CHAPTER 1

ON METHODS OF SOLUTION OF SOME CLASSES OF DIFFERENTIAL EQUATIONS

§ 1.1 Introduction

Integral transforms are reliable technique for solving differential equations. Usually integral transforms are defined by means of integrals over a range for a class of kernels and then the theory of these transforms is developed and applied in deriving the solution of a certain class of linear partial differential equations of Applied Mathematics and theoretical physics under certain boundary conditions.

Dutta and Debnath (1965) have developed a general scheme for defining a suitable integral transform for solving any given linear partial differential equation and developed a general theory of such transform by application of which the general solution of the given linear differential equation can be derived, under certain prescribed boundary conditions. Debnath (1964) considered applications of this scheme in a number of problems. Later Puri and Bhattacharya (1969) also have dealt with this theory of solution of partial differential equations.

Cinelli (1965) also obtained a suitable variant form of truncated Hankel transforms by which boundary value problems of spherical and cylindrical shells can be solved very concisely.
§ 1.2 On Transform Techniques

Generally transforms play an important role in obtaining a solution of boundary value problems. Usually to define a transform we first search for a kernel. Given a known function \( k(x, p) \) referred to as kernel, the integral transform of a function \( F(x) \) is defined by means of the integral

\[
\hat{F}(p) = \mathcal{T}\{F(x)\} = \int_a^b k(x, p) F(x) \, dx.
\]

provided the integral exists and the range of integral defining the transform may be finite, infinite or sometimes semi-infinite. The main problems of the theory of integral transforms are to investigate the function \( f(p) \) for suitable given kernel \( k(x, p) \) and to find range \((a, b)\) of the integral defining the transform and the converse case.

As it is known that transforms are Laplace, the Fourier and the Mellin transforms according as the kernels are

\[
k(x, p) = e^{-px}, \quad \frac{1}{\sqrt{2\pi}} e^{ipx}
\]

and \( x^{p-1} \) and the range of the integrals defining the transforms being \((0, \infty)\), \((-\infty, -\infty)\) and \((0, \infty)\) respectively, these are the transforms very common in use.

An integral transform is said to be truncated when the range of variation of the independent variable is not taken to be its full range but a proper subinterval contained in the range. As for example, in case the Laplace transform, the Mellin transform and the Mellin transform the intervals be taken as \((a, b)\), \(0 < a < b \leq \infty\) the transform is regarded to as truncated transform.
§1.3 Mathematical formulation of Cinelli's method

Cinelli (1965) mentioned a new form of finitely truncated Hankel transform with help of which a class of boundary value problem can be treated most conveniently.

Cinelli's method is mainly based on introduction of a new type of truncated finite Hankel transform. The method, in essence, is the following:

Consider the differential equation

\[
\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} + \left( \xi_i^2 - \frac{a^2}{x^2} \right) f = 0, \quad a \leq x \leq b
\]  

(1.1)

Let \( f(a) = f(b) = 0 \)

The general solution of (1.1) is given by

\[
f(x) = A J_\mu (\xi_i x) + B Y_\mu (\xi_i x)
\]  

(1.2)

where \( J_\mu \), \( Y_\mu \) are Bessel's functions of first and second kind respectively. Since we have \( f(a) = f(b) = 0 \), we may write

\[
f(x) = A \left[ J_\mu (\xi_i a) Y_\mu (\xi_i x) - J_\mu (\xi_i x) Y_\mu (\xi_i a) \right]
\]  

(1.3)

where \( \xi_i \)'s are positive roots of

\[
J_\mu (\xi_i b) Y_\mu (\xi_i a) - J_\mu (\xi_i a) Y_\mu (\xi_i b) = 0
\]  

(1.4)

Assuming that \( f(x) \) satisfies Dirichlet condition in the range \( a \leq x \leq b \) it can be expanded as an infinite series of the type in equation (1.3). Also if the weight function for the integral transform be \( x \), we have
The Hankel transform of $f(x)$ is written as $\mathcal{H}[f(x)]$ and defined by

$$\mathcal{H}[f(x)] = \int_a^b x \left[ J_\mu(\xi x) Y_\mu(\xi a) - J_\mu(\xi a) Y_\mu(\xi x) \right] dx$$

...(1.5)

where Hankel transform is defined by taking a Bessel's function as a kernel. For convenience of calculations here it is taken as a suitable linear combination of Bessel's functions of two different kinds. Usually in case of Hankel transforms, the range is $(0, \infty)$. Here it is taken as $0 < a < b < \infty$, so it is a truncated Hankel transform.

