CHAPTER III

ON THE UNSTEADY NON-NEWTONIAN ROTATING FLOW GENERATED

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§ 3.1 Introduction

Several authors including Greenspan and Howard (1963), Barrett (1969), Benton and Loper (1969, 1970), Jacobs (1971), Chawla (1973), Debnath (1972a, c, 1973) and others have studied the various aspects of the hydrodynamic and hydro-magnetic unsteady boundary layer flow in a rotating viscous fluid. From the initial value investigation of the torsionally generated hydrodynamic flow in a rotating system, Greenspan and Howard have made a critical analysis of the physical processes of the transient approach and ultimately studied the steady-state motion. It has been shown that the transient motion consists of the Ekman layer, spin-up and the decay of inertial oscillations by viscous diffusion. On the other hand, studies of the hydromagnetic problem reveal that the unsteady flow consists of Ekman, Hartmann and/or Stokes layer, hydromagnetic spin-up, and the generation of decaying inertial oscillations and the diffused hydromagnetic waves.

From the standpoint of geophysical applications, it is interesting to make an investigation of the torsionally generated unsteady boundary layer flow in a rotating elastico-
viscous fluid. Further, this investigation may be regarded as an extension of the hydrodynamic problem in a Newtonian viscous fluid, and hence deserves consideration on its own merit.

This paper is concerned with the study of the unsteady boundary layer flow generated in a semi-infinite expanse of rotating elastico-viscous fluid bounded by an infinite rigid plate which performs torsional oscillations with a given frequency. The analysis is carried out for the determination of the steady and unsteady flow fields, the structure of the associated boundary layers, the significant effects of the elastic parameter and rotation on the flow, and the Ekman suction velocity with physical significance. It has been shown that the analysis reveals the existence of two distinct Stokes-Ekman elastic boundary layers adjacent to the plate. It has also been demonstrated that the present study includes the solutions found by Debnath (1972 a, c; 1973), Greenspan and Howard (1963).

§ 3.2 Mathematical Formulation Of The Problem

We consider the unsteady motion in a semi-infinite expanse of an incompressible homogeneous visco-elastic fluid bounded by an infinite rigid plate. The whole system is in a state of solid body rotation with a uniform angular velocity \( \Omega \) about the z-axis normal to the plate.
In view of the inherent symmetry of the problem and the continuity equation, it is convenient to use the cylindrical polar coordinate \((r, \theta, z)\) where \(z\)-axis is along the axis of rotation, and \(r, \theta\)-axes are rotating with the system.

The equations of motion and continuity in a rotating frame of reference are

\[
\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} + 2\Omega \times \mathbf{q} = \nabla \mathcal{P} + \nabla (\nabla^2 \mathbf{q}) - k_0^* \left[ \frac{\partial}{\partial t} \nabla^2 \mathbf{q} + 2 (\mathbf{q} \cdot \nabla) \nabla^2 \mathbf{q} + \nabla \mathbf{q} \cdot \nabla \mathbf{q} \right],
\]

(3.1)

\[
\mathbf{u} \cdot \mathbf{q} = 0
\]

(3.2)

where \(\Omega\) is the rotation vector, \(\mathbf{q}\) is the velocity field, \(\mathcal{P}\) is pressure term including centrifugal force term, \(\nu = \frac{\eta_0}{\rho}\) is kinematic viscosity, \(k_0^* = \frac{k_0}{\rho}\) is the coefficient of elasticity and \(\rho\) is the density of fluid, \(\eta_0, k_0\) being the coefficients of absolute viscosity and elasticity.

