CHAPTER 1
The basic problem of theoretical seismology is the determination of the motion on the free surface and in the interior of a layered elastic solid due to an impulsive source located below the surface. After Lamb's fundamental work on a buried source in a half-space, the layered half-space has been treated by a large number of authors.

The problem of the disturbance due to a point source of P and S waves below the surface was considered by Lapwood (1949).

Newlands (1952) extended this to the case of a single layer overlying a half-space. The first attempt to provide complete theoretical seismograms for an explosive point source in a layered elastic half-space seems to be due to Pekeris, Alterman and Abramovici (1965). This work was based on the numerical evaluation of integrals and required the use of a large scale digital computer. Alterman and Karal (1968) solved the problem of a point source in a layered elastic half-space by using the finite difference method. Chopra (1957) discussed the disturbances due to a point source in a stratum within an infinite elastic medium. The problem of a point source in a layer overlying a half-space has also been discussed by Singh (1966).
In the present work we propose to derive expressions for the theoretical displacement generated by a compressional spherical point-source in a layer overlying a half-space and to examine the variation of the amplitude of different phases with epicentral distance, depth of source and frequency. Recently, Chakrabarty (1967) has discussed the stress equations of motion for a general type of source. These equations are used in the present investigation. The source is considered as a superposition of perpendicular couples without moment. A comparison of the calculated displacements with those recorded in seismograms should indicate the possibility of such a mechanism at the source. By our analysis, we are able to split the displacements into a succession of direct and reflected pulses having corresponding reflection coefficients. The Laplace Transforms of the solutions come first in the integral form and they are integrated along the modified Sommerfeld's contour and various reflected, refracted and diffracted waves are obtained.

1.1 Mathematical Analysis

The point source of compressional waves is situated in a solid layer $M_1$ of uniform thickness $H$ overlying a solid half-space $M_2$ of different elastic properties. Let $\rho_i$ be the density, $\lambda_i, \mu_i$ the Lamé's constants and $\alpha_i, \beta_i$ the velocities of compressional and distortional waves respectively in the layer.
The corresponding quantities for half-space are denoted by similar symbols with the suffix 1. We take the free surface as the xy-plane and the z-axis passing through the source and drawn into the medium (Fig. 1). Let h be the depth of source below the free surface. Such a source can be considered as a superposition of perpendicular couples without moment. We can take

\[ R = -\frac{2P\delta(x)}{\pi} \delta(z - h) \chi(t) \quad \alpha = 0 \quad z = \frac{P\chi(t)}{\pi} \delta(z - h) \chi(t) \]  

where \( R, \alpha, z \) are components of the body force \( F \), including those defining the force system operating at the source as defined by Chakrabarty (1967), and we take \( \chi(t) = e^{i\omega t} h(t) \).

Following Chakraborty (1967) taking \( p \) in place of \( \omega \), we have (as Laplace Transform variable) the following equations which are valid for both the medium \( M_1 \) and \( M_2 \):

\[ \left( \frac{\partial^2}{\partial z^2} - \omega^2 \right) \left( \bar{X} + \bar{Z} \right) = \frac{\bar{X}}{2\alpha z} N(\xi, \bar{Z}) - \frac{1}{\alpha z} \frac{\partial^2 \bar{Z}(\xi, \bar{Z})}{\partial \bar{Z}^2} \]  

\[ \left( \frac{\partial^2}{\partial z^2} - \omega^2 \right) \bar{Y} = \frac{\bar{Y}}{2\alpha z} M(\xi, \bar{Z}) \]  

\[ \left( \frac{\partial^2}{\partial z^2} - \omega^2 \right) \bar{Z} = \frac{1}{\alpha^2} \bar{Z}(\xi, \bar{Z}) - \frac{\partial^2 \bar{Z}}{\partial \bar{Z}^2} \frac{2}{\partial \bar{Z}} \left( \bar{X} + \bar{Z} \right) \]  

where \( X, Z, Y, W, Z_1 (\xi, Z), N(\xi, Z), Z_1 (\xi, Z) \) are defined.
in equations (0.4, 0.9, 0.12) on pages (26-27) with \( \gamma^2 = \xi^2 - \frac{\beta^2}{\alpha^2} \).

\[ \gamma^2 = \xi^2 - \frac{\beta^2}{\alpha^2} \]

The right hand terms in these equations are due to body force.

