CHAPTER VI

SHARP FRONTS AND NONLINEAR DISPERSIVE WAVES IN INTERSTELLAR MEDIUM

6.1 Introduction

Baade's (1944) observational results indicated that there is a strong correlation in space between young stars and relatively dense concentrations of interstellar matter. It lead to the suggestion that stars are in fact born out of the surrounding matter. It is observed that young stars form predominantly in groups. Hence two suggestions may be placed regarding the formation of these young stars. One, they form out of a single massive condensation of interstellar matter that fragments while collapsing. Two, they form out of small-scale, low-mass structures within a large scale condensation that undergo collapse almost simultaneously. The two processes described above need not mutually exclusive, especially since observations show a hot and tenuous interstellar medium in which there exist both cold 'clouds', ranging in mass from several to a few thousand solar masses and cloud complexes whose mass combined with the mass of the embedded young stars may be as large as $10^6$ $M_\odot$. Thus if stars are to be born from interstellar clouds, a theory of star formation must ultimately answer the following questions:
i) How do interstellar clouds and cloud complexes form?

ii) What are the initial conditions of the clouds?

iii) How do different types of instabilities (thermal, gravitational, modulational, magneto-gravitational etc) form?

iv) What are the equilibrium states, if any, accessible to the cloud?

v) How a cloud tends to gravitational collapse?

vi) What are the conditions of fragmentation processes?

vii) How a cloud transfer its angular momentum?

viii) How magnetic fluxes decrease in a cloud?

Although this is not a complete set of questions whose answers a theory of star formation must provide.

The detailed study of star formation is a complex problem, as well as a somewhat uncertain one. In the present chapter attempts are taken to study only two physical processes, such as formation of sharp fronts in ISM and modulational instability of the medium.

6.2 Sharp Fronts in ISM

Different types of physical processes occurring in interstellar medium (ISM), are not yet fully understood. Many authors have suggested that pressure disturbances may be an important phenomena in ISM (Kahn, 1971 and Spitzer, Jr., 1978). In this chapter, using Reimann analysis, it has been shown that pressure disturbances may lead to the formation of sharp fronts.
in ISM. It has been considered that all properties of the interstellar fluid will depend only on one space coordinate \( x \), and on the time \( t \). Also, the variation of \( \gamma \), the ratio of specific heats at constant pressure and volume, with the amplitude of sharp fronts has been discussed.

The change of the fluid velocity of the gas, \( v \), may be expressed by the usual momentum equation as,

\[
\left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = -\nabla p - \frac{1}{8\pi} \nabla B^2 + \frac{1}{4\pi} B \cdot \nabla B - \rho \nabla \phi \quad \text{ ...(6.1)}
\]

where \( \rho \), \( p \), \( B \) and \( \phi \) are density, pressure, magnetic field and gravitational potential respectively.

The density is determined by the continuity equation,

\[
\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \rho \nabla \cdot v = 0 \quad \text{ ...(6.2)}
\]

The equation of pressure may be written as

\[
p = k \rho \gamma \quad \text{ ...(6.3)}
\]

where \( k \) is a constant and \( \gamma \) equal to \( C_p / C_v \), the ratio of specific heats at constant pressure and constant volume.

The above two equations (6.1) and (6.2) can be linearized for smaller disturbances about an equilibrium condition. If \( B \) and \( \phi \) are ignored and \( \rho \) is assumed nearly uniform, the linearized equations may be combined to show that the pressure
disturbances propagate at the sound velocity, \( C_s \), given by

\[
C_s^2 = \frac{\partial p}{\partial \rho} \quad \ldots \quad (6.4)
\]

The equations (6.1) and (6.2) can be written, for a one-dimensional case

\[
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad \ldots \quad (6.5)
\]

and

\[
\frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{v}{\rho} \frac{\partial p}{\partial x} + \frac{\partial v}{\partial x} = 0. \quad \ldots \quad (6.6)
\]

Using equations (6.3) and (6.4), equations (6.5) and (6.6) may be written as,

\[
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{2}{\sqrt{\gamma - 1}} C_s \frac{\partial C_s}{\partial x} = 0 \quad \ldots \quad (6.7)
\]

and

\[
\frac{2}{\sqrt{\gamma - 1}} \frac{\partial C_s}{\partial t} + \frac{2}{\sqrt{\gamma - 1}} v \frac{\partial C_s}{\partial x} + C_s \frac{\partial v}{\partial x} = 0 \quad \ldots \quad (6.8)
\]

Now adding and subtracting equation (6.8) from equation (6.7), it can be written,

\[
\left\{ \frac{\partial}{\partial t} + (v \pm C_s) \frac{\partial}{\partial x} \right\} (v \pm \frac{2}{\sqrt{\gamma - 1}} C_s) = 0 \quad \ldots \quad (6.9)
\]

The generalized equation (6.9) can be used to investigate waves of finite amplitude.