We introduce a stream function \(\psi (r, z, t)\) such that the velocity vector \(\mathbf{q}\) can be written as

\[
\mathbf{q} = r \nabla (z, t) \hat{\theta} - \text{curl} \left\{ \hat{\theta} \psi (r, z, t) \right\}, \quad \psi = r \mathcal{J}(z, t)
\]

(3.3)

where \(\hat{\theta}\) is the unit vector along the \(\theta\)-direction of the coordinate system, and \(\mathbf{v}(z, t)\) is the transverse velocity.
We take the curl of equation (3.1) and then substitute (3.3) into it along with the assumption that the amplitude of the torsional oscillation of the plate is small compared with \( \Omega \) so that the non-linear terms in (2.9) can be neglected to obtain the equations

\[
\frac{\partial V}{\partial t} + 2\Omega \frac{\partial X}{\partial z} = 2\nu \frac{\partial^2 V}{\partial z^2} - \kappa_x \frac{\partial^3 V}{\partial t \partial z^2},
\]

(3.4)

\[
\frac{\partial^3 X}{\partial t \partial z^2} - 2\Omega \frac{\partial V}{\partial z} = 2\nu \frac{\partial^4 X}{\partial z^4} - \kappa_x \frac{\partial^5 X}{\partial t \partial z^4}.
\]

(3.5)

In view of the imposed oscillations on the boundary and no disturbance at infinity, equations (3.4) and (3.5) have to be solved subject to the boundary and initial conditions

\[
V(z,t) = \xi \tilde{F}(t) \quad \text{on} \quad z = 0, \quad t > 0, \quad (3.6)
\]

\[
\left( x, \frac{\partial x}{\partial z} \right) = (0,0) \quad \text{on} \quad z = 0, \quad t > 0, \quad (3.7)
\]

\[
\left( x, \frac{\partial x}{\partial z} \right) \rightarrow (0,0) \quad \text{as} \quad z \rightarrow \infty, \quad t > 0, \quad (3.8)
\]

\[
\nabla \cdot \nabla \times \nabla = \frac{\partial^2 \nabla}{\partial z^2} = \frac{\partial^4 \nabla}{\partial z^4} = \frac{\partial^6 \nabla}{\partial z^6} = 0 \quad \text{for all} \quad z > 0 \quad \text{at} \quad t = 0, \quad (3.9)
\]

where \( \xi \) is the amplitude of the imposed torsional oscillations which is assumed to be small compared to \( \Omega \) and \( \tilde{F}(t) \) is an arbitrary periodic function of time \( t \).
We choose a particular form of \( F(t) \) as

\[
F(t) = A e^{i \omega t} + B e^{-i \omega t}
\]  

where \( A, B \) are complex constants and \( \omega_3 \) is the frequency of oscillations so that \( V \) is real on the boundary.

In terms of the dimensionless flow variables defined by

\[
z' = \frac{z}{d} \quad \text{and} \quad t' = \frac{t}{\omega_3} \quad \text{and} \quad \nu' = \frac{\nu}{\nu d} \quad \text{and} \quad \kappa' = \frac{\kappa}{\nu d^2}
\]  

(3.11)

where \( d_0 = \left( \frac{\omega_3}{\Omega_3} \right)^{\frac{1}{2}} \)

equations (3.4) and (3.5) reduce to

\[
\frac{\partial V}{\partial t} + 2 \frac{\partial \nu}{\partial z} = \frac{\partial^2 V}{\partial z^2} - \kappa \frac{\partial^3 \nu}{\partial t \partial z^2} \quad (3.12)
\]

\[
\frac{\partial^2 \nu}{\partial t \partial z^2} - 2 \frac{\partial V}{\partial z} = \frac{\partial^4 \nu}{\partial z^4} - \kappa \frac{\partial^5 \nu}{\partial t \partial z^4} \quad (3.13)
\]

where \( \kappa = \frac{k_0^*}{d^2} \).

In terms of the non-dimensional variables (3.11), the form of the boundary and the initial conditions remain the same except (3.6) which takes the form

\[
V(z,t) = a e^{i \omega t} + b e^{-i \omega t} = f(t) \quad \text{on} \quad z=0, \quad t>0 \quad (3.14)
\]

where

\[
(a, b) = \frac{E}{\Omega_3 d} (A, B) \quad \text{and} \quad \omega = \frac{\omega_3}{\Omega_3} > 0
\]
3.3 The Solution Of Initial Valued Problem

The Laplace transformation

\[ \tilde{\varphi}(z, s) = \int_0^\infty e^{-st} \varphi(z, t) \, dt, \]  

(3.15)
can be applied to obtain the solution of the coupled boundary layer equations (3.12) and (3.13) with the boundary and the initial conditions.

