
\]
and when \( \theta < \theta' < \alpha \) both \( \gamma \) and \( \delta \) are negative, so that
the integrand vanishes as \( |M| \to \infty \), \( \text{Re} \ M > 0 \). Again if
\( \theta' < \theta < \alpha \), \( \theta \) and \( \theta' \) are interchanged and we have then
\[
\gamma = \ln \left( \frac{\tan \theta'/2}{\tan \theta/2} \right)
\]
and
\[
\delta = \ln \left( \frac{\tan \theta'/2 \cdot \tan \theta/2}{\tan \alpha/2 \cdot \tan \alpha'/2} \right)
\]
so that both \( \gamma \) and \( \delta \) are again negative. Hence in both the
cases the integrand tends to zero as \( |M| \to \infty \), \( \text{Re} \ M > 0 \).

Now, taking a large semicircle in the positive \( M \) plane
with \( L \) as its diameter, the only singularity of the integrand
is at \( M = \lambda \), and we see that the integrand in (3.4b) reduces
to (3.4a). So (3.4a) is written in terms of a contour integral
in (3.4b).

Substituting in (3.3) from (3.4b),
\[
\mathcal{V}_n(\theta) = -\frac{\lambda}{2i} \int \frac{\Gamma(s+1+M)}{\Gamma(s+1-M)} \frac{P_s^M(\cos \theta)}{\sin(s+1-M)\Pi} \left( \frac{M \, dM}{M^2 - \lambda^2} \right) F(M)
\]
where,

\[
F(M) = \alpha \int_{0}^{\alpha} \frac{f(\theta')}{\sin \theta'} \, d\theta'
\]

\[
x \left[ \frac{p_{s}^{-m}(\cos \theta') \cdot p_{s}^{-m}(-\cos \theta) - p_{s}^{-m}(-\cos \theta') \cdot p_{s}^{-m}(\cos \theta)}{p_{s}^{-m}(\cos \theta)} \right] \, d\theta,
\]

Hence

\[
\nu(\theta, \phi) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \nu_n \sin \lambda \phi
\]

But

\[
\frac{\sin m(\alpha - \phi)}{\sin m\alpha} = -\frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda \sin \lambda \phi}{\lambda^2 - \lambda^n}
\]

Therefore

\[
\nu(\theta, \phi) = \frac{1}{2i} \int \frac{\gamma(s+1+m) \cdot p_{s}^{-m}(\cos \theta)}{\gamma(s+1-m) \cdot \sin(s+1-m) \pi} \, dM
\]

\[
x \frac{\sin m(\alpha - \phi)}{\sin m\alpha} \cdot F(M) \cdot M \, dM
\]

As \( \nu(\theta, \phi) = f(\theta) \), when \( \phi = 0 \) we arrive at by

putting \( \phi = 0 \), as

\[
f(\theta) = \frac{1}{2\pi i} \int \frac{\gamma(s+1+m) \cdot \gamma(s-m) \cdot p_{s}^{-m}(\cos \theta) \cdot F(M) \cdot M \, dM}{L}
\]
where \( L \) is 
\[ \text{Re} M = c, -\lambda < s < c < \lambda. \]
As \( \text{Re} s > -1 \) and 
\[ \lambda = \frac{n\pi}{\beta} \quad (n = 1, 2, \ldots), \]
we may take \( c \to 0, \) so 
\[ f(\theta) = \int_{-i\infty}^{i\infty} \Gamma(s+1+M) \Gamma(M-s) P_s^{-M}(\cos \theta) F(M,s) M^s \, dM \]
--- (3.5)

where 
\[ F(M,s) = \int_0^\alpha \frac{f(\theta)}{\sin \theta} \]
\[ \left[ P_s^{-M}(\cos \theta) \cdot P_s^{-M}(-\cos \alpha) - P_s^{-M}(-\cos \theta) \cdot P_s^{-M}(\cos \alpha) \right] \]
\[ \frac{d\theta}{P_s^{-M}(\cos \alpha)} \]  
--- (3.6)

So, (3.6) gives the transform and (3.5) the inverse.

