CHAPTER 0

INTRODUCTION

§ 0.1 Preliminary Remarks

The present investigation has two main aspects, physical and mathematical. On the physical side the investigation concerns with the solutions of some problems on elastic, thermoelastic, poroelastic and fluid media. On mathematical side a suitable version of Hankel transformation for solving boundary value problems involving partial differential equation of the second order proposed by Cinelli (1965), has been studied and used in solving some problems. This method has been extended suitably for boundary value problems involving partial differential equation of any higher even order. In addition to this, study has been made about proper mathematical characterization of vibration and wavepropagation.

In theory of elasticity, problems of vibration of anisotropic and inhomogeneous cylindrical and spherical shells are solved here by a properly chosen variant of truncated Hankel transform technique. In case of inhomogeneous cylindrical and spherical shells, the most general law for Young's modulus has been found out, so that it embraces all forms of $E$, the Young's modulus generally found in recent papers of many investigators. It has also been shown how concisely a class of
boundary value problems can be solved by this transformation technique. This form of transform method for solving some boundary value problems involving second order differential equations is due to Cinelli (1965).

An extension of this transform method has been formulated so that certain type of physical problems which are generally reduced to differential equations of order higher than two, supplemented with a number of boundary conditions can be solved. Whereas Cinelli's method is applicable to problems involving second order differential equations only. The extended formula derived here is applied to solve certain problems of thermoelastic and poroelastic media.

In the latter part of the thesis, a critical discussion is made about mathematical characterization of vibration and wave propagation. Also problems on hydro-magnetic flow are discussed in some detail to determine the simultaneous influence of the coriolis force and electromagnetic forces on the flow phenomenon.

An unsteady analysis is carried out of the magnetohydrodynamic boundary layer flows, generated in a semi-infinite expanse of an incompressible, homogeneous, viscous, electrically conducting fluid bounded by an infinite flat plate which executes small amplitude harmonic oscillation in its own plane. It is shown the solution consists of the hydromagnetic boundary layers, current layers and the diffused *Alfven waves*.
Now in the present investigation the media under consideration have been considered as continua. This implies that every subregion however small is completely filled up with matter or in the precise language of mathematics a continuous body is a set, of which the class of all subsets (called the parts of the body) has one-to-one correspondence with the class of subsets of a set in Euclidean three-space. The functions representing this correspondence is bi-continuous and has sectionally smooth derivatives upto order four at least.

The continuum approach is essentially macroscopic in nature and hence does not take into account the motions of discrete particles like atoms, molecules or ions which constitute the media. Though the microscopic study seems to be more physical, in many cases microscopic consideration involves much complications. It is also seen that in a large number of physical problems, the properties and their interrelations may be优势ously investigated on the hypothesis that the body is a continuum without the introduction of any molecular hypothesis.

§ 0.2 On mathematical formulation of problems on continuum mechanics.

In continuum mechanics, materials of different types viz. elastic, poroelastic, fluid etc. are characterized by establishing some relations among the constitutive variables
which are called constitutive equations e.g. solid state of material is characterized by its elastic behaviour specified by certain constitutive equations, say stress-strain relations for elastic body. The selection of these variables and the construction of this constitutive equations are mainly based on the basic physical nature from analysis, from the point of views of mechanics, thermodynamics etc. So that it satisfies certain basic criteria generally referred to as some axioms of constitution. The basic laws expressed as field equations are same for all kinds of material whatsoever and these are:

(i) conservation of mass,
(ii) balance of momentum,
(iii) conservation of energy,
(iv) balance of moment of momentum,
(v) principle of entropy.

These five laws lead to five fundamental equations:

\( \frac{\partial \rho}{\partial t} + (\mathbf{f} \cdot \mathbf{v}_K)_K = 0 \) \quad \text{(conservation of mass)}

\( t_{k\ell} = f(t_k - \dot{v}_K) = 0 \) \quad \text{(balance of moments)}

\( \mathbf{f} \cdot \mathbf{E} = t_{k\ell} v_{K\ell} + q_{kK} + \mathbf{f} \cdot \mathbf{g} \) \quad \text{(balance of energy)}

\( \int (\dot{q} - \frac{\mathbf{E}}{T}) + \frac{1}{T} t_{k\ell} v_{K\ell} + \frac{1}{T} q_{kK} (\log T)_K \geq 0 \)

\quad \text{(entropy principle)}
$j$ is the mass density,

$v_i$ is the velocity component,

$\tau_k$ is the stress component,

$q_k$ is the heat flux,

$f_k$ is the component of external force,

$\varepsilon$ is the internal energy,

$\mathcal{Q}$ is the amount of heat absorbed,

$\mathcal{H}$ is the entropy density and $T$, temperature.