For a monatomic gas with no internal degrees of freedom

\( \gamma = 5/3 \). In a gas of atoms or molecules with excited energy levels \( C_p \) and \( C_v \) depend on the pressure as well as on temperature and \( \gamma \) falls below 5/3, when temperature is high enough
to permit appreciable excitation (Spitzer, Jr., 1978). The discussion here concerns with the interstellar gas for different values of $\gamma$.

For $\gamma = 1.50$ and $1.25$, the equation (6.9) can be written as,

$$\left\{ \frac{\partial}{\partial t} + (v + C_s) \frac{\partial}{\partial x} \right\} (v + 4C_s) = 0 \quad ...(6.10)$$

and

$$\left\{ \frac{\partial}{\partial t} + (v + C_s) \frac{\partial}{\partial x} \right\} (v - 8C_s) = 0 \quad ...(6.11)$$

respectively.

Using Reimann analysis it can be shown from equations 6.10 or 6.11 that different characteristics, $p$ or $p'$ $(v + 4C_s$ or $v + 8C_s)$ and $q$ or $q'$ $(v - 4C_s$ or $v - 8C_s)$ will be formed and will remain constant on the lines $dx/dt = v + C_s$ and $dx/dt = v - C_s$ respectively. The $p$ or $p'$ characteristic is the forward path of the front relative to the medium and $q$ or $q'$ characteristic is the backward path of the front.

The speed of the front is large at the 'crests'. The 'crests' systematically catch up the 'troughs' and the rise from a trough to the following crest steadily steepens. The time required by the front to become infinitely steep is.

$$t = \frac{L}{4C_r} = \frac{L}{2\sqrt{6}} \frac{k^{1/2}(\rho_c^{1/2} - \rho_t^{1/2})^{1/2}}{1/2} \quad \text{for } \gamma = 1.50$$

and

$$t' = \frac{L}{4C_{r'}^{1/2}} = \frac{L}{2\sqrt{5}} \frac{k^{1/2}(\rho_c^{1/2} - \rho_t^{1/2})^{1/2}}{1/2} \quad \text{for } \gamma = 1.25$$
where $L$ is the distance between a trough and the following crest at $t = 0$, or $C^r$, are the relative speeds of the crest and $\rho_c$ and $\rho_t$ are the densities at the crest and trough respectively.

6.3 Nonlinear Dispersive Waves in ISM

Hydromagnetic waves may play an important role to the heating and scattering of a medium via stable and unstable criteria. In this context it is worthy to point out that several authors have proposed various types of hydromagnetic wave propagation through the interstellar medium.

Inspite of the presentation of the afore said investigation on hydromagnetic wave propagation through ISM, there is still scope to study the same field from the standpoint of the nonlinear theory, since, in some cases strong excitement of the perturbed field variables contribute to the dynamical behaviour of ISM. It is well known that solar wind plasma leads to several types of hydromagnetic wave propagation in ISM (Parker, 1966, 1967 a,b, 1968).

In order to study the nonlinear behaviour of a medium, one may follow the concept of, (i) perturbation expansion method Tsytovich (1970), (ii) quantum mechanical approach of Haris (1969), (iii) Lagrangian formalism of Dougherty (1970), and (iv) couple mode technique (Louisell, 1960, Davidson, 1972, Pramanik, 1979). But in order to study the modulational
instability of the hydromagnetic waves in ISM, the perturbation technique of Krylov - Bogoliukov Mitropollosky as developed by Kakutani and Sugimoto (1974) has been followed.

Further we remember that the modulational effects presents some valuable information regarding the formation of the galaxy (Liang, 1979). The present section deals with the modulational instability of ISM. Also the maximum growth rate of the hydromagnetic waves in ISM is discussed.