II

PERSEVAL RELATIONS

Let \( F(M,s) \) and \( G(M,s) \) be the transforms of \( f(\theta) \)
and \( g(\theta) \) respectively. To obtain Perseval relation,
we write
\[ \int_{-i\infty}^{i\infty} \gamma(M,s) F(M,s) G(M,s) M^s \, dM = \]
\[ \int_{-i\infty}^{i\infty} \gamma(M,s) F(M,s) \int_0^{\pi} \frac{g(\pi-\theta)}{\pi-\alpha} \]
\[ \left[ P_s^{-M}(\cos \theta) \cdot P_s^{-M}(-\cos \alpha) - P_s^{-M}(-\cos \theta) \cdot P_s^{-M}(\cos \alpha) \right] \]
\[ \frac{d\theta}{P_s^{-M}(\cos \alpha)} \]  
where \( \gamma(M,s) \) is to be determined. Assuming the interchange
of the order of integration to be possible, we get 
--- (3.7)
\[
\begin{align*}
\int_0^\pi \frac{g(\pi-\theta)}{\sin \theta} d\theta & \int_{-i\omega}^{i\omega} \mathcal{M} \gamma(M,s) F(M,s) \overline{P_s^M}(\cos \theta) dM \\
+ \int_0^\pi \frac{g(\pi-\theta)}{\sin \theta} d\theta & \int_{-i\omega}^{i\omega} \mathcal{M} \gamma(M,s) F(M,s) \overline{P_s^M}(\cos \theta) \\
\times \frac{P_s^{-M}(-\cos \alpha)}{P_s^{-M}(\cos \alpha)} dM
\end{align*}
\]

--- (3.8)

Now taking \( \gamma(M,s) \) as the expression

\[
\frac{1}{2\pi i} \int_{-i\omega}^{i\omega} \frac{\Gamma(s+1+M) \cdot \Gamma(M-s)}{\sin \theta} d\theta
\]

we get

\[
\int_{-i\omega}^{i\omega} \mathcal{M} \gamma(M,s) F(M,s) G(M,s) dM
\]

\[
= \int_0^\pi \frac{1}{\sin \theta} \frac{P_s^{-M}(-\cos \alpha)}{P_s^{-M}(\cos \alpha)} f(\theta) g(\theta) d\theta
\]

\[
- \int_0^\pi \frac{1}{\sin \theta} g(\pi-\theta) f(\theta) d\theta
\]

--- (3.9)

As a particular case, when \( \mathcal{L} = \pi \),

\[
\frac{1}{2\pi i} \int_{-i\omega}^{i\omega} \mathcal{M} \gamma(s+1+M) \Gamma(M-s) F(M,s) G(M,s) dM
\]

\[
= \int_0^\pi \frac{1}{\sin \theta} g(\pi-\theta) f(\theta) d\theta
\]

which is established in (3.6)
We now define a particular integral with the Parseval relation, when $\alpha = \pi/2$. If

$$f(\theta) = (\cos \theta)^{\delta} \cdot (\sin \theta)^{3/2}$$

Then

$$F(M,s) = \frac{\Gamma(M-1/2) \cdot \Gamma \left( \frac{3+2M}{4} \right) \cdot \Gamma \left( \frac{1}{2} + \frac{M}{2} \right)}{\Gamma(1-M) \cdot \Gamma \left( \frac{5}{4} + \frac{M}{2} - \frac{M}{2} \right)}$$

$$\times \left[ \frac{\sin \pi (s-M)}{\sin \pi M} \cdot \frac{\Gamma(s-M+1)}{\Gamma(s+M+1)} \right]$$

$$+ \frac{\sin \pi (s-M)}{\sin \pi M} \cdot \frac{\Gamma(s-M+1)}{\Gamma(s+M+1)}$$

$$\times F_2 \left( \frac{s-M+1}{2}, \frac{s-M}{2}, 1-M, \frac{26-2M+5}{4}, 1 \right)$$

$$\times \frac{\sin \pi (s-M)}{\sin \pi M} \cdot \frac{\Gamma(s-M+1)}{\Gamma(s+M+1)}$$

$$\times F_2 \left( \frac{s+M+1}{2}, \frac{s+M}{2}, \frac{3+2M}{4}, 1-M, \frac{26+2M+5}{4}, 1 \right)$$