It is found from above, that the number of equations being insufficient to determine larger number of unknowns, a number of additional equations are needed to make a problem determinate. At this point the character of the material is brought into the formulation through appropriate constitutive equations and for each material the constitutive variables are restricted in their region of definition. The constitutive equations thus define an ideal material. In the selection of constitutive variables and in the formulation of constitutive equations certain physical and mathematical considerations will have to be satisfied. The most important axiom is the axiom of material indifference according to which response of a material is same for all observers. Other axioms are axiom of admissibility, axiom of reversibility etc.
Some writers Ericksen and Rudin (1955), Noll (1958), Jordan and Eringen (1964) have given constitutive theories to encompass a large number of physical phenomena.

§ 0.3 Historical Remarks

As already mentioned, deformation in elastic medium, response of the medium under elastic forces coupled with a thermal field, also deformation of porous material, lastly hydromagnetic flow are topics of investigations.

Attempts to see mechanics of different material as different forms of mechanics of continuum are comparatively recent. So in sketching the historical background mechanics of different materials have been treated separately.

Elasticity theory:

Main concerns of the mathematical theory of elasticity are the deformations and stresses in a strained elastic body. Hook's law and its generalization (1660) are of fundamental importance in the mathematical theory of elasticity. According to this law extensions of bodies, produced by the tensile forces are proportional to the forces.

Euler (1744) and D. Bernoulli (1751) obtained the differential equations of the lateral vibration of bar. Chaldfn (1802) made an important investigation on different modes of
vibration experimentally and also longitudinal and torsional vibrations of bars. The first attempt to deduce general equations of equilibrium and vibration of elastic solids was made by Navier (1785-1836) in a remarkable memoir read on May 14, 1921. He derived a set of three differential equations for the components of displacements in the interior of an isotropic elastic solid. His formulation is based on the simplified assumption of molecular interaction according to which the forces act along the lines joining two particles and are proportional to the change in distance between them. His equation contain only one constant. With an application of calculus of variation he deduced not only the differential equations previously obtained but also the boundary conditions that hold at the surface of the body. In the same year 1821 in which Navier's memoir came into light Fresnel (1821) published his work in which he gave an idea that "the observed facts in regard to the interference of polarised light could be explained only by the hypothesis of transverse vibration. He showed how a medium consisting of molecules connected by central forces might be expected to execute such vibration and to transmit waves". Cauchy (1820) and Poisson (1814) handled the theory of elasticity and in particular the problem of transmission of wave through an elastic medium. Cauchy's contribution in the field of elasticity is colossal. After Navier's work Cauchy (1789-1857) proceeded from different assumptions and gave a formulation of the linear theory of elasticity that remains virtually unchanged even to the present day. Cauchy showed
that the state of stress at an interior point of an elastic body can be determined by nine functions and when in particular the body is in equilibrium the number of functions is reduced to six by means of certain symmetry relation. The state of deformation determined by six functions are related to the components of displacements when the displacements are small. In case of elastic body are take only into account small deformations, and so it is justified to assume that the functions depicting the state of stress is linearly connected with the set of functions giving the deformation.

When the body is isotropic i.e. its elastic properties are same in every directions the above mentioned linear relationship contains two elastic constants whereas in Navier's work, we come across with only one constant.

In 1828 Cauchy obtained the stress-strain relation for the most general type of anisotropy, and it contains fifteen elastic constants. Later Green (1837) introduced the concepts of strain-energy and deduced basic equations of elasticity from the principle of virtual work. These, in essence, are the main contributions concerned with the formulation of foundation and general theories of elasticity.

A model of elastic body from continuum approach has been suggested by some modern investigators(cf. Sedov 1962). According to them the fundamental idea in elastic theory is the hypothesis of the reversibility of the processes, the original
model is in the form of a deformed solid considered as a material continuum, for whose small particles, the internal energy, free energy, entropy and other thermodynamic properties can be regarded as functions of the tensor of deformation, temperature and of the physical constant or variable parameters which characterize the thermal and mechanical properties and the state of the substances. It is suggested that the parameters which characterize the medium can be tensors.

The modern development of continuum mechanics is found in the recent works of Truesdell (1952, '53), Noll (1958), Eringen (1967), Sedov (1962) and many others.