The basic equations for hydromagnetic fluid in interstellar medium may be as follows:

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\nabla p - \frac{1}{B^2} \nabla B^2 + \frac{1}{\mu} \mathbf{B} \cdot \nabla \mathbf{B} - \rho \nabla \phi \quad \ldots (6.12a)
\]

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 \quad \ldots (6.12b)
\]

\[
\nabla^2 \phi = 4\pi G \rho \quad \ldots (6.12c)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{4\pi} \nabla^2 \mathbf{B} \quad \ldots (6.12d)
\]

and neglecting the radiation pressure, the gas pressure may be written as,

\[
p = \rho c_s^2 / \gamma \quad \ldots (6.12e)
\]

where \( \mathbf{v}, \rho, \phi, B \) and \( c_s \) are the fluid velocity of the gas, the density, the gravitational potential, magnetic field and sound speed respectively. We are considering the magnetic field along \( \mathbf{X} \)-direction, which is perpendicular to the galactic plane and the fluid velocity and streaming also along \( \mathbf{X} \)-direction.
The equations (6.12a) - (6.12d) may be written in nondimensional form, using the following parameters

\[ \bar{\rho} = \rho / \rho_0, \quad \bar{v} = v / v_0, \quad \bar{t} = t / w, \quad \bar{x} = x / w, \]

\[ \bar{B} = B / B_0, \quad \bar{\phi} = \phi / \phi_0 \]

where \( o \) script indicates unperturbed values. For weakly nonlinear systems we can use the following expansions,

\[
\begin{bmatrix}
\bar{\rho} \\
\bar{v} \\
\bar{B} \\
\bar{\phi}
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
+ \varepsilon
\begin{bmatrix}
\rho^{(1)} \\
v^{(1)} \\
B^{(1)} \\
\phi^{(1)}
\end{bmatrix}
+ \varepsilon^2
\begin{bmatrix}
\rho^{(2)} \\
v^{(2)} \\
B^{(2)} \\
\phi^{(2)}
\end{bmatrix}
+ \varepsilon^3
\begin{bmatrix}
\rho^{(3)} \\
v^{(3)} \\
B^{(3)} \\
\phi^{(3)}
\end{bmatrix}
+ \varepsilon^4
\begin{bmatrix}
\rho^{(4)} \\
v^{(4)} \\
B^{(4)} \\
\phi^{(4)}
\end{bmatrix}
\]

(6.14)

Let \( \rho^{(1)} \) represents the monochromatic plane wave, namely

\[ \rho^{(1)} = a \exp (i \psi) + \bar{a} \exp (-i \psi) \]

(6.15)

where 'a' is the amplitude and '\( \bar{a} \)' is its complex conjugate and \( \psi = (kx - wt) \) is its phase, \( k \) being the wave number and \( w \) the frequency. The complex amplitude 'a' is a slowly varying function of \( x \) and \( t \), this slow variation is given by

\[ \frac{\partial a}{\partial t} = \epsilon A_1 (a, \bar{a}) + \epsilon^2 A_2 (a, \bar{a}) + \epsilon^3 A_3 (a, \bar{a}) + \epsilon^4 A_4 (a, \bar{a}) \]

(6.16)

and

\[ \frac{\partial a}{\partial x} = \epsilon C_1 (a, \bar{a}) + \epsilon^2 C_2 (a, \bar{a}) + \epsilon^3 C_3 (a, \bar{a}) + \epsilon^4 C_4 (a, \bar{a}) \]

and their complex conjugates. The quantities \( A_1, C_1, A_2, C_2 \ldots \)
are yet unknown and are to be determined from the conditions that the perturbation scheme envisaged by equations (6.14) - (6.16) are from secularities. Equations (6.14) - (6.16) can be substituted into equations (6.12a) - (6.12d) and using (6.12e) and (6.13), the equations to different orders in $\xi$ is obtained. The $\xi^1$ order equations yield

$$\phi_x^{(1)} = -\frac{KV}{k^2} a \exp (i\psi) + b_1(a, \bar{a}) \psi + b_2(a, \bar{a}) + c.c$$

$$b_x^{(1)} = b_3(a, \bar{a}) \exp (i\psi) + b_4(a, \bar{a}) + c.c$$

$$v_x^{(1)} = -\beta a \exp (i\psi) + b_5(a, \bar{a}) + c.c$$

where $b_1 - b_5$ are constants with respect to $\psi$ but are functions of $a$ and $\bar{a}$ and $\beta = (K_1 k - w)/K_1 k$ and $K_1 = \nu_0/C_s$.

Also we obtain to the same order

$$\left(K_k - w\right) \frac{\partial v_x^{(1)}}{\partial \psi} = K_{II} k \frac{\partial \phi_x^{(1)}}{\partial \psi} - K_{IV} \frac{\partial \phi_x^{(1)}}{\partial \psi}$$

where $K_{II} = C_s/\nu_0 \nu^2$ and $K_{IV} = \Phi_0/C_s \cdot \nu_0$.

