CHAPTER - 6

A NOTE ON THE DEFORMATION OF CONSOLIDATING SPHERICAL SHELL

§ 6.1 Introduction

In the previous chapter, an extension of Cinelli's truncated Hankel transform method has been established. In this chapter an application of the method is made to solve a problem of deformation of consolidating spherical shell.

Jana and Sanayal (1971) have treated the same problem, using Laplace transform technique and obtained approximate solutions. The present procedure shows how concisely the solutions of the same problem can be achieved by the above mentioned method. This is to be noted that no approximation is needed in this case.

§ 6.2 Mathematical Formulation

Following Biot (1955) a physical assumption that the shell is purely elastic and contains a compressible viscous fluid is made.

Assume polar symmetry about the origin of a spherical polar co-ordinates \((r, \theta, \phi)\) and take the solid and fluid displacements as

\[
U_r = U(r, t), \quad U_\theta = 0, \quad U_\phi = 0.
\]

\[
V_r = V(r, t), \quad V_\theta = 0, \quad V_\phi = 0.
\]
The constitutive equations for an isotropic poroelastic medium are, (Biot, 1955)

\[ \sigma_{\tau \tau} = 2N\varepsilon_{\tau \tau} + A\varepsilon + Q\varepsilon \] ...

\[ \sigma_{\phi \phi} = 2N\varepsilon_{\phi \phi} + A\varepsilon + Q\varepsilon \] ...

\[ \sigma_{\tau \phi} = \sigma_{\phi \tau} = \sigma_{\phi \phi} = 0 > \sigma^* = Q\varepsilon + R\varepsilon \] ...

\[ e = \text{div } \mathbf{u}^* \quad \varepsilon = \text{div } \mathbf{v}^* \] ...

\( \sigma_{\tau \tau}, \sigma_{\phi \phi} \) etc. are stress components and \( \sigma^* \) is the fluid stress. Expressing the stresses in terms of the displacements \( u \) and \( v \) the dynamic equations are

\[ \nabla^2 \mathbf{u}^* + \text{grad} [\mathbf{C}_N + \mathbf{N} + \mathbf{Q} + \mathbf{R}] = \frac{\partial}{\partial t} \left( \int I_1 \mathbf{u}^* + \int I_2 \mathbf{v}^* \right) + \] ...

\[ 0 \] ...

\[ \text{grad} [Q\varepsilon + R\varepsilon] = \frac{\partial}{\partial t} \left( \int I_1 \mathbf{u}^* + \int I_2 \mathbf{v}^* \right) \] ...

\( A, N, Q, R \) are elastic constants.

The coefficients \( c \) is related to Darcy's coefficient of permeability \( K \) by

\[ c = \mu/\beta \] where \( \mu \) is the fluid viscosity and \( \beta \) the porosity. The coefficients \( I_1, I_2, \) etc. are mass coefficients (Biot, 1956).
Applying the divergence operator to equations (6.4) and (6.5) we find the equations

\[ \nabla^2 (\rho \varepsilon + \psi \varepsilon) = \frac{\partial^2}{\partial t^2} \left( f_{12} \varepsilon + s_{12} \varepsilon \right) + C \frac{\partial}{\partial t} \left( C \cdot \varepsilon \right) \]

\[ \ldots (6.6) \]

\[ \Delta^2 (\rho \varepsilon + \psi \varepsilon) = \frac{\partial^2}{\partial t^2} \left( f_{12} \varepsilon + s_{12} \varepsilon \right) - C \frac{\partial}{\partial t} \left( C \cdot \varepsilon \right) \]

\[ \ldots (6.7) \]

Substitute

\[ \varepsilon = \begin{cases} \rho^{-1/2} \varepsilon_1 (\tau, t) \\
\rho^{-1} \varepsilon_1 (\tau, t) \end{cases} \]