Thermoelectricity

An elasticity problem including the effect of temperature variation was studied first by J. Duhamel as early as in 1835, shortly after the basic formulation of elasticity. He established the fundamental equations of thermoelectricity, taking into account the coupling of temperature and strain fields. The generalized equations of heat conduction were deduced by W. Voigt (1910) and Jeffrey (1930). Previously in classical theory of thermoelectricity the facts that deformation occurring in elastic medium is accomplished by temperature variation and that the field of temperature produces strain in the body were neglected.
Recently (Cf. Biot 1956) formulation of the general theory of thermoelasticity in a variational form along with a minimum entropy principle and the concept of thermal forces are made. The realization came in the works that, in time dependent problems, the interaction of the thermal state and deformation of an elastic solid is such that the two effects can not be treated separately. A number of authors have attempted to rework the solutions of transient elastic and thermal stress problems using the cross-coupled thermoelastic equations. This rigorous approach to thermoelastic boundary value problems meets with severe analytical difficulties and to date one partial solution has been published by Paris (1958).

Other contributions in this direction are made by Leeson (1956), Chadwick and Sneddon (1958), Lockett (1958) and also by Deresiewicz (1958). Elastic wave propagation problem coupled with a thermal field has been studied by several authors like Leeson (1956, 1957), Chadwick and Sneddon (1958). These works are generally confined to infinite and semi-infinite media except for a paper published by Sneddon (1959) on propagation of thermal stresses in these metal rods.

Poroelasticity

Investigations on poroelasticity have been started recently because of its usefulness in river-research, irrigation and soil mechanics. In establishing the basic concepts
and relations generally used in soil-mechanics to describe the kinematical and mechanical behaviour of fluid-filled porous media, the fluid/solid system is regarded as a single continuum and the formalism of continuum mechanics is adopted.

Physicists and soilmechanics-engineers have experimented in the laboratory and field for many years in an attempt to derive the elastic properties of rocks and sediments. Significant works are done by Terzaghi (1948), Peck and Terzaghi (1948), Taylor (1948) and Tschelortarioff (1951). The summary of the result of their studies revealed that a sediment is porous and of loose structure which has elastic properties all of which can not be measured directly and some of which are not comparable to those of ideal elastic media. Also when these loose aggregates are placed under a compacting pressure either artificially or naturally as in earth's crust, the porosity is reduced, the grain-to-grain contact is increased and as the pressure increases the aggregate acts more and more as an ideal elastic body. Porosity is defined as the ratio of voids (pore space between the grains) to the total sample volume. A comprehensive study of the factors affecting porosity was made by Traser (1935). Biot (1956) investigated the theory of propagation of waves in fluid-saturated porous solid both in low frequency range and high frequency range. He expressed the stress-strain relation in a porous elastic solid in most general form.

Later significant works in this direction have been done by Deresiewicz (1960), Mashin (1963, 1964, 1965) and Mill (1963), (1964).
Hydrodynamics

The hydrodynamics is concerned with the motion of fluids which can be divided into three classes viz. viscous fluid, thin fluid and gaseous fluid. In this dissertation two problems of hydromagnetic viscous fluid motion have been dealt with. Here fluid is treated as continuum. The conditions under which fluids can be treated as continua have been discussed by Prandtl and hetzens(1957) in the opening chapter of their well known work "Fundamentals of hydro and aeromechanics". The conditions, in brief, is that the fluids, even viscous fluids can be regarded as continua if its molecular mean free paths are of negligible dimensions in comparison with the dimensions of the fluid region under investigation.

The effect of viscosity in modifying the motion of a fluid in contact with a vibrating solid was examined by Stoke. Rayleigh (1896) made use of Stoke's result in his investigation of the effect of boundary layers on the propagation of sound in tubes.

The study of hydromagnetic flow phenomenon draws the attention of a number of physicists particularly because of its geophysical interest. It is however accepted that the geomagnetic field is mainly due to hydromagnetic flow in the earth's liquid core. Hide (1956) and Hide and Robert (1960) gave detailed discussions of this geophysical phenomenon in their literature.
It has been suggested that the near coincidence between the geomagnetic and geographic poles is the result of the strong influence of Coriolis force, due to earth's rotation on the core motion. Lehnert (1954, 1955) has considered the effect of Coriolis forces on plane hydromagnetic waves in a perfectly conducting inviscid fluid. Chandrasekhar (1954, 1956, 1957) studied the theory of the thermal connection in a thin horizontal layer of a conducting fluid which rotates about a vertical axis, in the presence of a magnetic field.

Though a good deal number of papers have been appeared giving a discussions of hydromagnetic flow phenomenon, very few of them have so far dealt with the unsteady case. But only the study of unsteady flow can reveal the dependence of time on the solutions i.e. how the solutions grow or decay with the progress of time which is rather an important fact to be taken into consideration.