From equations (6.15), (6.17) and (6.18) we get the linear dispersion relation for hydromagnetic fluids in a dispersive medium as

$$D(k, w) = (K_1 k - w)^2 + K_{II} k^2 + K_{IV} K_V = 0$$

where $K_V = 4\pi G \rho_0/\Phi_0$. 
From equations (6.12a) to (6.12d), to order \( \varepsilon^2 \), we can eliminate \( v_x^{(2)} \), \( b_x^{(2)} \) and \( \phi_x^{(2)} \) and use equation (6.14) to obtain the equation for \( \rho^{(2)} \) viz,

\[
\frac{\partial^2 \rho^{(2)}}{\partial y^2} + \rho^{(2)} = \frac{k}{k_{IV} k_{V}} \left[ 2 \beta \alpha_1 + C_1 (2 \beta k_1 - \beta k_1 - K_{IV} k_{V}/k + \omega \alpha) \right] \exp(i\psi) + N \exp(i\eta) + c.c. \quad \text{(6.20)}
\]

where \( N = \frac{k}{k_{IV} k_{V}} (6 \beta^2 \alpha_k + 2 \alpha_k \omega - 2 \alpha_k k_{IV} k_{V}/k) \).

The \( \exp(\pm i\psi) \) terms on the right hand side give rise to resonant secular in the solution for \( \rho^{(2)} \) due to equation (6.19). The condition for the removal of this secularity is,

\[
\alpha_1 + V_g C_1 = 0 \quad \text{...(6.21)}
\]

where,

\[
V_g = \frac{\partial^2 \rho}{\partial y^2} = \frac{\partial k}{\partial \omega} = \frac{\beta k + \omega}{\beta}
\]

is the group velocity of the hydromagnetic fluid. The secular free solution of equation (6.20) is now given by

\[
\rho^{(2)} = \left[ \frac{N}{\alpha} \exp(2i\psi) + b_6(a, \bar{a}) \exp(i\eta) + c.c. \right] b_7(a, \bar{a}) \quad \text{...(6.22)}
\]

The solutions of the equations to order \( \varepsilon^2 \) are

\[
\phi_x^{(2)} = \left[ - \frac{K_{IV} \rho}{12k^2} \exp(2i\psi) + b_8(a, \bar{a}) \eta + c.c. \right] + b_9(a, \bar{a}) \quad \text{...(6.23a)}
\]

\[
v_x^{(2)} = \left[ \beta (a^2 - N/3) \exp(2i\psi) + k_1 C_1 \frac{\partial b_5}{\partial a} + c.c. \right] + b_{10}(a, \bar{a}) \quad \text{...(6.23b)}
\]
and,

\[ B_x^{(2)} = b_{11}(a, \tilde{a}) \exp(i\psi) - \frac{1}{K_1 k^2} \psi + \text{c.c.} + b_{12}(a, \tilde{a}) \]

... (6.23c)

where \( b_6(a, \tilde{a}) \) to \( b_{12}(a, \tilde{a}) \) are constants with respect to \( \psi \) but are functions of \( a \) and \( \tilde{a} \).

Now in equations (6.12a) - (6.12d), to order \( \varepsilon^3 \), \( v_x^{(3)} \), \( \rho_x^{(3)} \) and \( \Phi_x^{(3)} \) can be eliminated to give an equation in \( \rho^{(3)} \).

On using the expressions for \( \rho^{(2)} \), \( B_x^{(2)} \), \( v_x^{(2)} \), \( \Phi_x^{(2)} \), \( \rho^{(1)} \), \( B_x^{(1)} \), \( v_x^{(1)} \) and \( \Phi_x^{(1)} \) given above, the condition for the removal of resonant secularity in the equation for \( \rho^{(3)} \) is found to be

\[
i(A_2 + v_0 C_2) + \frac{1}{2} \frac{dv_0}{dk} \left( C_1 \frac{\partial C_1}{\partial a} + \bar{C}_1 \frac{\partial C_1}{\partial \bar{a}} \right) = \left[ K_1 k b_{10} - \frac{1}{2} \right]
\]

\[
\left( (K_1 k - 3w) + \frac{w^2}{K_1 k} \right) a
\]

... (6.24)

provided the condition

\[
K_1^2 K IV K V k - (K_1 k - w)^2 w = 0
\]

... (6.24a)

is to be satisfied.