Therefore for \( M_2 \)

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) (\ddot{x} + \ddot{z}) = -\frac{\xi^2}{2\alpha^2} \frac{\partial^2}{\partial z^2} N(\xi, z) - \frac{1}{2\alpha^2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial z} \left( \frac{\partial Z_1(\xi, z)}{\partial z} \right)
\]

\[ \ldots 1.5 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{y} = \frac{\xi^2}{2\rho_f} M(\xi, z)
\]

\[ \ldots 1.6 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{\omega} = -\frac{1}{\rho_f} \frac{\partial Z_1(\xi, z)}{\partial z} - \frac{\alpha_i^2}{\beta_i^2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial z} (\ddot{x} + \ddot{z})
\]

\[ \ldots 1.7 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{z} = -\frac{1}{\rho_f} \frac{\partial Z_1(\xi, z)}{\partial z} - \frac{\alpha_i^2}{\beta_i^2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial z} (\ddot{x} + \ddot{z})
\]

\[ \ldots 1.8 \]

For \( M_2 \)

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) (\ddot{x} + \ddot{z}) = 0
\]

\[ \ldots 1.9 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{y} = 0
\]

\[ \ldots 1.10 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{\omega} = -\frac{\alpha_i^2}{\beta_i^2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial z} (\ddot{x} + \ddot{z})
\]

\[ \ldots 1.11 \]

\[
\left( \frac{\partial^2}{\partial z^2} - \gamma^2 \right) \ddot{z} = -\frac{\alpha_i^2}{\beta_i^2} \frac{\partial^2}{\partial z^2} \frac{\partial}{\partial z} (\ddot{x} + \ddot{z})
\]

\[ \ldots 1.12 \]
For the compressional point source, the motion are symmetrical about the axis of \( Z \), so we have \( \lambda = 0 \).

From the equation \( \text{Eq. (07.7)} \) we have:

\[
\bar{R}_1 (\xi, z) = \int_0^\infty \pi R (\gamma, z) \ J_1 (\gamma \xi) d\gamma = -P_\xi \delta (z - h)
\]

\[
\bar{R}_2 (\xi, z) = \int_0^\infty \pi R (\gamma, z) \ J_1 (\gamma \xi) d\gamma = -P_\xi \delta (z - h)
\]

\[
\bar{\theta}_1 (\xi, z) = 0
\]

\[
\bar{\theta}_2 (\xi, z) = 0
\]

\[
Z_1 (\xi, z) = \int_0^\infty \pi J_0 (\gamma \xi) Z_1 (\gamma, z) d\gamma = P \xi (z - h)
\]

Therefore,

\[
M (\xi, z) = \bar{R}_1 (\xi, z) + \bar{R}_2 (\xi, z) = 0
\]

\[
N (\xi, z) = \bar{R}_1 (\xi, z) - \bar{R}_2 (\xi, z) = -2P_\xi \delta (z - h)
\]

The equations of motions in \( M_1 \) becomes

\[
\left( \frac{\delta^2}{\delta z^2} - \omega^2 \right) \left( \bar{x} + \bar{z} \right) = \frac{P \chi (\rho)}{\omega^2} \delta (z - h) - \frac{P \chi (\rho)}{\omega^2} \frac{\delta^2}{\delta z^2} \delta (z - h)
\]

\[
\left( \frac{\delta^2}{\delta z^2} - \omega^2 \right) \bar{\gamma} = 0
\]

\[
\left( \frac{\delta^2}{\delta z^2} - \omega^2 \right) \bar{\phi} = -\frac{P \chi (\rho)}{\beta^2} \delta' (z - h) - \frac{\alpha^2 - \beta^2}{\beta^2} \frac{\delta}{\delta z} \left( \bar{x} + \bar{z} \right)
\]

\[
\left( \frac{\delta^2}{\delta z^2} - \omega^2 \right) \bar{z} = -\frac{P \chi (\rho)}{\beta^2} \delta (z - h) - \frac{\alpha^2 - \beta^2}{\beta^2} \frac{\delta}{\delta z} \left( \bar{x} + \bar{z} \right)
\]
We have the following two types of differential equation

\[
\left( \frac{\partial^2}{\partial z^2} - \lambda z \right) (x + z) = \delta(z - h),
\]

\[
\left( \frac{\partial^2}{\partial z^2} - \lambda^2 \right) (x + z) = \delta'(z - h),
\]

and their solutions are

\[
x + z = \frac{1}{2\lambda'} \left[ e^{-\frac{\lambda}{\lambda'}(z - h)} H(z - h) + e^{-\frac{\lambda}{\lambda'}(h - z)} H(h - z) \right]
\]

\[
and \quad x + z = \delta(z - h) + \frac{\lambda}{\lambda'} \left[ e^{-\frac{\lambda}{\lambda'}(z - h)} H(z - h) + e^{-\frac{\lambda}{\lambda'}(h - z)} H(h - z) \right]
\]

respectively.

