CHAPTER 5

ON PROBLEMS OF LONGITUDINAL VIBRATION OF A CYLINDRICAL SHELL IN PRESENCE OF A THERMAL FIELD BY EXTENDED CINELLI'S METHOD

§ 5.1 Introduction

Thermoelectricity in general embraces varied number of theories e.g. Theory of heat conduction, theory of stress-strain due to heat-flow where there is a coupling between thermal expansions and elastic strains.

Cinelli (1965) himself obtained a general solution for the transient temperature produced in a finite hollow cylinder by an internal heat source when all four boundary surfaces are radiating. He applied the method as described in chapter 1 to solve this boundary value problem.

Nowacki (1962) dealt with a number of dynamic problems of coupled temperature and strain fields, problems of plane wave propagation taking into consideration the mutual influence of the temperature and strain fields. Suhaloi (1964) has handled a problem of longitudinal vibration of a circular cylinder coupled with a thermal field. He has considered the case of free vibration of an infinite circular cylinder whose lateral surface is free from stresses and being held at constant ambient temperature. He has obtained the frequency equations only.

To solve the same problem Cinelli's extended method as formulated in the chapter four has been applied and we have
been able to find out the exact solution of the problem by the method as formulated in the previous chapter.

§ 5.2 Mathematical formulation

We consider the boundary value problem for longitudinal vibration of a thick, isotropic, homogeneous circular cylindrical shell of infinite length coupled with a thermal field for the case when lateral surface of the cylinder is kept at ambient temperature.

The fundamental equations for a thermoelastic medium are

\[
\begin{align*}
(\lambda + 2\mu) \text{grad} \text{div} \vec{u} - \mu \text{rot} \text{rot} \vec{u} - \gamma \text{grad} \Theta &= J \frac{\partial^2 \vec{u}}{\partial t^2}, \\
K \text{div} \text{grad} \Theta - \frac{\gamma R}{\rho \alpha} \cdot \text{div} \vec{u} &= \frac{\partial \Theta}{\partial t}
\end{align*}
\]  

(5.1)

when \( \vec{u} \) is displacement vector, \( \lambda, \mu \) are Lamé's constants, \( \Theta \) is the temperature change about the equilibrium temperature \( T \), \( J \) is the density of the medium, \( K = \frac{R}{\rho \alpha} \) is the diffusivity, \( R \) is the thermal conductivity, \( \gamma = \alpha (\lambda + 2\mu) \) is the thermal expansion coefficient, \( \alpha \) is the specific heat per unit mass at constant strain.

If we refer the medium to cylindrical co-ordinates \((r, \phi, z)\), \( z \) being the axis of the cylinder and assume \( u_\phi = u_z = 0 \) and \( u_r \) and \( \Theta \) are functions of \( r \) and
t(time) only then equation (5.1) becomes,

\[
(\lambda + 2\mu) \frac{\partial \Delta}{\partial r} + \gamma \frac{\partial \phi}{\partial r} = \int \frac{\partial^2 \mu_r}{\partial t^2} \]

\[
K \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) - \frac{\gamma^2}{fc} \frac{\partial \Delta}{\partial t} = \frac{\partial \phi}{\partial t} \]

\ldots (5.2)

where

\[
\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{1}{r} u_r \]

\ldots (5.3)

By use of (5.3), (5.2) can be put in the following form

\[
(\lambda + 2\mu) \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + \gamma \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) = \int \frac{\partial^2 \mu_r}{\partial t^2} \]

\[
K \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) - \frac{\gamma^2}{fc} \frac{\partial \Delta}{\partial t} = \frac{\partial \phi}{\partial t} \]

\ldots (5.4)

Now assume that

\[
\Delta (r, t) = D (r) e^{ikt}, \quad \phi (r, t) = \Theta (r) e^{ikt} \]