The solution of the Laplace-transformed system are given by

\[ \bar{\varphi}(z, s) = \frac{1}{2} \tilde{\varphi}(s) \left[ e^{-\lambda_1^* z} + e^{-\lambda_2^* z} \right], \]  

(3.16)

\[ \bar{\chi}(z, s) = \frac{i}{2} \tilde{\varphi}(s) \left[ \left( \frac{1}{\lambda_1^*} - \frac{1}{\lambda_2^*} \right) - \left( \frac{e^{-\lambda_1^* z}}{\lambda_1^*} - \frac{e^{-\lambda_2^* z}}{\lambda_2^*} \right) \right], \]  

(3.17)

and

\[ \frac{d\bar{\chi}}{dz} = \frac{i}{2} \tilde{\varphi}(s) \left[ e^{-\lambda_1^* z} - e^{-\lambda_2^* z} \right], \]  

(3.18)

where \( \lambda_1^* \) and \( \lambda_2^* \) are given by

\[ \lambda_1^* = \left( \frac{s+2i}{1-sk} \right)^\frac{1}{2} \quad \text{and} \quad \lambda_2^* = \left( \frac{s-2i}{1-sk} \right)^\frac{1}{2}. \]  

(3.19)

The exact evaluation of (3.16) to (3.18) is not possible. Hence the approximate solutions are given below which give a fairly complete description of the steady as well as the unsteady flow.
§ 3.4 The Unsteady Motion For Small Values Of The Elastic Parameter

Allowing for the fact that the elastic parameter \( k \) is small, the solution \( \sqrt{V(z,s)} \) can readily be approximated in the form

\[
\sqrt{V(z,s)} = \frac{1}{2} \mathcal{F}^{-1}(s) \left[ \left( 1 - \frac{skm^2z}{2} \right) e^{-m_1z} + \left( 1 - \frac{skm^2z}{2} \right) e^{-m_2z} \right]
\]

(3.20)

where \( m_1 = (s+2i)^{\frac{1}{2}} \) and \( m_2 = (s-2i)^{\frac{1}{2}} \). (3.21)

Using the table of the inverse Laplace transform due to Campbell and Foster (1948), the transverse component of velocity is given by

\[
\sqrt{V(z,t)} = \frac{\alpha}{4} e^{j\omega t} \left[ \left( 1 - \frac{1}{2} \lambda_1 \omega k z \right) e^{-\lambda_1 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} - \lambda_1 \sqrt{t} \right) + \left( 1 + \frac{1}{2} \lambda_1 \omega k z \right) e^{\lambda_1 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} + \lambda_1 \sqrt{t} \right) \right]
\]

\[
+ \left( 1 - \frac{1}{2} \lambda_2 \omega k z \right) e^{-\lambda_2 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} + \lambda_2 \sqrt{t} \right) + \left( 1 + \frac{1}{2} \lambda_2 \omega k z \right) e^{\lambda_2 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} - \lambda_2 \sqrt{t} \right) \]

\[
+ \frac{b}{4} \frac{e^{-\omega t}}{\sqrt{t}}
\]

\[
x \left[ \left( 1 + \frac{1}{2} \lambda_1 \omega k z \right) e^{-\lambda_1 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} + \lambda_1 \sqrt{t} \right) + \left( 1 - \frac{1}{2} \lambda_1 \omega k z \right) e^{\lambda_1 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} - \lambda_1 \sqrt{t} \right) \right]
\]

\[
+ \left( 1 - \frac{1}{2} \lambda_2 \omega k z \right) e^{-\lambda_2 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} + \lambda_2 \sqrt{t} \right) + \left( 1 + \frac{1}{2} \lambda_2 \omega k z \right) e^{\lambda_2 z} \text{erfc} \left( \frac{Z}{2\sqrt{t}} - \lambda_2 \sqrt{t} \right) \]

\[
+ \sqrt{kz} \left[ \frac{(a+b)}{4 \sqrt{\pi t}} + \frac{i\omega(b-a)}{2 \sqrt{\pi t}} \right] \exp \left( -\frac{Z^2}{4t} \right) \cos \omega t ,
\]