Therefore, in Medium M1

\[
\bar{x} + \bar{z} = A_1 e^{\lambda' z} + A'_1 e^{-\lambda' z} \mathcal{P}_x(p) \delta(z - h)
\]

\[
- \frac{\mathcal{P}_x(p) p^2}{2 \lambda'^2} \left[ e^{-\frac{\lambda}{\lambda'}(z - h)} H(z - h) e^{-\frac{\lambda}{\lambda'}(h - z)} H(h - z) \right]
\]

1.15a

In a similar way we have

\[
\bar{y} = B_1 e^{\lambda' z} + B'_1 e^{-\lambda' z} \quad \ldots 1.15b
\]

\[
\bar{w} = \frac{A_1' \lambda'^2 - \lambda'^2}{P^2} e^{-\lambda' z} + \frac{A_1 \lambda^2 - \lambda^2}{P^2} e^{\lambda' z} - \frac{\gamma e^{-\lambda' z}}{\lambda'} \]

\[
- \frac{\mathcal{P}_x(p) p^2}{2 \lambda'^2} \left[ e^{-\frac{\lambda}{\lambda'}(z - h)} H(z - h) e^{-\frac{\lambda}{\lambda'}(h - z)} H(h - z) \right]
\]

1.15c
The boundary conditions are (page 31)

\( i) \) at \( Z = 0 \):

\[ \rho_j \kappa_j^2 \left( \bar{X} + \bar{Z} \right) - 2 \beta_i^2 \bar{P}_i \bar{X} = 0 \]

\[ \left( \frac{\partial \bar{X}}{\partial \bar{Z}} - \bar{X}^2 \bar{\omega} \right) = 0 \]

\[ \frac{\partial \bar{Y}}{\partial \bar{Z}} = 0 \]...1.17

and

\( ii) \) at \( Z = H \)

\[ \begin{bmatrix} \rho_j \kappa_j^2 \left( \bar{X} + \bar{Z} \right) - 2 \beta_i^2 \bar{P}_i \bar{X} \end{bmatrix}_{M_1} = \begin{bmatrix} \rho_j \kappa_j^2 \left( \bar{X} + \bar{Z} \right) - 2 \beta_i^2 \bar{P}_i \bar{X} \end{bmatrix}_{M_2} \]

\[ \begin{bmatrix} \partial \bar{X} \bar{Z} - \bar{Z}^2 \bar{\omega} \end{bmatrix}_{M_1} = \begin{bmatrix} \partial \bar{X} \bar{Z} - \bar{Z}^2 \bar{\omega} \end{bmatrix}_{M_2} \]

\[ \bar{X}_{M_1} = \bar{X}_{M_2} \]
which give

\[(i) \quad \frac{2 \xi^2 \beta^2}{\beta_1^2} \partial^2 \gamma_1 (A_1' - A_1) + \left( 2 \xi^2 - \frac{\beta^2}{\beta_1^2} \right) (C_t + C_t') + \frac{P_x(p)}{\alpha t} \partial^2 \gamma_1 e^{-3h} = 0\]

\[(ii) \quad \left( 2 \xi^2 - \frac{\beta^2}{\beta_1^2} \right) (A_1 + A_1') + \frac{2 \mu t}{\alpha t} \partial_1 (C_t - C_t') - \frac{P_x(p)}{2 \alpha t^t} e^{-3h} (2 \xi^2 - \frac{\beta^2}{\beta_1^2}) = 0\]

\[(iii) \quad B_1 \partial_1' - B_1' \partial_1 = 0\]

\[(iv) \quad \frac{A_1 \alpha t^2 \xi^2}{\beta^2} e^{2h} + \frac{A_1 t^2 \xi^2}{\beta^2} q e^{-2h} + C_t' \partial_1^2 e^{2h} - C_t \partial_1 e^{2h} \]

\[- \frac{P_x(p)}{2 \alpha t^t} e^{2h} = \frac{A_2 \alpha t^2 \xi^2}{\beta^2} e^{-2h} + C_t' \partial_1 e^{-2h} \]

\[(v) \quad C_t e^{2h} + C_t' e^{-2h} - \frac{P_x(p)}{2 \alpha t^t} e^{2h} - \frac{A_2 \alpha t^2 \xi^2}{\beta^2} e^{-2h} = 0\]

\[(vi) \quad B_1 e^{2h} + B_1' e^{-2h} = B_1 e^{2h}\]