--- (3.10)
Taking

\[ g(\theta) = (\cos \theta)^6 \cdot (\sin \theta)^{3/2} \]

\[ G(M, s) = \frac{2^{M-1}}{\Gamma\left(\frac{3+2M}{4}\right)} \cdot \frac{\Gamma\left(\frac{1}{2} + \frac{\beta}{2}\right)}{\Gamma\left(1-M\right) \cdot \Gamma\left(\frac{s}{4} + \frac{s}{2} - \frac{M}{2}\right)} \]

\[ \times \left[ 1 + \frac{\sin \pi (s-M)}{\sin \pi M} \cdot \frac{\Gamma(s-M+1)}{\Gamma(s+M+1)} \right] \]

\[ \times 3F_2 \left( \frac{s-M+1}{2}, \frac{s-M}{2}, 1-M, \frac{26 - 2s + s}{4}, 1 \right) \]

\[ + \frac{2^{M-1}}{\Gamma\left(\frac{3-2M}{4}\right)} \cdot \frac{\Gamma\left(\frac{1}{2} + \frac{\gamma}{2}\right)}{\Gamma\left(1+M\right) \cdot \Gamma\left(\frac{s}{4} + \frac{s}{2} + \frac{M}{2}\right)} \]

\[ \times \frac{\sin \pi (s-M)}{\sin \pi /4} \cdot \frac{\Gamma(s-M+1)}{\Gamma(s+M+1)} \]

--- (3.11)
Then the integral is given by

\[ I = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+1+m)\, \Gamma(m-s)\, \Gamma(m,s)\, \Gamma(m,s)}{\Gamma(\frac{s+6'+1}{2})} \, dm \]

\[ = \left[ 1 - (-1)^{2s+6'} \right] \cdot \frac{\Gamma\left(\frac{6+6'+1}{2}\right)}{2\Gamma\left(\frac{6+6'+1}{2} + 1\right)} \]

\[ = \left[ 1 - (-1)^{2s+6'} \right] \cdot \frac{1}{6+6'+1} \]

--- (3.12)
In this note, we derive an integral involving an associated Legendre function and an \( E \) function in the range 0 to \( \pi/2 \). The formula to be established is:

\[
\int_{0}^{\pi/2} \left( \frac{\sin \theta}{\cos \theta} \right)^{M+1} P_V^M(\cos \theta) \ \mathrm{d}\theta
\]

\[
\times \sum_{r=0}^{q} \left( \frac{\cos \theta}{\sin \theta} \right)^{r+2} \left( \frac{\sin \theta}{\cos \theta} \right)^{r+1} \frac{1}{r!} \Gamma(r+1) \Gamma(r+2)
\]

\[
= (2n)^{M-1} \sum_{r+2}^{q+2n} \left( \frac{x}{\beta_1, \ldots, \beta_5, \gamma_1, \ldots, \gamma_2n} \right)
\]

--- (3.13)

where

\[
\delta = \frac{1-6+2k}{2n}, \quad \delta = \frac{-6+2k}{2n}
\]

---

\[ Y_{k+1} = \frac{-\ell + \nu + M + 2k}{2n}, \quad Y_{n+k+1} = \frac{1 - \ell - \nu + M + 2k}{2n} \]

\( k = 0, 1, \ldots, (n-1) \), and \( n \) is a positive integer.

This result is valid under the following conditions:

i) \( 0 \leq \rho \leq s \), \( 0 \leq \nu \leq \gamma < s \)

(or \( \gamma = s \) and \( |x| < 1 \))

\[ \text{Re} \left( 1 + \ell + 2k + 2n\beta_j \right) > 0 \quad (j=1,2,\ldots,p) \]

ii) \( 0 \leq \nu \leq \gamma \), \( 1 \leq \rho \leq s < \gamma \),

\[ 2(\rho + \nu) > \gamma + s, \]

\[ |\arg x| < (\rho + \nu - \frac{1}{2} \gamma - \frac{3}{2} s) \pi \]

and

\[ \text{Re} \left( 1 + \ell + 2k + 2n\beta_j \right) > 0 \quad (j=1,2,\ldots,p) \]