(3.22)
Similarly we obtain

\[ \mathcal{A}(z, t) = \frac{ia}{2} e^{i\omega t} \left[ (1 - i\omega k \frac{b}{2}) \left\{ \frac{\text{erf}(\lambda' V E)}{\lambda'-1} - \frac{\text{erf}(\lambda V E)}{\lambda'} \right\} + i \frac{k}{\sqrt{\pi t}} e^{-i\omega t} \sin 2t \right] \\
+ \frac{ib}{2} e^{-i\omega t} \left[ (1 + i\omega k \frac{b}{2}) \left\{ \frac{\text{erf}(\lambda' V E)}{\lambda'-1} - \frac{\text{erf}(\lambda V E)}{\lambda'} \right\} + i \frac{k}{\sqrt{\pi t}} e^{i\omega t} \sin 2t \right] \\
- \frac{i \alpha^2}{2} \left\{ 1 - \frac{i k \omega}{2} (1 + \lambda' V E) \right\} \left\{ e^{-\lambda V E} \text{erf}(\frac{z}{2 V E} - \lambda V E) - e^{\lambda' V E} \right\} \\
x \left\{ e^{-\lambda' V E} \text{erf}(\frac{z}{2 V E} + \lambda' V E) \right\} \\
+ \frac{ib}{4 \lambda'} e^{-i\omega t} \left\{ 1 + \frac{i k \omega}{2} (1 + \lambda' V E) \right\} \left\{ e^{-\lambda V E} \text{erf}(\frac{z}{2 V E} - \lambda V E) - e^{\lambda' V E} \right\} \\
- e^{\lambda' V E} \text{erf}(\frac{z}{2 V E} + \lambda' V E) \right\} \\
x \left\{ e^{-\lambda V E} \text{erf}(\frac{z}{2 V E} - \lambda V E) - e^{\lambda' V E} \text{erf}(\frac{z}{2 V E} + \lambda' V E) \right\} \\
+ \frac{1 k}{2 \sqrt{\pi t}} \left( 1 + \frac{z}{2 t} \right) \exp \left( - \frac{z^2}{4 t} \right) \sin 2 \gamma \exp (-b - a). \quad (3.24) \]

Solutions (3.22) and (3.24) describe the general features of the unsteady boundary layer flow in a visco-elastic fluid for small values of the elastic parameter \( k \). These solutions also include the inertial oscillations of dimensional frequency \( \omega \).
It is worth noting that when \( k = 0, \omega \neq 0 \), equations (3.22) and (3.24) are exactly identical with those of Debnath (1972, 1973), and when \( k = 0 \) and \( \omega = 0 \), the solutions of Greenspan and Howard (1963) follow as special cases of the above results.

§ 3.5 The Ultimate Flow Field And The Structure Of The Associated Boundary Layers

According to the Laplace transform theory, the solution for the final flow field as \( t \to \infty \) follows from the behaviour of the transform solution as \( s \to 0 \) for fixed elastic parameter \( k \). Invoking the approximate expressions for \( \lambda^*_r, \gamma = 1, 2 \) as

\[
\lambda_1^* \sim (1 + 2ik) s + 2i, \quad \lambda_2^* \sim (1 - 2ik) s - 2i \quad \text{as} \quad s \to 0,
\]

(3.25)

into equations (3.16) to (3.18) and carrying out the inversion, we obtain

\[
\chi(z, t) = \frac{a}{4} e^{i\omega \xi} \left[ e^{-\lambda_2 z} \exp \left( \frac{z}{2} \sqrt{\frac{\sigma_1}{E_1}} - \lambda_3 \sqrt{\frac{E_1}{\sigma_1}} \right) + e^{\lambda_3 z} \left( \frac{z}{2} \sqrt{\frac{\sigma_1}{E_1}} + \lambda_3 \sqrt{\frac{E_1}{\sigma_1}} \right) \right]
\]

\[
\cdot \frac{b}{4} e^{i\omega \xi} \left[ e^{-\lambda_4 z} \exp \left( \frac{z}{2} \sqrt{\frac{\sigma_2}{E_2}} - \lambda_4 \sqrt{\frac{E_2}{\sigma_2}} \right) + e^{\lambda_4 z} \left( \frac{z}{2} \sqrt{\frac{\sigma_2}{E_2}} + \lambda_4 \sqrt{\frac{E_2}{\sigma_2}} \right) \right]
\]