Write for the irrotational velocity \( \frac{\sqrt{\lambda + 2\mu}}{S} = C \), and for the equivoluminal velocity \( \sqrt{\frac{\mu}{S}} = C_L \). Introducing the dimensionless quantities

\[
\xi = q, \quad \frac{\kappa}{C_1} = \omega, \quad \frac{\kappa}{C_2} = \beta \omega, \quad \beta = \frac{\sqrt{\lambda + 2\mu}}{\mu} \]

and differential operator

\[
L \equiv \frac{d^2}{ds^2} + \frac{i}{\xi} \frac{d}{ds} \]

(5.4) can be written as
Eliminating $\Theta$ we get

$$(L + \lambda_1^2)(L + \lambda_2^2)\Theta = 0$$

where

$$\lambda_1^2 + \lambda_2^2 = \nu^2 - 1 - (1 + \frac{iC_1\omega}{kq}) - i\omega$$

$$\lambda_1^2 \lambda_2^2 = i\omega - (\omega^2 - 1)(1 + \frac{iC_1\omega}{kq})$$

Thus the longitudinal vibration of a thick, isotropic homogeneous cylindrical shell of infinite length in this case satisfies the following boundary value problem:

$$\left( L + \omega^2 - 1 \right) D - \frac{\nabla^2}{\lambda + 2\mu} (L - 1) \Theta = 0$$

$$\frac{iC_1\omega}{kq} \frac{\sigma_r}{c_c} D - (L - 1 - \frac{iC_1\omega}{k}) \Theta = 0, \quad \lambda \leq r \leq b, \quad t > 0$$

where $a$ and $b$ are the inner and outer radii of the shell.

Boundary conditions are

$$\left. \sigma_r \right|_{r = a} = \lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} - \gamma \frac{\partial \gamma}{\partial r} = A(t)$$

$$\left. \sigma_r \right|_{r = b} = \lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} - \gamma \frac{\partial \gamma}{\partial r} = B(t)$$
and $\theta = 0$ at $r = a$ and $r = b$

A and B are radial stresses on the inner and outer boundary respectively and prescribed.

Let

$$
\Delta \bigg|_{r=a} = \Delta_a, \quad \Delta \bigg|_{r=b} = \Delta_b
$$

Then

$$
\frac{\partial u_r}{\partial r} \bigg|_{r=a} = u'_a, \quad \frac{\partial u_r}{\partial r} \bigg|_{r=b} = u'_b
$$

Then

$$
\Delta_a = \frac{1}{\lambda} \left( A - 2 \mu u'_a \right), \quad \Delta_b = \frac{1}{\lambda} \left( B - 2 \mu u'_b \right)
$$

$$(5.7)
$$

From (5.6) we get

$$
(L + \lambda_1^2)(L + \lambda_2^2) = 0.
$$

Following the method of Cinelli as described in chapter 1 and the extended method formulated in chapter 4 we get

$$
H \left[ L (L \theta) \right] = B_1 - \xi_i A_1 + \xi_i^4 \delta
$$

where $H$ stands for Cinelli's modified Hankel transform operator, $\xi_i$ the transform parameter, and in this case are the positive root of

$$
J_0 (\xi a) Y_0 (\xi b) - J_0 (\xi b) Y_0 (\xi a) = 0
$$

$$
\delta = H (\theta).
$$
Hence we get from (5.6)

\[ \overline{D} = -\frac{A_1 + B_1}{\xi_i^4 - (\lambda_{1\nu} + \lambda_{2\nu}) \xi_i^2 + \lambda_{1\nu} \lambda_{2\nu}} \]

where

\[ A_1 = \frac{2}{n} \left\{ \frac{\int \left( \xi_i^a \right)}{\int \left( \xi_i^b \right)} - \left[ K_2 \left\{ i \omega b - (\omega^{r+1}) \right\} \right] \right\} - \left[ K_1 \left\{ i \omega b - (\omega^{r+1}) \right\} \right] \]

\[ \beta_1 = \frac{2}{n} \left[ \frac{\int \left( \xi_i^a \right)}{\int \left( \xi_i^b \right)} K_2 - K_1 \right] \]