In the proof we require the definition of Meijer G function as available in Erdelyi [1953], as

\[ G_{\nu_s}^{\rho q} \left( x \middle| \begin{array}{c} \alpha_1 \ldots \alpha_r \\ \beta_1 \ldots \beta_s \end{array} \right) = \]
\[
\prod_{j=1}^{p} \frac{\Gamma \left( \beta_j - \frac{s}{2} \right)}{\Gamma \left( 1 - \beta_j + \frac{s}{2} \right)} 
\prod_{j=1}^{q} \frac{\Gamma \left( 1 - \alpha_j + \frac{s}{2} \right)}{\Gamma \left( \alpha_j - \frac{s}{2} \right)}
\]

Where \( L \) is a suitable contour. The integral
\[
\int_{0}^{\pi/2} \left( \cos \theta \right)^{\delta} \left( \sin \theta \right)^{-M+1} \rho \left( \cos \theta \right) d\theta
\]

\[
= \frac{\Gamma \left( \frac{1}{2} + \frac{\delta}{2} \right) \cdot \Gamma \left( 1 + \frac{\delta}{2} \right)}{\Gamma \left( 1 + \frac{1}{2} \delta - \frac{1}{2} \nu - \frac{1}{2} M \right) \cdot \Gamma \left( \frac{1}{2} \delta + \frac{1}{2} \nu - \frac{1}{2} M + \frac{3}{2} \right)}
\]

\( \text{Re} \, M < 1, \, \text{Re} \, \delta > 1 \)

\( \cdots \) \( (3.14) \)

and
\[
\prod_{k=0}^{m-1} \Gamma \left( z + \frac{k}{m} \right) = \left( 2 \pi \right)^{\frac{1}{2} m} \cdot \Gamma \left( m z \right)
\]

as given in Erdelyi [1953].
To prove (3.13), we substitute the expression for $G$-function from (3.14) in the integrand of (3.13), change the order of integration (which we assume to be permissible without loss of generality), and make use of (3.15) to evaluate the inner integral. Then the value of the integral is given by

$$
\frac{1}{2\pi i} \int_{L} \prod_{j=1}^{S} \Gamma(1-\beta j + \xi) \cdot \prod_{j=1}^{r} \Gamma(1-\alpha j + \xi) \cdot \prod_{j=1}^{p} \Gamma(\beta j - \xi)
$$

which is equivalent to

$$
\frac{\Gamma \left( \frac{1}{2} + \frac{s}{2} + \eta \xi \right)}{\Gamma \left( \frac{1}{2} + \frac{s}{2} - \nu \xi \right)} \cdot \frac{\Gamma \left( \frac{1}{2} + \frac{s}{2} + \eta \xi \right)}{\Gamma \left( \frac{1}{2} \xi + \frac{1}{2} \nu - \frac{1}{2} M + \frac{3}{2} + \eta \xi \right)} \cdot x^{\xi} d\xi
$$

--- (3.16)
The contour $L$ is a loop starting and ending at $+\infty$ and encircling all poles $\Gamma(\beta_j + \frac{\ell}{2})$ ($j = 1, \ldots, p$) once in the negative direction, but none of the poles of $\Gamma(1 - \delta_j + \frac{\ell}{2})$ ($j = 1, \ldots, q$). Again as discussed in Erdelyi [1953], we have

$$G \frac{p}{q} \left( \begin{array}{c} \alpha_1 \ldots \alpha_r \\ \beta_1 \ldots \beta_s \end{array} \right) = G \frac{q}{p} \left( \begin{array}{c} 1 - \beta_1 \ldots 1 - \beta_s \\ 1 - \delta_1 \ldots 1 - \delta_r \end{array} \right)$$
On taking \( p = 1, q = r = 1, \alpha = m + 1, \beta_1 = 0 \) and writing \( 1 - \beta_{n-1} \) and \( 1 - \alpha_j \) for \( \beta_n \) and \( \alpha_j \) respectively \((n = 2, \ldots, m + 1, j = 1, \ldots, l)\) and using the relation c.f. Erdelyi \([1953]\), i.e.