\[
+ \frac{a}{4} e^{i\omega \xi} \left[ e^{-\lambda_6 z} \exp \left( \frac{z}{2} \sqrt{\frac{\sigma_1}{E_1}} - \lambda_6 \sqrt{\frac{E_1}{\sigma_1}} \right) + e^{\lambda_6 z} \left( \frac{z}{2} \sqrt{\frac{\sigma_1}{E_1}} + \lambda_6 \sqrt{\frac{E_1}{\sigma_1}} \right) \right]
\]

\[
+ \frac{b}{4} e^{i\omega \xi} \left[ e^{-\lambda_5 z} \exp \left( \frac{z}{2} \sqrt{\frac{\sigma_2}{E_2}} - \lambda_5 \sqrt{\frac{E_2}{\sigma_2}} \right) + e^{\lambda_5 z} \left( \frac{z}{2} \sqrt{\frac{\sigma_2}{E_2}} + \lambda_5 \sqrt{\frac{E_2}{\sigma_2}} \right) \right]
\]

(3.26)
where \( \lambda_3 = \left\{ \frac{i(2\omega - 2k\omega)}{\sqrt{\sigma_1}} \right\}^{\frac{1}{2}} \), \( \lambda_4 = \left\{ \frac{i(2\omega + 2k\omega)}{\sqrt{\sigma_2}} \right\}^{\frac{1}{2}} \),
\[
\begin{align*}
\lambda_5 &= \left\{ \frac{i(2\omega - 2k\omega)}{\sqrt{\sigma_1}} \right\}^{\frac{1}{2}}, \\
\lambda_6 &= \left\{ \frac{i(2\omega + 2k\omega)}{\sqrt{\sigma_2}} \right\}^{\frac{1}{2}},
\end{align*}
\]
\( \sigma_1 = (1 + 2ik) \) and \( \sigma_2 = (1 - 2ik) \),

and
\[
\chi(z, t) \sim \frac{ia}{q} e^{\frac{iqt}{\sigma_1}} \left[ \frac{\text{erf} \left( \frac{(p_1 + i\omega)^{\frac{1}{2}}}{\sqrt{\sigma_1}} \right)}{\sqrt{\sigma_1} (p_1 + i\omega)^{\frac{1}{2}}} - \frac{\text{erf} \left( \frac{(p_2 + i\omega)^{\frac{1}{2}}}{\sqrt{\sigma_2}} \right)}{\sqrt{\sigma_2} (p_2 + i\omega)^{\frac{1}{2}}} \right]
\]
\[
+ \frac{ib}{q} e^{\frac{iqt}{\sigma_1}} \left[ \frac{\text{erf} \left( \frac{(p_1 - i\omega)^{\frac{1}{2}}}{\sqrt{\sigma_1}} \right)}{\sqrt{\sigma_1} (p_1 - i\omega)^{\frac{1}{2}}} - \frac{\text{erf} \left( \frac{(p_2 - i\omega)^{\frac{1}{2}}}{\sqrt{\sigma_2}} \right)}{\sqrt{\sigma_2} (p_2 - i\omega)^{\frac{1}{2}}} \right]
\]
\[
- \frac{1}{\mu_1 \sqrt{\sigma_1}} \left\{ \exp \left( -z \mu_1 \sqrt{\sigma_1} \right) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_1}} - \mu_1 \sqrt{\sigma_1} \right) - \exp(z \mu_1 \sqrt{\sigma_1}) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_1}} + \mu_1 \sqrt{\sigma_1} \right) \right\}
\]
\[
- \frac{1}{\mu_2 \sqrt{\sigma_2}} \left\{ \exp \left( -z \mu_2 \sqrt{\sigma_2} \right) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_2}} - \mu_2 \sqrt{\sigma_2} \right) - \exp(z \mu_2 \sqrt{\sigma_2}) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_2}} + \mu_2 \sqrt{\sigma_2} \right) \right\}
\]
\[
- \frac{ib}{q} e^{-\frac{iqt}{\sigma_1}} \left[ \frac{1}{\mu_2 \sqrt{\sigma_1}} \left\{ \exp \left( -z \mu_2 \sqrt{\sigma_1} \right) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_1}} - \mu_2 \sqrt{\sigma_1} \right) - \exp(z \mu_2 \sqrt{\sigma_1}) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_1}} + \mu_2 \sqrt{\sigma_1} \right) \right\}
\]
\[
- \frac{1}{\mu_2 \sqrt{\sigma_2}} \left\{ \exp \left( -z \mu_1 \sqrt{\sigma_2} \right) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_2}} - \mu_1 \sqrt{\sigma_2} \right) - \exp(z \mu_1 \sqrt{\sigma_2}) \text{erfc} \left( \frac{z}{2 \sqrt{\sigma_2}} + \mu_1 \sqrt{\sigma_2} \right) \right\}
\]