Expanding the denominator in (1.5)

$$\int_a^b x \left[ J_\mu(\xi x) Y_\mu(\xi a) - J_\mu(\xi a) Y_\mu(\xi x) \right] dx =$$

$$= \int_a^b x \left[ J_\mu(\xi x) Y_\mu(\xi a) - J_\mu(\xi a) Y_\mu(\xi x) \right] dx - 2 \int_a^b x J_\mu(\xi x) Y_\mu(\xi a) dx$$

From Jahnke and Emeše(1945) we can evaluate the integrals, using their results we get
Now we know the relation
\[
\begin{vmatrix}
J_\mu(x) & Y_\mu(x) \\
J_\mu'(x) & Y_\mu'(x)
\end{vmatrix} = -\frac{2}{\pi x}.
\]

By virtue of this and relation (1.5) we get
\[
\int_a^b \left[ J_\mu(\xi_i x) Y_\mu(\xi_i a) - J_\mu(\xi_i a) Y_\mu(\xi_i x) \right]^2 \, dx = -\frac{2}{\pi x} \cdot \frac{J_\mu^2(\xi_i a) - J_\mu^2(\xi_i b)}{J_\mu^2(\xi_i b)}.
\]

Hence finally by series expansion we get
\[
f(x) = \frac{\eta}{2} \cdot \sum_{\xi_i} \frac{\xi_i \cdot J_\mu^2(\xi_i b) \cdot \bar{f}(\xi_i)}{J_\mu^2(\xi_i a) - J_\mu^2(\xi_i b)} \left[ J_\mu(\xi_i x) Y_\mu(\xi_i a) - J_\mu(\xi_i a) Y_\mu(\xi_i x) \right].
\]

Or
\[
f(x) = \frac{\eta}{2} \cdot \sum_{\xi_i} \frac{\xi_i \cdot J_\mu^2(\xi_i a) \cdot \bar{f}(\xi_i)}{J_\mu^2(\xi_i a) - J_\mu^2(\xi_i b)} \left[ J_\mu(\xi_i x) Y_\mu(\xi_i b) - J_\mu(\xi_i b) Y_\mu(\xi_i x) \right].
\]
Since \( f(x) \) can also be written as

\[
f(x) = \beta \left[ J_\mu (\xi_i x) Y_\mu (\xi_i b) - J_\mu (\xi_i b) Y_\mu (\xi_i x) \right]
\]

in this case kernel of transform will be

\[
J_\mu (\xi_i x) Y_\mu (\xi_i b) - J_\mu (\xi_i b) Y_\mu (\xi_i x)
\]

and as before

\[ \xi_i 's \] are positive root of

\[
J_\mu (\xi_i b) Y_\mu (\xi_i a) - J_\mu (\xi_i a) Y_\mu (\xi_i b) = 0 .
\]

Either of the two expressions can be taken to be the expression for \( f(x) \) which \( f(x) \) satisfies Dirichlet's conditions at \( x = a \) and \( x = b \) i.e. values of \( f(x) \) at these two points are prescribed. Thus we find that a method can be so constructed to give the solution of a differential equation of Bessel's form, satisfying Dirichlet-type boundary condition at two end-points. Similar procedures have been adopted to give expressions for \( f(x) \) while \( f(x) \) satisfies say Neumann condition at one end-point and Dirichlet on the other. In that case kernel only is to be chosen to suit the conditions which are \( f'(x) = 0 \) at \( x = a \) and \( f(x) = 0 \) at \( x = b \) say.

The kernel in that case will be

\[
J_\mu (\xi_i x) Y_\mu (\xi_i a) - J_\mu (\xi_i a) Y_\mu (\xi_i x)
\]

and \( \xi_i 's \) are positive roots of

\[
J_\mu (\xi_i b) Y_\mu (\xi_i a) - J_\mu (\xi_i a) Y_\mu (\xi_i b) = 0
\]

and

\[
f(x) = \frac{n^2}{2} \sum \frac{\xi_i^2 J_\mu^2 (\xi_i b) f(\xi_i)}{\left[ J_\mu (\xi_i a)^2 - [1 - \left( \frac{\xi_i}{\xi_i a} \right)^2] J_\mu^2 (\xi_i b) \right]} \times
\]

\[
\times \left[ J_\mu (\xi_i x) Y_\mu (\xi_i a) - J_\mu (\xi_i a) Y_\mu (\xi_i x) \right]
\]
If on the other hand $f(x)$ satisfies Cauchy or mixed type boundary condition at one end $x = a$ say and Dirichlet type boundary condition on the other end $x = b$. The kernel will accordingly be changed. For different type of conditions satisfied at the end-points, Cinelli has given expressions for $f(x)$ with properly chosen kernels.

§ 1.4 Conclusion

Cinelli (1966) used this method to find out the transient displacements and stresses in thick elastic cylindrical and spherical shells when the surfaces are subjected to dynamic loads. In the next two chapters application of this method has been made to solve two problems of elasticity which ensures the utility as well as the convenience of the method proposed by Cinelli.