\[(vii) \quad - \frac{A_1 \alpha t^2}{\beta^2} \left( 2 \xi^2 - \frac{\beta^2}{\beta_1^2} \right) \left( A_1 e^{2h} + A_1' e^{-2h} \right) + 2 \mu t \partial_1 e^{2h} C_t \]

\[- 2 \mu_1 \partial_1 e^{2h} C_t + \frac{A_1 \alpha t^2}{2 \alpha t^t} \frac{P_x(p)}{\beta^2} e^{2h} (2 \xi^2 - \frac{\beta^2}{\beta_1^2}) = \frac{A_2 \alpha t^2}{\beta^2} \left( 2 \xi^2 - \frac{\beta^2}{\beta_1^2} \right) A_1' e^{-2h} - 2 \mu_2 \partial_1 e^{-2h} C_t' e^{-2h}\]
We have \( \triangle = 0 \) as the periodic equation, where \( \triangle \) is the six order determinant given by Press et al (1957, p192).

\[
\triangle = \frac{\phi \phi_0 e^{\alpha \phi - \frac{\beta}{\phi} - \frac{\gamma}{\phi^2}}}{p^6} \left[ D_1 + D_2 e^{2\phi_1} + (D_3 + D_4) e^{-(\phi_1 - \phi_2)} + D_5 e^{2\phi_1} + D_6 e^{2(\phi_1 + \phi_2)} \right]
\]

where

\[
\begin{align*}
D_1 &= \Phi \Delta_{12} \\
D_2 &= \Phi \Delta_{14} \\
D_3 &= \Phi \Delta_{15} \\
D_4 &= \Phi \Delta_{14} \\
D_5 &= \Phi \Delta_{23} \\
D_6 &= \Phi \Delta_{34}
\end{align*}
\]

\[
\Delta_{12} = \begin{vmatrix} -\xi_2 & \xi_1 & \xi_1 & \xi_1 & \xi_1 \\ \xi_1 & -1 & \xi_1 & \xi_1 & \xi_1 \\ -2 \xi_1 \xi_3 & (2 \xi_1 - \frac{\beta}{\phi^2}) & -\frac{\mu_2}{\mu_1} (2 \xi_1 - \frac{\beta}{\phi^2}) & -\frac{\mu_2}{\mu_1} (2 \xi_1 - \frac{\beta}{\phi^2}) \\ (2 \xi_1 - \frac{\beta}{\phi^2}) & -2 \xi_1' & -\frac{\mu_2}{\mu_1} (2 \xi_1 - \frac{\beta}{\phi^2}) & -2 \mu_1 \xi_1' \\ \end{vmatrix}
\]
\[ \Delta_{24} = \begin{vmatrix} -\xi_1^2 & -\xi_2^2 & \xi_5^2 & \nu_2' \\ -\xi_1 & \xi_2 & \nu_2 & 1 \\ 2\xi_5 & -2\xi_5^2 & -2\mu_2 \nu_2 \xi_5 & -\mu_2 (2\xi_5^2 - \frac{\mu_2^2}{\beta_5^2}) \\ (2\xi_5^2 - \frac{\mu_2^2}{\beta_5^2}) & (2\xi_5 - \frac{\mu_2}{\beta_5}) & -\mu_2 (2\xi_5^2 - \frac{\mu_2^2}{\beta_5^2}) & -\mu_2 (2\xi_5 - \frac{\mu_2}{\beta_5}) \end{vmatrix} \]
\[ \Delta_{34} = \begin{vmatrix} -z_2 & -2z_1 & -z_1 & -z_1 \\ -z_1 & -2z_1 & -2z_1 & -z_1 \\ z_1 & z_1 & z_1 & z_1 \\ z_1 & z_1 & z_1 & z_1 \end{vmatrix} \]

Since displacements tend to zero as \( Z \to \infty \) the real parts of \( \nu_1, \nu_1', \nu_2, \nu_2' \) are all positive and hence if \( H \) does not tend to zero all the terms in \( \Delta_1 \) (Equation (1.19)) except the first term \( F(\zeta_1) \Delta_{12} \) will be small and may be neglected and we can take

\[ \Delta \approx \alpha_1 \alpha_2 \beta_1 \beta_2 e^{(\nu_1 + \nu_1' - \nu_2 - \nu_2')H} F(\zeta_1) \Delta_{12}. \]