(3.27)
where
\[ \rho_1 = \frac{2i}{\sigma_1}, \quad \mu_1 = \left( \frac{2i}{\sigma_1} + i\omega \right)^{1/2}, \quad \mu_2 = \left( 1 - \frac{2i}{\sigma_1} - i\omega \right)^{1/2}. \]
\[ \rho_2 = -\frac{2i}{\sigma_2}, \quad \mu_1' = \left( \frac{2i}{\sigma_2} + i\omega \right)^{1/2}, \quad \mu_2' = \left( 1 - \frac{2i}{\sigma_2} - i\omega \right)^{1/2}. \]

It is clear that the effect of elasticity is reflected in the boundary layer solutions (3.26) and (3.27).

Proceeding to the limit time \( t \to \infty \) and using asymptotic representation of the complementary error function in (3.26) and (3.27), we obtain the ultimate steady-state solution in the form
\[ V(z,t) \sim \frac{a}{2} e^{i\omega t} \left( e^{-\lambda_3 z} + e^{-\lambda_5 z} \right) + \frac{b}{2} e^{-i\omega t} \left( e^{-\lambda_4 z} + e^{-\lambda_6 z} \right) \]
(3.28)

and
\[ \chi(z,t) \sim \frac{ia}{4} e^{i\omega t} \left[ \frac{1}{\sqrt{\sigma_1} (\rho_1 + i\omega)^{1/2}} - \frac{1}{\sqrt{\sigma_3} (\rho_2 + i\omega)^{1/2}} \right] 
+ \frac{ib}{4} e^{-i\omega t} \left[ \frac{1}{\sqrt{\sigma_1} (\rho_1 - i\omega)^{1/2}} - \frac{1}{\sqrt{\sigma_2} (\rho_2 - i\omega)^{1/2}} \right] 
- \frac{ia}{2} e^{i\omega t} \left[ \frac{1}{\sqrt{\sigma_1} \mu_1 \rho_1 \pm 1} \exp \left( -z / \mu_1 \sigma_1 \right) \right] 
- \frac{ib}{2} e^{-i\omega t} \left[ \frac{1}{\sqrt{\sigma_2} \mu_2 \rho_2 \pm 1} \exp \left( -z / \mu_2 \sigma_2 \right) \right]. \]
(3.29)
These results enable us to determine the ultimate steady-state velocity field as
\[
\begin{align*}
\frac{u}{r} = \frac{\partial \eta}{\partial z} & \sim \frac{i a}{2} e^{i \omega t} \left[ \exp (-z \mu_1 \tau_{\sigma_1}) - \exp (-z \mu_2 \sqrt{\sigma_2}) \right] \\
& + \frac{1}{2} b e^{-i \omega t} \left[ \exp (-z \mu_1 \sqrt{\sigma_1}) - \exp (-z \mu_2 \sqrt{\sigma_2}) \right], \quad (3.30)
\end{align*}
\]

\[
\frac{v}{r} = V(z, t) \sim \frac{a}{2} e^{i \omega t} \left[ e^{-\lambda_3 z} + e^{-\lambda_4 z} \right] + \frac{1}{2} e^{i \omega t} \left[ e^{-\lambda_4 z} + e^{-\lambda_5 z} \right], \quad (3.31)
\]