Axford (1960), Kakutani (1958), Prasad and Sukla (1969) Hide and Robert (1960), Rajvanshi (1969) and others have discussed only the steady problem. A few authors like Hall and Debnath (1972), Debnath (1971) have discussed the unsteady problems.
§ 0.4 Objectives and Motivation of the Present Investigation

A number of physical problems from the field of continuous media have been solved by application of a suitable transform technique recently proposed by Cinelli (1965). By this application the problems are solved very concisely and it has become possible to have an exact solution in each case. Whereas several authors in handling similar type of problems have not been able to derive exact solutions. In dealing with vibration problems they have obtained the frequency relation only. Presentation of the formulation of this method to point out its usefulness in solving a good number of problems of continuous media is one of the objectives of this dissertation.

For mathematical simplification elastic bodies are often referred to as isotropic and homogeneous but this simplified assumption often leads to results inadequate to explain physical relations. A law of variation of Young's modulus has been established, the law of variation whenever is satisfied, Cinelli's transform method can be utilised to obtain solution of vibrations associated with the cylindrical and spherical shell of inhomogeneous material. The general law established there in embraces a good many forms generally found in many recent papers.

An emphasis is then given to formulation of a method which is a modification of Cinelli's integral transform method for solving certain class of boundary value problems involving partial differential equations of order higher than two. Its
application to a problem of thermoelasticity and poroelasticity confirms the usefulness and applicability of the method, introduced.

In the latter part of the thesis, well-known Laplace transform technique is used to solve partial differential equations encountered in problems of fluid mechanics.

These problems have been discussed, firstly, to understand the influence of Coriolis and electromagnetic forces on the flow phenomenon. The velocity and magnetic modes associated with the problems are analysed.

An unsteady problem of boundary layer flows has been solved to reveal the dependence of time factor on the growth of the solution. The unsteady velocity distribution and the induced magnetic field are determined explicitly. The structure of the boundary layers and current layers including the flows outside these layers are examined. The initial and final states of the motion of the fluid are investigated with physical implications.
§ 0.5 Brief Reviews of Relevant Developments used in the Recent Investigations.

As already mentioned though the main work is mainly concerned with different problems of continuum mechanics, some works also have been done in mathematical technique.

The method of Cinelli for solving boundary value problem of second order differential equations connected with problems of cylindrical and spherical shells (reported in Chapter I) has been extended suitably for solving boundary value problems involving higher order differential equations.

Elasticity

Cinelli (1966) used his method to find out the transient displacements and stresses in elastic cylindrical and spherical shells when the surfaces are subjected to dynamic loads for the following problems:

(a) Pure radial and torsional motion of an infinitely long circular cylindrical shell;

(b) radially symmetric motion of a spherical shell.

The loads applied to both surfaces of the cylindrical and spherical shells are completely arbitrary functions of time. He proceeded in the following manner. The boundary value problem for the radial motion of a thick isotropic homogeneous cylindrical shell of infinite length subjected to loads on both surfaces is
\[ \frac{\partial u}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (ru) - \frac{u}{\nu^2} = \frac{1}{\nu^2} \frac{\partial^2 u}{\partial t^2}, \quad a \leq r \leq b, \quad t > 0 \quad \ldots (0.1) \]

and \( \nu^2 = \frac{\lambda + 2\mu}{\rho} \), \( a \) and \( b \) are the inner and outer radii of the cylinder. Initially it is assumed that

\[ u = \frac{\partial u}{\partial t} = 0, \quad t = 0, \quad a \leq r \leq b \]

Boundary conditions are that the stresses \( \sigma_r (r, t) \)'s are prescribed at the boundaries i.e.

\[ \sigma_r (r, t) \bigg|_{r=a} = (\lambda + 2\mu) \frac{\partial u}{\partial r} + \frac{\lambda}{r} u = A(t) \quad r=a, \quad t > 0 \quad \ldots (0.2) \]

\[ \sigma_r (r, t) \bigg|_{r=b} = (\lambda + 2\mu) \frac{\partial u}{\partial r} + \frac{\lambda}{r} u = B(t) \quad r=b, \quad t > 0 \quad \ldots (0.3) \]

where \( u \) is the radial displacement of the cylinder, \( \nu \) is the compressional wave velocity, \( \lambda, \mu \) are the Lamé's constants for an isotropic, homogeneous medium, \( \rho \) is the density of the medium, \( A, B \) are radial stresses on inner and outer surfaces respectively, \( h \) and \( k_r \) are defined by the following

\[ h = \frac{\lambda/a}{\lambda + 2\mu}, \quad k_r = \frac{\lambda/b}{\lambda + 2\mu} \quad \ldots (0.4) \]
Then the boundary conditions are seemed to be of mixed type, accordingly kernels for Hankel transform is written from Cinelli (1965). Also if $H[f]$ denotes new finite Hankel transform of $f$

$$H \left[ \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{f}{r^2} \right]$$

Can be written down.