From definition of \( q(\gamma, z) \) and \( \omega(\gamma, \bar{z}) \) given in equation (6.10a+b)

\[ q(\gamma, z) = \int_0^\infty \mathcal{J}_1(\gamma \zeta) \, \bar{X}(\zeta, z) \, d\zeta. \]

\[ \omega(\gamma, \bar{z}) = \int_0^\infty \zeta \mathcal{J}_0(\gamma \zeta) \, \bar{\omega}(\zeta, \bar{z}) \, d\zeta. \]

Then inserting the values of \( A_1, A_1', C_1, Q_1' \) in the expression of \( \bar{X}_{11} \) and \( \bar{\omega}_{11} \) considering the approximation (\( \gamma, \bar{z} \))
\[ q(r, z, p) = \frac{P^2(q)}{2\alpha^2} \int_0^\infty J_1(\gamma \xi) \left[ \frac{\alpha^2}{\xi^2} \right] \left[ \left( e^{\gamma^2 (z^2 - p^2)} \right) H_1(z \gamma) + e^\gamma H_2(z \gamma) \right] \]

\[ + R_{pp} e^{-\gamma^2 (r^2 + z^2)} + R_{ps} e^{-\gamma^2 (r^2 + z^2)} + R_{pp} e^{-\gamma^2 (r^2 + z^2)} \]

\[ + R_{ps} e^{-\gamma^2 (r^2 + z^2)} + R_{pp} e^{-\gamma^2 (r^2 + z^2)} + R_{ps} e^{-\gamma^2 (r^2 + z^2)} \]

\[ + R_{ps} e^{-\gamma^2 (r^2 + z^2)} + R_{ps} e^{-\gamma^2 (r^2 + z^2)} \]

\[ + R_{ps} e^{-\gamma^2 (r^2 + z^2)} \]

\[ \int d\xi \]

where

\[ f(\xi) = \left( 2 \xi^2 - \frac{p^2}{\beta^2} \right)^2 + 2 \xi \xi', \]

\[ g(\xi) = \left( 2 \xi^2 - \frac{p^2}{\beta^2} \right)^2 - 2 \xi \xi', \]

\[ R_{pp} = -\frac{\xi}{F(\xi)} \]

\[ R_{ps} = \frac{\xi'}{F(\xi)} \]

\[ R_{pp} = \frac{\Delta_{12}}{\Delta_{12}} \]

\[ R_{ps} = \frac{\Delta_{12}}{\Delta_{12}} \]

\[ R_{ss} = \frac{\Delta_{14}}{\Delta_{14}} \]

\[ 1.23 \]

\[ 1.24a \]
\[ R_{ppp} = -\frac{f}{F} \frac{A_{23}}{A_{12}} = R_{pp} \cdot R_{pp} \]

\[ R_{pPS} = \frac{2\gamma_{\mu} f}{2} \frac{A_{24}}{A_{12}} = R_{pp} \cdot R_{ps} \]

\[ R_{ppS} = -\frac{A_{23}}{A_{12}} \left( 2 \gamma_{\mu}^2 - \frac{\beta^2}{\beta^2} \right) R_{pp} \cdot R_{ps} \]

\[ R_{pSS} = -\frac{A_{24}}{A_{12}} \left( 2 \gamma_{\mu}^2 - \frac{\beta^2}{\beta^2} \right) R_{ps} \cdot R_{ss} \]

\[ R_{pSP} = -\frac{A_{24}}{A_{12}} \left( 2 \gamma_{\mu}^2 - \frac{\beta^2}{\beta^2} \right) R_{ps} \cdot R_{sp} \]

\[ \omega(r, z, \theta) = \frac{P\tilde{r}(p)}{2\alpha^2} \left[ \frac{1}{\xi^z} \int_0^\infty \xi F_\mu (\gamma, \xi) \left\{ \left[ e^{-\gamma_1(h-\lambda)} \tilde{h}(h-\lambda) - e^{-\gamma_1(h-\lambda)} \delta(h-\lambda) \right] + \right. \right. \]

\[ -\frac{\gamma_1}{\xi^z} \left\{ \int R_{pp} e^{-\gamma_1(h-\lambda)} + R_{pSP} e^{-\gamma_1(h-\lambda)} \right\} \]

\[ + \left. \left[ R_{pp} e^{-\gamma_1(h-\lambda)} - \gamma_1 \delta(h-\lambda) \right] \right\} \]
The results are agreement with Singh (1966).