Then from (6.6) and (6.7) we get

\[ L \left[ \rho \varepsilon_1 \right] + L \left[ \psi \varepsilon_1 \right] = f_{11} \frac{\partial^2 \varepsilon_1}{\partial t^2} + f_{12} \frac{\partial \varepsilon_1}{\partial t} + \]

\[ + C \frac{\partial}{\partial t} \left( \varepsilon_1 \cdot \varepsilon_1 \right) \]

\[ \ldots (6.8) \]

\[ L \left[ \rho \varepsilon_1 \right] + L \left[ \psi \varepsilon_1 \right] = f_{12} \frac{\partial \varepsilon_1}{\partial t} + f_{22} \frac{\partial \varepsilon_1}{\partial t} - C \frac{\partial}{\partial t} \left( \varepsilon_1 \cdot \varepsilon_1 \right) \]

\[ \ldots (6.9) \]

where \( L \) denotes the operator

\[ L \equiv -\frac{\partial^2}{\partial \tau^2} + -\frac{1}{\tau} - \frac{1}{\tau^2} \]

Equations (6.8) and (6.9) can be rewritten as

\[ (\mathcal{P} L - f_{11} \frac{\partial^2}{\partial t^2} - C \frac{\partial}{\partial t}) \varepsilon_1 + (\mathcal{Q} L - f_{12} \frac{\partial}{\partial t} + C \frac{\partial}{\partial t}) \varepsilon_1 = 0 \]

\[ \ldots (6.10) \]

\[ (\mathcal{Q} L - f_{12} \frac{\partial}{\partial t} + C \frac{\partial}{\partial t}) \varepsilon_1 + (\mathcal{R} L - f_{22} \frac{\partial}{\partial t} - C \frac{\partial}{\partial t}) \varepsilon_1 = 0 \]

\[ \ldots (6.11) \]
Eliminating $\varepsilon_i$, we have
\[(\rho R - \sigma^\nu) L^2 \varepsilon_i + \{ (2 \sigma_i^{\nu, \nu} - \kappa) f_{1,1} - \rho f_{2,2} \} \frac{\partial^2 \varepsilon_i}{\partial t^2} - (2 \sigma + P + \kappa) \]
\[\times \frac{\partial^2 \varepsilon_i}{\partial t^2} \} L \varepsilon_i + \{ (f_{1,1} f_{1,1} - f_{1,1} \nu) \frac{\partial^2 \varepsilon_i}{\partial t^2} + \sigma (f_{1,1} f_{1,1} + f_{1,1} \nu) \} \frac{\partial^3 \varepsilon_i}{\partial t^3} \} \varepsilon_i = 0 \]
\[\text{..(6.12)}\]

when there is no relative motion between fluid and solid,$\int f_{1,1} f_{1,1} + f_{1,1} \nu \text{ can be taken as total mass of the fluid -solid aggregate per unit volume.}\]

We now consider a region occupied by a thick spherical shell of inner and outer radii $a$ and $b$ respectively of porous material under the following boundary conditions:
\[\sigma^i = f_1(t), \quad \sigma^{i, r} = f_2(t) \quad \text{on} \quad r = a \quad t > 0 \]
\[\text{..(6.13)}\]
\[\sigma^i = f_3(t), \quad \sigma^{i, r} = f_4(t) \quad \text{on} \quad r = b \quad t > 0 \]
\[\text{..(6.14)}\]

Initially, at $t = 0$, we assume the displacements, pressure, dilatation and stresses are zero.

Boundary conditions taken here are same as those taken by Jana and Sanyal (1971).