If $H[u]$ is denoted by $\bar{u}$ we get from Cinelli (1965)

$$\frac{1}{v^2} \frac{\partial^2 \bar{u}}{\partial t^2} = \frac{2}{\pi (\lambda + 2\mu)} \left\{ \left[ \frac{\xi_i J_i'(\xi_i a) + k J_i(\xi_i a)}{\xi_i J_i'(\xi_i b) + k J_i(\xi_i b)} \right] \times \Phi(t) - A(t) \right\}$$

Taking Laplace transform and then on inversion we have

$$\bar{u} = \frac{2}{\pi \int_{\xi_i}} \left\{ \left[ \frac{\xi_i J_i'(\xi_i a) + k J_i(\xi_i a)}{\xi_i J_i'(\xi_i b) + k J_i(\xi_i b)} \right] \times \Phi(t) - A(t) \right\} \times \int_{\xi_i} \Phi(t - \tau) d\tau$$

And hence

$$u(r, t) = \frac{2}{\pi} \sum_{\xi_i} \eta_i \left[ \frac{\xi_i J_i'(\xi_i b) + k J_i(\xi_i b)}{\xi_i J_i'(\xi_i a) + k J_i(\xi_i a)} \right] \bar{u}(\xi_i, t) \frac{\eta_i r}{\eta_i}$$

$\xi_i$'s are positive roots of the equation

$$\left[ \xi y_i'(\xi a) + k y_i(\xi a) \right] \left[ \xi y_i'(\xi b) + k y_i(\xi b) \right] = \left[ \xi y_i'(\xi b) + k y_i(\xi b) \right] \left[ \xi y_i'(\xi a) + k y_i(\xi a) \right]$$
Thus \( u \) and consequently stresses \( \sigma_{rr}, \sigma_{\theta \theta}, \sigma_{r \theta} \) etc. have been found out.

Cinelli in the same literature has given the solution for corresponding spherical shell problems also proceeding exactly in the similar fashion.

At the present time, the theory of shells is one of the more active branches of the theory of elasticity which is receiving everywhere a great deal of attention. A number of authors have so far dealt with the problem of vibration in different ways. Chakravorty (1958), Ghosh (1968), Sur (1964) have discussed the problems of vibration of shells but they have not derived the exact solutions but obtained the frequency equations only. Cinelli's method is more convenient in this respect because with the help of his method he has been able to get exact solutions of the problem. Also choice of
kernel depends on the type of boundary conditions of the problem. Absorbing these conditions kernels and corresponding solutions can at once be written down in a very neat form where lies the fineness of Cinelli's method.

**Thermoelasticity**

Cinelli's method has been introduced to solve a class of problems involving second order differential equations only. An extension of the method has been established in this dissertation so that it can now be used for solving certain type of problems involving higher-order-differential equation, order higher than two. Its use is made in the field of thermoelasticity.

Thermoelasticity in general embraces varied number of theories e.g. theory of heat conduction, theory of stress-strain due to heat-flow when there is a coupling between temperature and strain. Cinelli(1965) himself obtained a general solution for the transient temperature produced in a finite hollow cylinder by an internal heat source with radiation taking place on all four surfaces. He assumed that the properties of the material of the cylinder are independent of temperature.

The boundary value-problem for this case is:

$$\frac{\partial T}{\partial t} = \mathcal{R}\left(\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial T}{\partial \phi} + \frac{\partial^2 T}{\partial \phi^2}\right) + \varphi(r, \theta, \phi, t),$$

$$a \leq r \leq b, -\pi \leq \theta \leq \pi,$$
and \( 0 \leq \frac{q}{\ell} \leq \ell \)

Initially \( T(r, \theta, \frac{q}{\ell}, 0) = F(r, \theta, \frac{q}{\ell}) \) say