\( (L_2) \) and \( (L_3) \) being equal to \( K_1 [i \omega b - (\omega^{r+1})] \) and \( K_2 [i \omega b - (\omega^{r+1})] \) respectively and are obtained from (5.5)

\[ D = \frac{n^2}{2} \sum \frac{\xi_i^a \left( \frac{\xi_i^a}{\int \left( \xi_i^a \right)} - \int \left( \xi_i^b \right) \left( \xi_i^b \right) \right)}{\int \left( \xi_i^b \right) - \int \left( \xi_i^a \right)} \left[ \int \left( \xi_i^a \right) y_0 \left( \xi_i^a \right) - \int \left( \xi_i^b \right) y_0 \left( \xi_i^b \right) \right] \overline{D} \]

Hence

\[ \Delta = D e^{i\mu t} \]

Consequently

\[ U_r = \frac{1}{r} \frac{n^2}{2} \int \sum \frac{\xi_i^a \left( \frac{\xi_i^a}{\int \left( \xi_i^a \right)} - \int \left( \xi_i^b \right) \left( \xi_i^b \right) \right)}{\int \left( \xi_i^b \right) - \int \left( \xi_i^a \right)} \left[ \int \left( \xi_i^a \right) y_0 \left( \xi_i^a \right) - \int \left( \xi_i^b \right) y_0 \left( \xi_i^b \right) \right] \overline{D} e^{i\mu t} dr. \]
Stresses are given by
\[
\sigma_{rr}(r, \xi, \theta) = \lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} + \left( \lambda + 2\mu \right) \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial}{\partial r} \left[ \int \sum_{i=1}^{\infty} \frac{J_0^2(\xi_i a)}{J_0(\xi_i a) - J_0(\xi_i b)} \cdot x \left[ J_0(\xi_i r) Y_0(\xi_i b) - J_0(\xi_i b) Y_0(\xi_i r) \right] e^{i\beta t} dr \right]
\]
\[+ \lambda \frac{\partial}{\partial r} \int \sum_{i=1}^{\infty} \left\{ \frac{J_0^2(\xi_i a)}{J_0(\xi_i a) - J_0(\xi_i b)} \right\} \left[ J_0(\xi_i r) Y_0(\xi_i b) - J_0(\xi_i b) Y_0(\xi_i r) \right] \frac{\partial}{\partial r} \theta.
\]
From (5.5) \( \theta = \frac{1}{2} \sum_{i=1}^{\infty} \frac{J_0^2(\xi_i a)}{J_0(\xi_i a) - J_0(\xi_i b)} \left[ J_0(\xi_i r) Y_0(\xi_i b) - J_0(\xi_i b) Y_0(\xi_i r) \right] \frac{\partial}{\partial r} \theta.
\]
where
\[
\tilde{\Theta} = (A_i + B_i) i \omega \xi \gamma^T / (1 + \xi_i \gamma^T + i c \omega q) \kappa q \kappa c \times \left( \xi_i \gamma - (\lambda_1 \gamma + \lambda_2 \gamma) \right) \frac{\partial}{\partial r} \theta.
\]
If \( T = 0 \) then values of \( \lambda_1 \) and \( \lambda_2 \) to be denoted by \( \lambda_1^* \) and \( \lambda_2^* \) respectively are given by
\[
\lambda_1^* = \omega^L - 1 \quad \text{and} \quad \lambda_2^* = - \left( 1 + \frac{ic \omega^*}{kq} \right)
\]
and these correspond to the case in which elastic wave and heat conduction equations are not coupled.

Amplitudes of the displacement are given by the real part of the expression of \( U_\gamma S \).
5.3 Conclusion

Thus with the application of extended Cinelli's method, the response of a thick isotropic, homogeneous cylindrical shell in presence of a thermal field has been found for some particular boundary conditions. The solutions have been obtained by means of Cinelli's finite Hankel transform method after suitable modification in a manner which is direct and very concise.

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