\[
G_{\ell}^{l} m+1 (x | 1 - \alpha_1, \ldots, 1 - \alpha_l) \oint x^{1-\beta_1} \ldots 1 - \beta_m \]

\[
= E \left\{ \ell; \alpha_r : m; \beta_s : \frac{1}{x} \right\}
\]

we find that

\[
M_2 \int (\cos \theta)^{\ell} (\sin \theta)^{M+1} P_n^M E \left\{ \ell; \alpha_r : m; \beta_s : \frac{x}{\cos \theta} \right\} d\theta
\]

\[
= (2\pi)^{M-1} E \left\{ \ell + 2n; \alpha_r : m + 2n; \beta_s : x \right\}
\]

where

\[
\alpha_{\ell+K+1} = \frac{1 + 6 + 2K}{2n}, \quad \alpha_{l+n+k+1} = \frac{2 + 6 + 2K}{2n}
\]

\[
\beta_{m+k+1} = \frac{2 + 6 - \nu - M + 2K}{2n}, \quad \beta_{m+n+k+1} = \frac{6 + \nu - M + 3 + 2K}{2n}
\]

\( K = 0, 1, \ldots (n-1) \)
and

$$R(\xi) > -1,$$

$$|\arg x| < \frac{1}{2} (l-m+1) \pi, \text{ if } l > m+1;$$

$$x \neq 0 \text{ if } l < m+1$$

and $$|x| > 1 \text{ if } l = m+1$$
SECTION - II

(i) FAULTS IN DIGITAL CIRCUITS

(ii) ON USE OF LAPLACE TRANSFORM IN QUANTITATIVE ANALYSIS OF PRIORITY-NAND FAILURE LOGIC
Different types of faults

Digital systems may suffer two classes of fault:

a) Temporary faults or intermittent faults:

These occur due to noise and the nonideal transient behaviour of switching components. Usually in switching theory the switching elements are considered ideal and signal propagation time is assumed zero. Actually, however, the delays associated with switching components cause non-instantaneous changes of states which in turn result in hazards. Consider the following switching function

\[ f(x, y, z) = x' y + x z = \sum(2, 3, 5, 7) \]

Two minimal implementations: Diode and contact network, their Karnaugh map are shown in fig 3.5 (a), (b) and (c) respectively. Now suppose \( y z = 11 \) and \( x \) changes from 0 to 1. Clearly output should not change (both entries in column \( y z = 11 \) in map are crossed). As \( x \) changes the transmission changes from path \( P_1 \) to \( P_2 \). (fig 3.5(b)). Let delay of gates/path 1 and 2 be \( \Delta_1 \) and \( \Delta_2 \) respectively, and let \( \Delta_1 < \Delta_2 \). Thus for an interval \( \Delta_2 - \Delta_1 \) output ceases to be '1' even though the function realised remains unaffected. Thus output temporarily becomes zero during this interval though for rest of time the output is 1. This type of fault can be eliminated by including a redundant path, \( y z \) here.

FIG. 3-5 EXAMPLE OF INTERMITTENT FAULTS

FIG. 3-6 IMPROVED NETWORK WITH NO INTERMITTENT FAULTS.
Similarly consider the network of fig - 3.6 corresponding to the previous situation, \( yz = 11 \) and \( x \) changing from 0 to 1. The path \( P_1 \) in the contact network or the diode network is opened and path \( P_2 \) is established but during this interval the path \( P_3 \) is closed and thus no intermittent fault occur at the output.

b) Permanent Faults

There are broadly three types of permanent faults. They are

i) Stuck-at faults

ii) Short circuit faults

iii) Bridging faults

**Stuck-at faults**

Consider the simple diode network shown in fig - 3.7, the truth table for which is given below in Table - 3.1.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Z</th>
<th>( z_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table - 3.1

Consider diode \( D_1 \) open, let the corresponding output be \( z_0 \). Obviously \( z_0 = B \), i.e. same as \( A \) and is permanently = 0. Thus this situation can be called line '1' is stuck at 0 denoted by \( s.a. \). Now consider further a positive logic and gate (diode resistance) as shown in fig - 3.8.
The corresponding truth-table for which is given below in Table - 3.2.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Z</th>
<th>Z₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table - 3.2

Let diode \( D₁ \) be open. This situation is same as \( A = 1 \) and the fault is termed line 1 stuck at 1 or \( 1/1 \). In general an open diode (emitter) constitute a s.e.o fault for an OR or NOR gate, and an open diode (emitter) constitute a s.a.1 fault for an AND or NAND gate. Thus s.e.o and s.a.1 faults, together are called struck at faults represented as \( 1/0 \) where 1 represents the particular and \( 0 \in (0,1) \).