From the expressions for \( q(r, z, p) \) and \( \omega (r, z, p) \), we can easily separate the parts of displacement produced by a definite reflected wave directly by noting the exponential factor.

\[
R_{pp} (R_{ps}) = \text{plane wave reflection coefficient}
\]

when a P or S wave is derived from an incident P wave by reflection at the free surface (Singh (1966)).

\[
R'_{pp} (R'_{ps}) = \text{plane wave reflection coefficient when a P or S wave is derived from an incident P-wave by reflection at the interface.}
\]

We now proceed to calculate the integrals in equations (1.23 + 1.25) and finally the values of \( q(r, z, t) \) and \( \omega (r, z, t) \).
1.2 The evaluation of the integrals (1.23) and (1.25) will therefore depend on the evaluation of integrals of the following type:

\[ I = 2 \int_0^\infty F(\xi) J_1(\nu \xi) \, d\xi \]

where \( F(\xi) \) is an even function of \( \xi \) and of the form

\[ \frac{\psi(\xi)}{\Phi(\xi)} \exp \left\{ -v_1 a - v_1' b \right\} \]

where \( a, b \) contain \( h, H, z, \) and other constants but not \( \xi \).

When \( \xi \) is regarded as a complex variable, the branch points are given by

\[ \xi = \pm p_1, \pm p_2, \pm p_3, \pm p_4, \pm p_5 \]

where

\[ p = \frac{h}{d_i} e^{is} \]

Since displacements tend to zero as \( z \to \infty \), \( \text{Real} \nu_1 \geq 0 \), \( \text{Real} \nu_1' \geq 0 \), \( \text{Real} \nu_2 \geq 0 \) and \( \text{Real} \nu_2' \geq 0 \),

where

\[ \nu_1 = \sqrt{\xi^2 - \beta_1^2} \]
\[ \nu_1' = \sqrt{\xi^2 - \beta_1'^2} \]
\[ \nu_2 = \sqrt{\xi^2 - \beta_2^2} \]
\[ \nu_2' = \sqrt{\xi^2 - \beta_2'^2} \]

Therefore the branch cuts are given by \( \text{Real} \nu_1 = 0 \), \( \text{Real} \nu_1' = 0 \), \( \text{Real} \nu_2 = 0 \), and \( \text{Real} \nu_2' = 0 \). We take \( \beta_1, \beta_2, \beta_1', \beta_2', \) and \( p = c - is \), where \( c > 0 \), \( s > 0 \).

Transforming the path of integration from the real axis to the Sommerfeld contour we have.
Fig. 1.2 - Deformation of the path of integration in the complex $\Sigma$-plane $\text{Re}(\beta) > 0$, $\text{Im} \beta < 0$, $\beta_1 < \beta < \beta_2$, $\alpha_1 < \alpha_2$. 

(52)
\[
I = \int F(\xi) H_{\nu}(\xi) d\xi = \int_{L_{\alpha_1}} + \int_{L_{\beta_1}} + \int_{L_{\gamma_1}} + \int_{L_{\delta_1}} - 2\pi i \sum \text{Res}
\]

Evaluation of such integrals along \(L_{\alpha_i}, \ldots\) are thoroughly discussed by Singh (1966). Therefore we shall only give the final results.

We have taken \(\lambda(t) = e^{i\omega t} H(t)\) hence

\[
I(\gamma, z, t) = \frac{1}{2\pi} \int_{\alpha - \omega}^{\alpha + \omega} I(\gamma, \zeta, \eta) e^{ip\eta} d\eta
\]

1.3. Corresponding to the first, second and third integrals \(q\) parts of displacement \(q(r, z, \eta)\) we have

\[
q_1(\gamma, z, \eta) = \frac{p \hat{X}(p)}{2\alpha_1^2} \int_0^\infty -\frac{1}{2\eta} e^{-2\eta(\lambda - 2)} J_1(\gamma \eta) d\eta
\]

\[
q_2(\gamma, z, \eta) = \frac{p \hat{X}(p)}{2\alpha_1^2} \int_0^\infty \Re \hat{J}(p) e^{-2\eta(\lambda + 2)} J_1(\gamma \eta) d\eta
\]

\[
q_3(\gamma, z, \eta) = \frac{p \hat{X}(p)}{2\alpha_1^2} \int_0^\infty \Re \hat{K}(p) e^{-2\eta(\lambda - 2)} J_1(\gamma \eta) d\eta
\]

We denote the part of \(q_1(\gamma, z, \eta)\) obtained through integration along the loop \(L_{\alpha_1}, L_{\beta_1}, L_{\gamma_1}, \ldots\), and we
easily have,
\[ q_{1,1}(r, z, t) = -\frac{P}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right], \]

where \( t_p = \frac{\gamma}{\alpha_1} + \frac{(h+z)^2}{2 \alpha_1} \) \( \gamma, h, z \) are small compared to \( r \).