\[
\omega = -\frac{\partial \eta}{\partial t} \sim \frac{i a}{2} e^{i \omega t} \left[ \frac{1}{\tau_{\sigma_2} (\rho_1 + i \omega)^{1/2}} - \frac{1}{\tau_{\sigma_1} (\rho_1 + i \omega)^{1/2}} \right] \\
& + \frac{1}{2} e^{-i \omega t} \left[ \frac{1}{\tau_{\sigma_2} (\rho_2 - i \omega)^{1/2}} - \frac{1}{\tau_{\sigma_1} (\rho_2 - i \omega)^{1/2}} \right] \\
& + i a e^{i \omega t} \left[ \frac{1}{\mu_1 \tau_{\sigma_1}} \exp (-z \mu_1 \sqrt{\sigma_1}) - \frac{1}{\mu_5 \tau_{\sigma_2}} \exp (-z \mu_5 \sqrt{\sigma_2}) \right] \\
& + i b e^{-i \omega t} \left[ \frac{1}{\mu_1 \tau_{\sigma_1}} \exp (-z \mu_1 \sqrt{\sigma_1}) - \frac{1}{\mu_5 \tau_{\sigma_2}} \exp (-z \mu_5 \sqrt{\sigma_2}) \right], \quad (3.32)
\]

In order to investigate the structure of the boundary layers associated with the flow field, it is convenient to express \( \lambda_\tau = \alpha_\tau + i \beta_\tau \) \((\tau = 3, 4, 5, 6)\) so that
Substituting $\alpha_r + \beta_r$ for $\alpha_r$ into (3.30) to (3.32), it follows that the velocity fields consist of two distinct Stokes-Ekman-elastic boundary layers of thicknesses of the orders $\left(\frac{3}{4}\right)^{\gamma_2} \frac{1}{d_r}$, $\gamma = 3, 4$. In the limit $k \to 0$ these boundary layers correspond to the combined Stokes-Ekman layers of thicknesses of the orders $\left(\frac{3}{2} \Omega + \frac{\omega}{2}\right)^{\gamma_2}$ in dimensional unit.

It may easily be verified that results (3.30) to (3.32) do not depend on the order of the double limit $t \to \infty$, $z \to \infty$ and lead to a unique answer. Consequently, the only axial component of the velocity field is found to be non-zero and is given by

$$L_m l_m \omega(z, t) = \frac{ia}{2} e^{i\omega t} \left[ \frac{1}{\sqrt{\sigma_2 (p_2 + i\omega)^{\gamma_2}}} - \frac{1}{\sqrt{\sigma_1 (p_1 + i\omega)^{\gamma_2}}} \right]$$

$$+ \frac{ib}{2} e^{-i\omega t} \left[ \frac{1}{\sqrt{\sigma_2 (p_2 - i\omega)^{\gamma_2}}} - \frac{1}{\sqrt{\sigma_1 (p_1 - i\omega)^{\gamma_2}}} \right],$$

(3.34)
where the real part of the right hand side of (3.34) is to be taken.

This may be recognized as the Ekman suction velocity in non-Newtonian rotating fluids which represents the generation of an axial inflow towards the Stokes-Ekman-Elastic boundary layers. Thus these boundary layers play an important role in the transient process by inducing a purely vertical secondary flow throughout the main body of the fluid by the boundary layer suction.

The characteristic features revealed in the above analysis are in accord with those of Debnath (1972, 1973) when $k = 0$, $\omega = 0$.

Hence a further elaboration about these special cases would be redundant.
3.6 Concluding Remarks

The above analysis appears to have some geophysical interest and can be considered to be useful for the study of the motion of the earth's liquid core. The nature of the multiple boundary layers and the Ekman suction velocity seem to play important roles on the spin-up mechanism in the non-Newtonian rotating fluid flow.

This chapter is based on one of my papers jointly written with Professor L. Debnath [Debnath and Basu (1975c)] and the paper has been accepted for publication in Nuovo Cimento (Italy) in series B.

* The paper has already been published in

Nuovo Cimento (Italy), 1975, 27B, p 31-44