ii) Short Circuit faults

For this fault the output is unaffected but the input connected to the faulty diode and all fanout leads from that input are forced to a value dependent on the output of the faulty gate. Consider the network shown in fig - 3.9 assuming positive logic, where the diode connected to be started, and the truth table showing \( Z₁ \) is given in Table-3.3 as follows.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Z₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table - 3.3
Now, let $A = 0$, then $z_1 = 0$, $\alpha$, $\beta$ both forced to '0' independent of $B$. If $A = 1$, $z_1 = B$, also $\alpha$, $\beta = B$. Thus $z_1$ remains unaffected. For a correctly working network $z_2 = BC$, which is independent of $A$. However, for short circuit fault as shown, $z_2$ is now dependent on $A$ and is given by $z_2 = ABC$. A shorted diode fault has property that it may be undetectable in circuits for which all single stuck-at type faults are detectable. Further, occurrence of a shorted diode fault may cause previously detectable s.a. faults now undetectable.

iii) Bridge faults

is said to exist in a network whenever any two wires are shorted. Existence of a feed back loop makes the situation complex, i.e. one of the leads shorted is an output to a gate $G_1$ and the other is an output of a gate $G$ which is dependent on $G_1$. Avoiding such situations, let us consider only shorting between input leads. It can be shown that depending on whether positive or negative logic is being used, this fault has the effect respectively of AND-ing or OR-ing. Consider the simple two input diode OR gate shown in fig - 3.10, the truth table for which is shown in Table - 3.4 below.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$Z$</th>
<th>$Z_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table - 3.4
G 3-9 DIODE NETWORK WITH SHORTS

G 3-10 DIODE NETWORK WITH BRIDGE FAULTS

G 3-11 AND GATE WITH BRIDGE FAULTS
The column \( Z_f \) means the output, when the short circuit faults, A, B shorted exist. Thus for input combinations \( AB = 01 \), current will flow from B to A through short circuit and logic value at short circuit is '0'. Thus \( AB = 00 \), 01 and 10 are all equivalent to \( AB = 00 \) and correspondingly produce output \( Z_f = 0 \), \( Z_f = 1 \) only for \( AB = 11 \), thus showing that for positive logic the result of a shorted input is that of AND-ing. Similarly OR-ing for negative logic can be provided by considering the gate without fan-out using negative logic as shown in fig - 3.11.

VI

DETECTION OF FAULTS

The simplest way of detecting whether a particular circuit is faulty is to apply to it all possible input combinations and to note for the desired output. If yes, the circuit is O.K., otherwise faulty. A fault-table has \( 2^n \) rows corresponding to all possible combinations and \( N \) columns corresponding to all possible faults. A 'X' is entered at the intersection of the \( i \)th row and \( j \)th column, if the fault corresponding to the \( j \)th column is detected by the \( i \)th test. A set of test constitute a fault detection experiment, which is shown in Table - 3.5 below as an example.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output (Faults)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 ) ( x_2 ) ... ( x_m )</td>
<td>( n_0 ) ( n_1 ) ... ( n_N )</td>
</tr>
<tr>
<td>0 0 ... 0</td>
<td>X ... X X</td>
</tr>
<tr>
<td>0 0 ... 1</td>
<td>. . . . .</td>
</tr>
<tr>
<td>... ... ...</td>
<td>. . X . .</td>
</tr>
<tr>
<td>1 1 ... 1</td>
<td>. . . . .</td>
</tr>
</tbody>
</table>

Table - 3.5
The problem of finding a minimum number of tests is equivalent to the detection of set of minimum number of rows of fault-table such that every column has an 'X' in at least one row. We will illustrate the technique by means of an example - 3.1.

Example - 3.1

Consider the function given by

\[ f(x, y, z) = xy + z \]

It is assumed that any of lines 1, m, n or p can be s.a.o or s.a.1 writing \( 1/0 \rightarrow 1_0 \), \( 1/1 \rightarrow 1 \), etc.