This expression gives the part of displacement due to direct wave \( P \). Similarly we have
\[ q_{1,0}(r, z, t) = -\frac{P}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

In a similar way we have
\[ \omega P(r, z, t) = \frac{2c_1^4}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

Again from Eq. (1,3) we have
\[ q_{1,1}(r, z, t) = -\frac{P}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

where \( t_p = \frac{\gamma}{\alpha_1} + \frac{(h+z)^2}{2 \alpha_1} \)

\[ R(r, z, h) = \frac{2c_1^4}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

\[ Q(r, z, h) = \frac{2c_1^4}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

\[ q_{1,1}(r, z, t) \] thus represents the part of displacement due to the reflected wave \( P \cdot P \). Similarly we have
\[ q_{1,1}(r, z, t) = \frac{P}{\pi \alpha_1^2} \frac{1}{r \alpha_1} \left[ \delta(t-t_p) + j\omega e^{j\omega(t-t_p)} H(t-t_p) \right] \]

where \( t_p = \frac{\gamma}{\beta_1} + \frac{(h+z)^2}{c_1} \) \( \beta_1, c_1 \) are small compared to \( r \).

From Fig. (1,3) \( OA = R_1, AB = R_2, BC = R_3, \frac{\delta \omega 2}{\alpha_1} = \frac{1}{\beta_1} \)
Since $a_1 > \beta_1$, $\sin \theta$ exceeds unity for a range of real values of $a_1$, $\beta_1$. In this case, instead of refracted waves, we have a surface movement whose amplitude diminishes exponentially with increasing distance from the boundary and which travels along the free surface with the velocity of shear waves. This wave is known as the diffracted wave.

Similarly,

$$Q_{b, \alpha_2}(r, z, t) = -\frac{\omega}{2a_1^2} \frac{c_f}{a_2^2} \frac{\Delta_{23,0}^x}{\Delta_{23,0}^f} \left( \frac{\Delta_{13,0}^x}{\Delta_{13,0}^f} + \frac{\Delta_{12,0}^x}{\Delta_{12,0}^f} \right) \frac{1}{\gamma^2} \times \exp \left[ i \omega \left( t - t_{pp,p} - t' \right) \right] H(t - t_{pp,p})$$

where

$$t_{pp,p} = \frac{\gamma}{\alpha_2} + \frac{(2\gamma - h - z)}{c_f}$$

$$\frac{1}{c_i} = \left( \frac{1}{\beta_i} - \frac{1}{a_i^2} \right) h$$

$$\frac{1}{c_f} = \left( \frac{1}{\alpha_f} - \frac{1}{a_f^2} \right) h$$
\[
\frac{1}{c_1} = \left( \frac{1}{\beta_1^2} - \frac{1}{\alpha_2^2} \right)^{1/2} \quad \frac{1}{c_1} = \left( \frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2} \right)^{1/2} \quad \frac{1}{c_1} = \left( \frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2} \right)^{1/2}
\]

\[
\Delta_{\beta_1,0} = -\mu \left[ \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16} \alpha_2^2} \right) \alpha_2^2 - \mu_2 \right] \left( \frac{1}{\alpha_2^2} + \frac{1}{c_{15} c_{16}} \right) - \frac{2 \mu_2}{\mu_1} \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16}} \right) \frac{1}{\alpha_2^2} - \frac{1}{c_{15} c_{16}} \frac{1}{\alpha_2^2} \frac{1}{c_{15} c_{16}}
\]

\[
\Delta_{\alpha_1,0} = -\mu \left[ \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16} \alpha_2^2} \right) \alpha_2^2 + \frac{2 \mu_2}{\mu_1} \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16}} \right) \frac{1}{\alpha_2^2} - \frac{1}{c_{15} c_{16}} \frac{1}{\alpha_2^2} \frac{1}{c_{15} c_{16}} \right]
\]

\[
\Delta_{\alpha_1,0} = -\mu \left[ \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16} \alpha_2^2} \right) \alpha_2^2 + \frac{2 \mu_2}{\mu_1} \left( \frac{1}{c_{17}^2} + \frac{1}{c_{15} c_{16}} \right) \frac{1}{\alpha_2^2} - \frac{1}{c_{15} c_{16}} \frac{1}{\alpha_2^2} \frac{1}{c_{15} c_{16}} \right]
\]