Boundary conditions are the following:

\[ \frac{\partial T}{\partial r} + k_1 T = A_1 (\theta, \frac{q}{\ell}, t) \quad \text{on } r = a, \ t > 0 \]
\[ \frac{\partial T}{\partial r} + k_2 T = A_2 (\theta, \frac{q}{\ell}, t) \quad \text{on } r = b, \ t > 0 \]
\[ \frac{\partial T}{\partial \frac{q}{\ell}} + k_3 T = A_3 (r, \theta, t) \quad \text{on } \frac{q}{\ell} = 0, \ t > 0 \]
\[ \frac{\partial T}{\partial \frac{q}{\ell}} + k_4 T = A_4 (r, \theta, t) \quad \text{on } \frac{q}{\ell} = \ell, \ t > 0 \]

where \( R \) is constant equal to \( \frac{K}{\rho C} \), \( K \) being the thermal conductivity of the material, \( \rho \) is the density, \( C \) the specific heat constant, \( \phi \) the internal heat source \( F \), initial temperature distribution in the cylinder, \( T \) the temperature distribution in the cylinder. \( A_1, A_2, A_3, A_4 \) are temperature distribution of the surrounding media, \( h_1, h_2, h_3, h_4 \) are constants. With help of the result of Kaplan and Sonneman (1959) and also by means of newly constructed finite Hankel transform method Ginelli solved the above problem.

As stated earlier Duhamel (1837) first took into account the coupling of temperature and strain fields, Nowacki (1962) dealt with a number of dynamic problems of coupled temperature and strain-fields, problems of plane waves propagation.
taking into consideration the mutual influence of the temperature and strain-fields. Suhabo(1964) has handled a problem of longitudinal vibration of a circular cylinder coupled with a thermal field. He has considered the case of free vibration of an infinite circular cylinder whose lateral surface is free from stresses and being held at constant ambient temperature. He obtains the frequency equation for elastic wave in this case. He proceeds as follows:

The basic equation for thermoelastic medium can be written from Nowacki(1962) as

\[(\lambda + 2\mu) \nabla \nabla \vec{u} - \mu \nabla \times \nabla \times \vec{u} - \gamma \nabla \nabla \vec{C} = \int \frac{\partial^2 \vec{u}}{\partial t^2} \]

\[\mathcal{R} \nabla \nabla \vec{C} + \frac{\gamma}{\mathcal{C}} \frac{\partial \vec{u}}{\partial t} \cdot \nabla \nabla \vec{u} = \frac{\partial \theta}{\partial t} \]

where \(\vec{u}\) is the displacement vector, \(\lambda, \mu\) are Lame's constants, \(\theta\) is the temperature change from the equilibrium temperature \(T\), \(\mathcal{C}\) is the density of the medium, \(\mathcal{R} = \frac{k}{\mathcal{C}}\) is the constant of the diffusity, \(k\) is thermal conductivity, \(\gamma = (3\lambda + 2\mu)\) the thermal expansion coefficient, \(c\) is the specific heat.

Referring the medium to cylindrical co-ordinates \(r, \phi, z\) and then writing the equations with \(u_r = 0, u_\phi, u_z\) depending on \(r\) and \(z\) only he eliminates one of the variables. There are equations in variables \(T\) and \(\vec{u}\) only, equations are of the form

\[(L + \lambda_1^2)(L + \lambda_2^2) \vec{u} = 0\]

where \(L\) is an operator.
of the form \( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \cdot \overrightarrow{\nabla} \) is related to \( \text{div} \overrightarrow{\mathbf{u}} \). Now considering the case that the lateral surface of the cylinder kept at the ambient temperature, boundary conditions reduce to

\[
\sigma^r = r_{rj} = 0, \quad \theta = 0 \quad \text{for} \quad r = a
\]

where \( \sigma^r, r_{rj} \) are thermal stress components.

Suhabi from the further equations with the help of boundary conditions has only been able to derive the frequency equation in this case. But this work does not give the exact solutions i.e. the temperature field associated in this case or stresses produced by the temperature field.

Cinelli's finite Hankel transform when extended suitably which is done in the fourth chapter, makes it possible to obtain the exact solutions for the above case.

**Poroelasticity**

As we have stated earlier that while treating fluid-filled porous medium, the fluid/solid system is regarded as a single continuum.

Biot (1956) developed a theory for the propagation of stress-waves in a porous elastic solid. In this connection he established the most general stress-strain relation in a porous elastic solid containing a fluid from the strain-energy function. Jana and Sanayal (1971) recently have dealt with the
problem on deformation of a consolidating spherical shell of porous material. Using Laplace transform method they have obtained the solutions for the problem. For inversion of transform they have taken longtime and short-time approximation.