Approach

To detect a s.a.o (s.a.1) faults apply 1 (0) to the line under test and apply '1' to the remaining inputs of the AND or NAND gates and '0' to all remaining inputs to OR or NOR gates. Thus to detect \( 1_0 \) apply \( x = 1, y = 1 \) and \( z = 0 \) i.e. \( z = 1 \). If line 1 is not s.a.o i.e. normal then the output is 1, i.e. \( f = 1 \). If the line 1 is s.a.o i.e. \( 1_0 \) exists, application of \( x = 1 \) does not affect the input to the AND gate 1 which is 0 and \( n \rightarrow 0 \), \( z' = 0 \) give \( f = 0 \). Thus the input combination \( xyz = 111 \) produces output \( f = 1 \) if line 1 is not s.a.o and \( f = 0 \) if 1 is s.a.o and in this way detects 1_0. If \( n_0 \), the same test as above will detect it. Also if \( n_0 \), there is no way to distinguish between 1_0, n_0, p_0. These faults are thus indistinguishable.

We construct below the corresponding fault table in Table - 3.5, where 'X' 's are entered in row \( xyz = 111 \) and columns 1_0, n_0 and p_0 similarly.
Similarly others can be detected and thus the fault table can be completed. A single entry in column means that particular row is essential for the fault represented by the column. These 'X' are encircled, which imply that these tests are essential. Other approaches are path sensitization, algebraic equivalent normal form etc., and detection of other types of faults can also be performed by applying different types of available techniques.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>l₀</th>
<th>m₀</th>
<th>n₀</th>
<th>p₀</th>
<th>l₁</th>
<th>m₁</th>
<th>n₁</th>
<th>p₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

Table 3.5
ON AN APPLICATION OF INVERSE LAPLACE TRANSFORM IN THE QUANTITATIVE ANALYSIS OF PRIORITY - NAND FAILURE LOGIC

We shall first illustrate the graphical example of Priority - Nand logic by means of fig - 3.12, where the inputs $A_1, A_2, \ldots, A_{m-1}, A_m$ arrive sequentially. The following notations will be used throughout this subsection.

$\lambda_i$ := failure rate of input event $i$

$m$ := number of the inputs to the Priority - Nand logic

$L^{-1}$ := inverse Laplace transform

$\eta$ := vector $[N, N-1, N-2, \ldots, 1]$

$s$ := the Laplace transform variable

t := time

$f_i(t_i)$ := $\lambda_i \exp (-\lambda_i t_i)$

EXACT SOLUTION

The following assumptions are made,

a) The input events are $e$ - independent

b) The input events have an exponential failure distribution with known failure rate

c) The input events are non-repairable

* The paper is to be published in the Journal of the Institution of Electronics & Telecommunication Engineers, India
FIG: 3-12
Thus event $N$ must occur first, then event $N - 1$, ..., and finally event 1 occurs at the last. Therefore, the probability of the failure of the output of Priority - Nand logic gate with $N$ inputs arriving sequentially with known probability of failure may be given by,

$$p_Y^*(N) = 1 - \int_0^t \int_0^{t_1} \int_0^{t_2} \left[ 1 - \int_0^{t_3} f_3(t_3) \right] \cdot \left[ 1 - \int_0^{t_2} f_2(t_2) \right] \cdot \left[ 1 - \int_0^{t_1} f_1(t_1) \right] dt_1 dt_2 dt_3 \ldots$$

$\cdots \int_0^{t_{n-3}} f_{n-3}(t_{n-3}) \left[ 1 - \int_0^{t_{n-2}} f_{n-2}(t_{n-2}) \right] \left[ 1 - \int_0^{t_{n-1}} f_{n-1}(t_{n-1}) \right] dt_{n-1} dt_{n-2} \ldots$

$\ldots dt_3 \right) dt_2 \right) dt_1$

$$= 1 - \left\{ \sum_{m=1}^{N} p_{ym}(N) \right\} \quad \text{(3.18)}$$

Where

$$p_{ym}(N) = \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{m-1}} f_m(t_m) dt_m \ldots dt_2 dt_1 \quad \text{(3.19)}$$
But we have,
\[
\int_{0}^{t_{i-1}} f_{i}(t_{i}) \, dt_{i} = L^{-1} \left\{ \frac{\lambda_{i}}{s(s+\lambda_{i})} \right\}
\]

Therefore
\[
P_{Y_{m}}(N) = L^{-1} \left\{ \frac{1}{s} \prod_{j=1}^{m} \left( \frac{\lambda_{j}}{s-a_{j}} \right) \right\}
\]