This is the most general case that has been considered. It is not, however, realised how one can prescribe different stresses on the fluid and solid parts of the same boundary of a poroelastic material.
Now boundary conditions can be rewritten as
\[
f_2(t) - \frac{\partial}{\partial r} f_1(t) = \left(2N - A - \frac{\alpha_0}{R}\right) \frac{\partial u}{\partial r} + \left(N - \frac{\alpha_0}{R}\right) \frac{2u}{r} \quad \text{on} \quad r = a, \ t > 0
\]
and
\[
f_4(t) - \frac{\partial}{\partial r} f_3(t) = \left(2N \cdot A - \frac{\alpha_0}{R}\right) \frac{\partial u}{\partial r} + \left(N - \frac{\alpha_0}{R}\right) \frac{2u}{r} \quad \text{on} \quad r = b, \ t > 0.
\]
If \( A = N \) (corresponds to Poisson’s condition of elasticity) we can write
\[
X \phi_1(t) = \frac{\partial u}{\partial r} + \frac{2u}{r} \quad \text{on} \quad r = a, \ t > 0 \quad \ldots \ldots \ (6.15)
\]
and
\[
X \phi_2(t) = \frac{\partial u}{\partial r} + \frac{2u}{r} \quad \text{on} \quad r = b, \ t > 0. \quad \ldots \ldots \ (6.16)
\]
where
\[
X = \frac{1}{N} - \frac{\alpha_0}{R}.
\]
and
\[
\phi_1(t) = f_2(t) - \frac{\partial}{\partial r} f_1(t) \quad \ldots \ldots \ (6.17)
\]
\[
\phi_2(t) = f_4(t) - \frac{\partial}{\partial r} f_3(t) \quad \ldots \ldots \ (6.18)
\]
and hence
\[
\epsilon_1 = a^{\frac{N}{2}} X \phi_1(t) \quad \text{on} \quad r = a, \ t > 0 \quad \ldots \ldots \ (6.19)
\]
\[
\epsilon_1 = b^{\frac{N}{2}} X \phi_2(t) \quad \text{on} \quad r = b, \ t > 0. \quad \ldots \ldots \ (6.20)
\]
Now following Cinelli (1965) if ‘H’ stands for truncated Hankel transform and we denote \( H[\epsilon] = \tilde{C} \) we have from (6.12)
\[(PR - \Omega^2) \# \left[ L_1^2 \psi_i \right] + \left\{ (2 \Omega S_{12} - \partial S_{11} + \partial \mathbf{S}_{22}) \right\} \frac{\partial^2 \psi_i}{\partial \xi^2} - \left( \Omega - \Omega_0 \right)^2 \psi_i + \left( \Omega - \Omega_0 \right)^2 \psi_i \right\} \frac{\partial^2 \psi_i}{\partial \xi^2} = \Omega \psi_i \quad \psi_i \in \{ \psi_1, \psi_2 \} \quad \text{...(6.21)}

Now
\[
H \left[ L \psi_i \right] = \frac{2}{\pi^2} b^{\nu} X \phi_2 (t) - \frac{2}{\pi} \frac{J_{\nu + 1} (\xi a)}{J_{\nu} (\xi a)} \alpha \nu \phi_1 (t) - \xi^2 \phi_i
\]
where \( \xi_i \) is a positive root of
\[
J_{\nu} (\xi a) Y_\nu (\xi b) - J_{\nu} (\xi b) Y_\nu (\xi a) = 0 \quad \text{...(6.22)}
\]

\( J(\xi) \) and \( Y(\xi) \) are Bessel functions of first and second kind respectively.

Now
\[
a = f_1 (t) \quad \text{on } r = a, \ t > 0
\]
and
\[\psi_i = \xi \phi_i (t)\]

\[\psi_i = \frac{1}{R} \left[ f_1 (t) - \xi \phi_1 (t) \right] = \lambda_1 (t) \quad \text{may on } r = a, \ t > 0.
\]

and
\[
\psi_i = \frac{1}{R} \left[ f_3 (t) - \xi \phi_2 (t) \right] = \lambda_2 (t) \quad \text{may on } r = b, \ t > 0
\]
where
\[
\lambda_1 (t) = \frac{1}{R} \left[ f_1 (t) - \xi \phi_1 (t) \right],
\]
\[
\lambda_2 (t) = \frac{1}{R} \left[ f_3 (t) - \xi \phi_2 (t) \right].
\]
Hence \[ E = a \mu \lambda_1(t) \quad \text{on} \quad r = a, \quad t > 0, \]
\[ E = b \mu \lambda_2(t) \quad \text{on} \quad r = b, \quad t > 0. \]