The constants of integration \( b_7 \) and \( b_{10} \) can be determined from the condition of removal of resonant secularity in the equation for \( \rho^{(4)} \). They are obtained as follows

\[
b_7 = \frac{2(k^2 + 1) K_4 k^3}{(w - K_1 k)(2K_1 + w - K_1 k)w^2} a \bar{a}
\]
\[ b_{10} = \frac{[K_4 k^3 - 2(k^2 + 1)w + K_4 k(w^2 + 1) - w^3]}{(w - K_1 k)(2k_1 + w - K_1 k)w^2} \]

Now equation (6.24) may be written as,

\[ i(A_g + V C_2) + P(C_1 \frac{\partial C_1}{\partial a} + C_1 \frac{\partial C_1}{\partial a}) = \frac{Q}{a^2/a} \]

where,

\[ P = \frac{1/2. dV}{dk} \quad \text{and} \quad Q = \frac{(K_4 k + k^2 + 1)K_4 k^2}{(K_4 k - 2k_1 - w)w} \]

The slow variations of the amplitude 'a' with respect to space and time are governed by the conditions (6.21) and (6.24). On defining the new space and time variables,

\[ t_2 = \epsilon t_1, \quad t_1 = \epsilon t, \quad x_2 = \epsilon x_1, \quad x_1 = \epsilon x \]

equation (6.21) can be written as

\[ \frac{\partial a}{\partial t_1} + V \frac{\partial a}{\partial x_1} = 0. \]

This equation indicates that in the slow scale \( t_1 \) and \( x_1 \) the amplitude 'a' propagates with the group velocity \( V \) without any change of form. Equation (6.24) can now be written as

\[ i(\frac{\partial a}{\partial t_2} + V \frac{\partial a}{\partial x_2}) + P \frac{\partial^2 a}{\partial x_1^2} = \frac{Q}{a^2/a} \]

On using the coordinate transformations

\[ \xi = (x_1 - V_g t), \quad \tau = \epsilon^2 t = \epsilon t_1 = t_2. \]
equation (6.28) reduces to

\[ 1 \frac{\partial a}{\partial t} + P \frac{\partial^2 a}{\partial \xi^2} = Q/a^2/a . \]  

\[ \text{...}(6.29) \]

This is the nonlinear Schrodinger equation for hydromagnetic fluid in interstellar medium.

Hydromagnetic stability or instability of interstellar medium may be studied using the nonlinear Schrodinger equation. It may be remarked that, for small values of \( k \) the hydromagnetic waves will be unstable if \( PQ > 0 \), otherwise it will be stable (Hasegawa, 1972). Thus the medium leads to the unstable or stable configuration according as,

\[ \frac{K_1 kw + w^2 + 1}{(2K_1 + w - K_1 k)k} > 0. \]  

\[ \text{...}(6.30) \]

In case of ISM, the value of \( K_1 \) may be taken as 2.8. From relation (6.30) it is found that the modulational instability in ISM attains maximum value as \( k \) tends to very small value and the medium becomes stable for large values of \( k \). Thus it may be remarked that the streaming in ISM governs the instability process.

Maximum growth rate of hydromagnetic waves may be studied from equation (6.29). Following Hasegawa (1972) we substitute

\[ a = a^{1/2} \exp \left\{ i \int \xi \, d\xi / \mathcal{P} \right\} \]
and expanding $a$ and $\chi$ as,

\[
\begin{bmatrix}
a \\
\chi
\end{bmatrix} = \begin{bmatrix}
a_0 \\
\chi_1
\end{bmatrix} + \begin{bmatrix}
a_1 \\
\chi_0
\end{bmatrix} \exp \left\{ i (k\xi - \omega \tau) \right\}.
\]

In equation (6.29), a dispersion relation leads to

\[\Omega^2 = \frac{P^2}{Q} (k^2 - \frac{Qa_0}{P})^2 - \frac{Q^2 a_0^2}{P} \]

and accordingly the maximum growth rate obtained from

\[T_m = \frac{I_m \Omega}{\omega} \quad \text{as} \]

\[T_m = \frac{Q}{a_0} \quad \cdots (6.31)
\]

Using relation (6.31), the variation of maximum growth rate ($T_m$) vs. wave number ($k$) may be studied for different values of $K_1$, $w$ and $a_0$. It may be found that as the amplitude of hydromagnetic waves decreases, $T_m$ is switched on from lower values for a particular value of $k$.

6.4 Discussion

The information that may be obtained from the study of sharp fronts in ISM are summarised as follows:

i) As the value of $\gamma$ decreases, the amplitude of the sharp fronts decreases.

ii) It may not necessarily be possible for a fluid to find
steady flow patterns in ISM, even given an arbitrary choice of reference frames.

iii) The pressure waves tend to steepen in ISM.

Since the transport of heat and momentum play dominant role in interstellar shock region, the momentum and continuity equations will nonlonger be valid throughout the flow region. From the study of nonlinear dispersive waves in ISM it may be remarked that the modulational instability will lead to condensations in the medium. Three dimensional study of this problem may give details information about the condensation processes in ISM.
REFERENCES