Where
\[
a_{0} = 0, \quad a_{p} = -\sum_{j=1}^{p} \lambda_{j} \quad \text{for} \quad p > 0
\]

So, (3.18) can be written as
\[
P_{Y}^{*}(N) = 1 - L^{-1} \left[ \frac{1}{s} \cdot \frac{\lambda_{1}}{s-a_{1}} \right] - L^{-1} \left[ \frac{1}{s} \cdot \frac{\lambda_{1}}{s-a_{1}} \cdot \frac{\lambda_{2}}{s-a_{2}} \right] - \ldots - L^{-1} \left[ \frac{1}{s} \cdot \frac{\lambda_{1}}{s-a_{1}} \cdot \ldots \cdot \frac{\lambda_{n}}{s-a_{n}} \right]
\]

from we get $P_{Y}^{*}(N)$ as
\[
= 1 - \left\{ L^{-1} \left[ \frac{1}{s} \cdot \prod_{i=1}^{n} \frac{\lambda_{i}}{s-a_{i}} \right] + \right\}
\]
Finally using Heaviside's expansion formula, we establish the formula for $P_x^*(N)$ as,

$$P_x^*(N) = 1 - \left\{ \sum_{m=1}^{N} \left[ \prod_{i=1}^{m} \lambda_i \cdot \sum_{k=0}^{m} \left( \prod_{j=0}^{m} \frac{\exp(a_k t)}{(a_k - a_j)^{j-k}} \right) \right] \right\}$$

--- (3.22)

IX

APPROXIMATE SOLUTION

We have from (3.18)

$$P_y \{ N \} \simeq \frac{1}{N!} \prod_{i=1}^{N} \lambda_i^k$$

and so, $P_x^* \{ N \}$ can be written as,
\[ P^*_Y \{ N \} = 1 - \left[ P_Y(1) + P_Y(2, 1) + \cdots + P_Y(n, n-1, \ldots, 1) \right] \]

\[ \approx 1 - \left[ \lambda_1 t + \frac{\lambda_1 \lambda_2}{2!} t^2 + \frac{\lambda_1 \lambda_2 \lambda_3}{3!} t^3 + \cdots + \frac{\lambda_1 \lambda_2 \cdots \lambda_N}{N!} t^n \right] \]

\[ = 1 - \sum_{n=1}^{N} \frac{1}{N!} \prod_{i=1}^{n} \lambda_i t \]

\[ (3.23) \]

Now, the exact solution for two-event input to the Priority-Nand logic necessitates reduction of eq (3.22) to

\[ P^*_Y \{ 2, 1 \} = \exp(-\lambda_1 t) - \left[ \frac{\lambda^2}{\lambda_1 + \lambda_2} \exp(-\lambda_1 t) + \frac{\lambda_1}{\lambda_1 + \lambda_2} \exp(-\lambda_1 + \lambda_2 \cdot t) \right] \]

\[ (3.24) \]

and approximate solution for two-event input to the Priority-Nand logic may be given by

\[ P^*_Y \{ 2, 1 \} = 1 - \left[ \lambda_1 t + \frac{1}{2} \lambda_1 \lambda_2 t^2 \right] \]

\[ (3.25) \]
COMPARISON OF APPROXIMATION
WITH EXACT SOLUTION

A time dependent comparison of these two expressions are given in Table - 3.7 for two-event input to the Priority - Nand failure logic.

<table>
<thead>
<tr>
<th>$\lambda_1 t$</th>
<th>Ratio for $P_T^*(2,1)$ for two-input events, $\lambda_2/\lambda_1 = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-3}$</td>
<td>0.99999916</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>1.00000505</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>1.0053789</td>
</tr>
<tr>
<td>$0^+$</td>
<td></td>
</tr>
</tbody>
</table>

Table - 3.7

CONCLUSION

In the first half of the section a brief discussion is made of the common type of faults encountered in digital networks, and also some of their properties and standard techniques for detecting them has been studied.

The second half of the section is devoted to describe a methodology for establishing probability of failure of output for a priority - Nand logic gates with known probability of failure of inputs arriving sequentially. The
method prescribed in this part has advantages over the existing similar techniques for Priority-And logic, because Nand is an universal gate, which has already been discussed in the introduction to this chapter.