From (6.8) and (6.9) we have
\[
(PR - q^r) L e_1 = (R s_{11} - q s_{12}) \frac{\partial^2 e_1}{\partial t^2} + (R s_{12} - q s_{22}) \frac{\partial^2 E_1}{\partial t^2} +
+ (a + R) c \frac{\partial}{\partial t} (e_1 - e_1).
\]
Hence
\[
Le_1 \bigg|_{r = a} = a \mu \left\{ (R s_{11} - q s_{12}) \lambda_1''(t) +
+ (R s_{12} - q s_{22}) \lambda_1'(t) + (a + R) c (X \phi_1'(t) - \lambda_1(t)) \right\}
\]
where
\[
Z = \frac{1}{PR - q^r} \quad \text{and} \quad \lambda_1(t) = (R s_{11} - q s_{12}) \chi \phi_1''(t) -
- (R s_{12} - q s_{22}) \lambda_1''(t) + (a + R) c (X \phi_1'(t) - \lambda_1(t))
\]
\[
Le_1 \bigg|_{r = b} = b \mu \left\{ Z \chi \phi_1''(t) - (R s_{11} - q s_{12}) \lambda_1''(t) +
+ (a + R) c (X \phi_1'(t) - \lambda_1(t)) \right\}
\]
Hence by extended Cinelli's method Sen (1972) we have
\[
H \left[ L^2 e_1 \right] = H \left[ L (L e_1) \right] = \frac{2}{\mu_1} b \mu \chi \phi_1''(t) -
- \frac{2}{\mu_1} a \mu Z \chi \phi_1''(t) - \frac{2}{\mu_1} \frac{\partial}{\partial t} \left( \phi_1''(t) \right) \chi \phi_1''(t)
\]
\[
\times a \mu Z \chi \phi_1''(t) - \frac{2}{\mu_1} \frac{\partial}{\partial t} \left( \phi_1''(t) \right) \chi \phi_1''(t)
\]
\[
\chi \phi_1''(t) - \frac{2}{\mu_1} \frac{\partial}{\partial t} \left( \phi_1''(t) \right) \chi \phi_1''(t)
\]
where as before \( \phi_1''(t) \) is a positive root of (6.23).
Therefore from (6.22)
\[ H \left[ L^2 \mathbb{E}_1 \right] = \frac{2}{\pi} \times \left\{ A \left( \xi_1, t \right) - \frac{\xi_1^2}{\eta_1} \beta \left( \xi_1, t \right) \right\}^2 + \xi_1^4 \dot{\mathbb{E}}_1 \]

where
\[ A \left( \xi_1, t \right) = \zeta \left( b^{1/4} \varphi_2 (t) - a^{1/4} \frac{J_{\nu} (\xi, b)}{J_{\nu} (\xi, a)} \varphi_1 (t) \right) \]
\[ \beta \left( \xi_1, t \right) = \chi \left( b^{1/4} \varphi_2 (t) - a^{1/4} \frac{J_{\nu} (\xi, b)}{J_{\nu} (\xi, a)} \varphi_1 (t) \right) \]

therefore from (6.21)
\[ (PR - Q^2) \left[ \frac{2}{\eta_1} \times \left\{ A \left( \xi_1, t \right) - \frac{\xi_1^2}{\eta_1} \beta \left( \xi_1, t \right) \right\}^2 + \xi_1^4 \dot{\mathbb{E}}_1 \right] + t \left( 2 \mathcal{R}_{12} - R \mathbb{S}_1 + p \mathcal{P}_{22} \right) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} - (2 \mathcal{R} + p \mathbb{R}) C \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \left\{ \mathbb{E}_1 \right\} \]
\[ \times \left[ \frac{2}{\eta_1} \beta \left( \xi_1, t \right) - \xi_1^2 \dot{\mathbb{E}}_1 \right] + \left( \mathbb{S}_1 \mathcal{S}_{22} + \mathcal{S}_{12} \mathbb{S}_{22} \right) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \phi_1 \dot{\mathbb{E}}_1 \]
\[ - \cdot C \left( \mathbb{S}_1 + 2 \mathcal{S}_{12} + \mathcal{S}_{22} \right) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \phi_1 \dot{\mathbb{E}}_1 \]
\[ \mathbb{E}_1 = 0 \]

Taking Laplace transform

Let \( \mathbb{E}_1, A_1, \beta_1 \) denote Laplace transform of \( \mathbb{E}_1, A, \beta \) respectively.

Let \( \rho \) denote the transform parameter.

Then
\[ (PR - Q^2) \left[ \frac{2}{\eta_1} \times \left\{ \mathbb{H} - \frac{\xi_1^2}{\eta_1} \beta + \xi_1^4 \dot{\mathbb{E}}_1 \right\}^2 + \left( 2 \mathcal{R}_{12} - R \mathbb{S}_1 + p \mathcal{P}_{22} \right) \right] \]
\[ - (2 \mathcal{R} + p \mathbb{R}) C \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \left\{ \mathbb{E}_1 \right\} + \left( \mathbb{S}_1 \mathcal{S}_{22} + \mathcal{S}_{12} \mathbb{S}_{22} \right) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \phi_1 \dot{\mathbb{E}}_1 \]
\[ - \cdot C \left( \mathbb{S}_1 + 2 \mathcal{S}_{12} + \mathcal{S}_{22} \right) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \phi_1 \dot{\mathbb{E}}_1 \]
\[ \mathbb{E}_1 = 0 \]