The constitutive equations for an isotropic poroelastic medium are (Biot, 1955)

\[
\sigma_{rr} = 2 N e_{rr} + A e + \xi E,
\]

\[
\sigma_{\theta\theta} = \sigma_{\phi\phi} = 2 N e_{\theta\theta} + A e + \xi E,
\]

\[
\sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\phi\theta} = 0, \quad \sigma^0 = \xi e - R E.
\]

where \(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \sigma_{r\phi}, \sigma_{\phi\theta}\) etc. are stress components, \(\sigma^0\) the fluid stress and \(A, N, Q, R\) are elastic constants. With the help of prescribed boundary conditions the equations of equilibrium for poroelastic medium have been solved by taking Laplace transformation. Expressions for displacement components are obtained from inversion of Laplace-transformation. For inversion long-time and short-time approximations have been made. The boundary conditions they have chosen are the following:

\[
\begin{cases}
\sigma = \phi_1(t), \quad \sigma_{rr} = \phi_2(t) \quad \text{on } r = a \quad t > 0 \\
\sigma^0 = \phi_3(t), \quad \sigma_{r\theta} = \phi_4(t) \quad \text{on } r = b
\end{cases}
\]

Where \(\sigma^0\) is the fluid stress and \(\sigma_{rr}\)'s are stress-components.

But it is not realizable how one can prescribe different stresses
on the fluid and solid parts respectively on the same boundary.

Hydrodynamics

Hydromagnetic flow phenomenon has attracted attention of a number of authors specially because of the close relation with Geophysics. The works on hydromagnetic flow which have been dealt with in this text have their stems in Hide and Robert's published work of 1960. Hide and Robert (1960) consider an idealized model consisting of a semi-infinite expanse of electrically conducting rotating fluid, bounded by an infinite horizontal rigid disk. They have considered the steady problem only.

Stoke examined a particular simple case (Rayleigh 1896), to study the effect of viscosity on the motion of a fluid in contact with a vibrating solid. He considered an infinite plane at \( z = 0 \) to execute harmonic vibration in a direction \( (x, \text{say}) \) parallel to itself, and that the fluid in contact with this plane to occupy the whole region \( z < 0 \) and also it is at rest for very large values of \( z \), \( \nu \) the coefficient of kinematical viscosity which is constant and it is assumed that no slip occurs between the fluid and the surface in contact. He has shown that if the velocity of the vibrating plane is \( (u_0 \cos \omega t, 0, 0) \), where \( t \) denotes time, \( u_0 \) and \( \omega \) which are constants by assumption are the velocity and angular frequency of the vibration; the velocity at any point in the
fluid

\[ \mathbf{\bar{u}} = (u_x, u_y, u_z) \]

is given by

\[ u_x = u_0 \exp \left[ -\left( \frac{(\omega/\nu)^2}{2} \right) \right] \cos \left[ \omega t - \left( \frac{(\omega/\nu)^2}{2} \right) \right] \]

\[ u_y = u_z = 0 \]

The present authors consider the case of a conducting fluid with electrical conductivity \( \sigma \) in the presence of an impressed uniform magnetic field of strength \( B_0 \) in the \( z \)-direction. It is seen that the overall behaviour of the system now depends on three parameters \( \alpha, \beta, \gamma \) where

\[ \alpha = \frac{B_0^2}{\omega \nu \mu}, \quad \beta = \frac{\lambda}{\nu}, \quad \gamma = \frac{u_0}{(\nu
u)^{1/2}} \]

where \( \mu \) is the magnetic permeability and \( \lambda \) is sometimes referred to as electromagnetic viscosity. The fluid motion is no longer of the form as given by Stokes. In the presence of magnetic field \( u_x \) is seemed to consist of two parts which are termed as 'velocity mode' and 'magnetic mode'.

The relative amplitude of these modes, the induced magnetic and electric fields depend on \( \alpha, \beta, \gamma \). These quantities also depend on the boundary conditions which are determined by the electrical properties of the space \( z < 0 \) not occupied by the fluid and also the conditions at \( y = \pm \infty \).
Solutions have been found for a number of cases corresponding to different values of $\kappa / \beta$, $\gamma$. The results show quantitatively the interrelation between hydromagnetic and viscous effects. The effects of uniform rotation and a uniform magnetic field acting simultaneously on the flow of a conducting fluid due to oscillation of a rigid plane with which it is in contact, have been studied by Hide and Robert as follows.