From analysing the travel time of \( t_{\beta_1,0} \), it is found that \( \alpha_1, \alpha_2 \) represents the part of the displacement due to the critically refracted wave which traverses the interface at the higher velocity \( \alpha_2 \) and the remainder of the path at the lower velocity \( \alpha_1 \) and the angle of incidence at the interface is equal to the critical angle \( \theta = \sin^{-1} \frac{a_2}{a_1} \left( 0 < \theta < \frac{\pi}{2} \right) \)

since \( \alpha_1 < \alpha_2 \). Therefore the integral along the loop \( L_{\alpha_2} \)
gives rise to critically refracted i.e. head wave \( q_{\beta_1,0} \)
1.4 Numerical Results

For purpose of comparison, we have assumed the values similar to those of Singh (1969), viz.,

\[
\begin{align*}
\alpha_1 &= 6.2 \text{ Km/sec} \\
\beta_0 &= 3.75 \text{ Km/sec} \\
\alpha_2 &= 8.2 \text{ Km/sec} \\
\beta_2 &= 4.05 \text{ Km/sec} \\
\frac{\mu}{\mu_1} &= 1.395
\end{align*}
\]

The amplitude of the direct P wave is denoted by \(A_p\), that of the reflected PP wave is denoted by \(A_{pp}\), and so on.

**Table 1**

\[
\begin{array}{cccccccccc}
\frac{\omega}{\mu} & 40 & 60 & 80 & 100 & 120 & 140 & 160 \\
\frac{2\alpha_1^2}{P} & 24.2 & 16.1 & 12.1 & 9.88 & 8.07 & 6.91 & 6.05 \\
\frac{2\alpha_2^2}{P} & 16.0 & 11.8 & 9.61 & 8.02 & 6.66 & 6.00 & 5.31 \\
\frac{2\alpha_1^2}{P} & 0.36 & 0.16 & 0.0901 & 0.0577 & 0.04 & 0.0294 & 0.0225 \\
\end{array}
\]
Fig. 1.5. Variation of amplitude with epicentral distance.

Fig. 1.6. Variation of amplitude with frequency.
Variation of amplitude with the depth of focus.

Fig. 1.7. Variation of amplitude with the depth of focus.
Table II

\[ r = 200\text{km} \quad h = 5 \text{ km} \quad H = 30 \text{ km} \quad z = 0 \]

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2\alpha^2}{p} \times 10^3 )</td>
<td>4.03</td>
<td>8.06</td>
<td>16.1</td>
<td>24.2</td>
<td>32.2</td>
</tr>
<tr>
<td>( \frac{2\beta^2}{p} \times 10^3 )</td>
<td>5.32</td>
<td>6.91</td>
<td>11.2</td>
<td>16.0</td>
<td>20.9</td>
</tr>
<tr>
<td>( \frac{2\gamma^2}{p} \times 10^3 )</td>
<td>0.00086</td>
<td>0.000036</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Table III

\[ \omega = 30 \quad r = 200\text{km} \]

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>0.0025</th>
<th>0.004</th>
<th>0.005</th>
<th>0.01</th>
<th>0.02</th>
<th>0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2\alpha^2}{p} \times 10^3 )</td>
<td>23.1</td>
<td>22.1</td>
<td>21.9</td>
<td>20.0</td>
<td>17.2</td>
<td>16.0</td>
</tr>
</tbody>
</table>

1.5 Discussion of Results

From the above discussion we have some idea about the disturbances in the layer associated with the point source.
Here we only consider the part of displacement $q(r,z,t)$ due to direct P, surface reflected PP and interface reflected PP' wave.

The evaluation of the integrals along the loops $L_{d1}$ gives rise to reflected waves, that along the loops $L_{d2}$ gives rise to head waves and that along the loops $L_{p1}$ and $L_{p2}$ gives rise to diffracted waves. From Table I, it is seen that the amplitude of direct P and surface reflected PP is large compared to that of head wave PPBP'. Since the amplitude of the diffracted wave PSP contains the decay factor $\exp \left\{ -\frac{\omega}{c} (4+2) \right\}$, its contribution is insignificant in comparison to the direct or reflected and head wave which are shown in Table II.

It is seen from Fig.1.5 that the amplitude of each phase decreases with the increase of epicentral distance but increases with $\omega A$ for fixed $h$ and $H$. An increase in depth of focus, however, causes decrease in amplitude (Fig.1.7)

1.6 Conclusion

It has been found that the results obtained by considering the body forces due to the compressional point source, are in agreement with the results done by several authors who had solved the problem by considering wave potential. For harmonic source, it has been shown that the contribution due to the diffracted waves is small as compared to that due to the minimum-
time-path waves which have been shown by Singh (1966). Out of the minimum-time path waves, the main contributors to the response curve are the direct and reflected waves at large epicentral distance.