So
\[ \mathbb{E}_1 = \frac{2}{\pi} \times \frac{1}{\alpha} \left( \frac{\xi_1^2 \beta^2 - 1 \cdot \frac{\xi_1^2}{\eta_1} \beta + (PR - Q^2) \frac{2}{\eta_1} \frac{\partial}{\partial \xi_1} \phi_1 \dot{\mathbb{E}}_1}{\rho^4 + \frac{2}{\alpha} \rho^3 + \frac{2}{\alpha} \rho^2 + \frac{2}{\alpha} \rho + \frac{2}{\alpha} \rho^4 + \frac{2}{\alpha} \rho^3 + \frac{2}{\alpha} \rho^2 + \frac{2}{\alpha} \rho + \frac{2}{\alpha} \rho^4} \right) \]
where \( \alpha = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \) 
\( \beta = c(\xi_{1u} + 2\xi_{1z} - \xi_{1w}) \), \( \gamma = -(2\alpha s_{12} - R s_{1u} + \rho s_{1w}) \) 
\( \delta = -c(2\alpha \cdot t \cdot R + s) \).

Hence

\[
\tilde{e}_1 = \frac{2}{\alpha} \int_0^T \left[ \sum_{i=1}^4 \frac{[\gamma^2(\lambda_i) + \delta (\lambda_i) + (pR-q) \xi_i^2]}{\prod_{i \neq j} (\lambda_i - \lambda_j)} \right] \chi e^{-\lambda_i (t-r)} \beta(t) + \sum_{i=1}^4 \frac{(pR-q) e^{\lambda_i (t-r)} \lambda_i ^3 \beta(t)}{\prod_{i \neq j} (\lambda_i - \lambda_j)}.
\]

where

\[
\rho^4 + A \cdot \rho^3 + \frac{\beta}{\alpha} \rho^2 + \frac{\gamma}{\alpha} \rho + \frac{\delta}{\alpha} + \frac{pR-q}{\alpha} \xi_i^4.
\]

Therefore

\[
\tilde{e}_1 = \frac{\eta \cdot \sum_{\xi_i} \xi_i^2 J_{\nu}^\tau (\xi_i a) \tilde{e}_1}{\xi_i J_{\nu}^\tau (\xi_i a) - J_{\nu}^\tau (\xi_i b) \left[ J_{\nu}^\tau (\xi_i r) \gamma_{\nu} (\xi_i b) - J_{\nu}^\tau (\xi_i b) \gamma_{\nu} (\xi_i r) \right]}.
\]
\( \bar{c}_1 \) is given by (6.24).

We have
\[
\frac{1}{r^2} c_1 = -\frac{2u}{r} - \frac{2u}{r^2},
\]
or
\[
r^{5/2} c_1 = \frac{2}{r} (-u^2 u).
\]

Hence
\[
U = \int_{-1}^{1} r^{5/2} c_1 \, dr - C.
\]

\[
= \frac{\pi^2}{2} \int_{-1}^{1} \sum_{\xi_i} \frac{J_{\frac{2}{\xi_i}} (\xi_i a) \xi_i}{J_{\frac{2}{\xi_i}} (\xi_i a) - J_{\frac{2}{\xi_i}} (\xi_i b)} \left[ J_{\frac{2}{\xi_i}} (\xi_i a) Y_{\frac{2}{\xi_i}} (\xi_i b) - J_{\frac{2}{\xi_i}} (\xi_i b) Y_{\frac{2}{\xi_i}} (\xi_i a) \right] \, dr - C.
\]

\( C \) is the constant of integration which is determined from relation (6.25).

§ 6.3 Conclusion

Cinelli's method has so long been used for finding the solution of vibration and heat conduction problems as done by Cinelli (1966), Sen (1972, 1973). But the present note shows that the method is also helpful for solving the problem of deformation of a consolidating sphere. The procedure appears to be simpler and direct than the existing ones.

This chapter is based on a paper by Sen (1973) communicated to Jour. Appl. Mech.