The governing equations are
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \mathbf{\Omega} \times \mathbf{u} = - \nabla p + \nabla \cdot \mathbf{b} + \nabla \times \mathbf{B} + \nabla \nabla \cdot \mathbf{u}
\]

The equation of continuity gives $\text{div} \mathbf{u} = 0$ neglecting displacement currents. Maxwell's equations are
\[
\begin{align*}
\text{Curl } \mathbf{B} &= \mu_0 \mathbf{j} \\
\text{Curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\text{div } \mathbf{B} &= 0
\end{align*}
\]

Also from generalized Ohm's law
\[
\mathbf{j} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B})
\]

$\mathbf{p}$ denotes the hydrostatic pressure, $\mathbf{E}$ the electric field, $\mathbf{J}$ the electric current density, $\mathbf{B}$ is assumed to be consisted of two parts $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ being the induced field.

Prescribed boundary conditions are chosen as
\[
\mathbf{u} \big|_{z \to \infty} = 0 \quad , \quad \mathbf{u} \big|_{z = 0} = (u_0 \cos \omega t, 0, 0)
\]
Also since it is an assumption that the region \( \xi < 0 \) is an insulator, \( \vec{j} = 0 \) there, also \( \vec{J} = 0 \) at \( \xi \to \infty \) and therefore \( \vec{B}/\xi \to \infty = 0 \), \( \vec{b}/\xi = 0 = 0 \).

Since the fields on the plate depend only on \( z \) and \( t \) equations for \( \vec{u} \) and \( \vec{b} \) are
\[
\left[ \frac{\partial}{\partial t} - \frac{1}{\sigma \mu} \frac{\partial^2}{\partial z^2} \right] \vec{b} = \mathcal{B}_0 \frac{\partial \vec{u}}{\partial \xi},
\]
\[
\left[ \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2} \right] \vec{u} + 2 \vec{\xi} \times \vec{u} = \frac{\mathcal{B}_0}{\mu_f} \frac{\partial \vec{b}}{\partial \xi}.
\]

Eliminating \( \vec{b} \) from above equations, an equation in \( \vec{u} \) is obtained, \( \vec{u} \) and \( \vec{b} \) have been supposed to be of the forms

\[
(\vec{u}, \vec{b}) \propto e^{i\omega t - qz/L},
\]
where \( L = \left( \frac{\gamma}{\omega} \right)^{1/2} \).

With the help of boundary conditions stated before, solutions for \( \vec{u} \) and \( \vec{b} \) are obtained. \( \vec{u} \) is obtained as a combination of two parts, one depending on magnetic field and the other independent of magnetic field. These two parts have been termed as 'magnetic mode' and 'velocity mode' respectively.

A generalization of this problem considering motion of fluids in between two parallel plates has been discussed in chapter 8, where the velocity distribution and the induced magnetic fields are explicitly calculated for the investigation of the structure of the flow.
The 9th chapter is concerned with the unsteady analysis of the hydromagnetic boundary layer flows.

Kakutanî (1958) has made a steady state analysis of the magnetohydrodynamic flows induced by an oscillating flat plate in an incompressible viscous electrically conducting fluid in the presence of transverse magnetic field. He has calculated the velocity field, the induced magnetic field, the current density and the electric field. It has been found that the velocity field and the other related physical quantities consist of two different but decaying oscillations.

Axford (1960) has considered the steady motion of an incompressible electrically conducting fluid generated by an infinite horizontal non-conducting plate which executes simple harmonic oscillation in its own plane with a given frequency. It has been shown that the velocity distribution consists of hydromagnetic boundary layers and the hydromagnetic waves which decay due to the finite electrical conductivity of the field.

Both Kakutanî and Axford have considered the steady state motion and completely neglected the transient effects on the flow phenomena. An unsteady analysis is presented in ninth chapter to follow up the decay and growth of the solution with time.
§ 0.6 Original and significant points of the main results of the thesis: A summary with specification of characteristic features.

In this article we describe the original and significant points of main results of our investigations with their specific characteristic features. Our investigation is mainly concerned with dealings of some problems in continuous medium, we have selected a number of fields from physical world which can be treated as continua, such as elasticity, thermoelasticity, poroelasticity and fluid mechanics. The vibration problems with mathematical formulation which is quite general and physically realistic has been discussed extensively throughout the text. With a view to the nature of the problems and also the method used the content of the dissertation is divided into three parts.

In the first chapter of Part I description and formulation of Cinelli's method are reported. In second and third chapter of this part we have utilized Cinelli's newly formulated finite Hankel transform to problems of vibration of anisotropic and inhomogeneous bodies together with remarks on the advantages of the method, over the existing ones.

In the second part in chapter four an extension of Cinelli's method is established. In chapter five and six of this part, some problems of physical interests viz. a problem of thermoelasticity and a problem of poroelasticity have been
very concisely solved by this extended method.

In the third part in chapter seven a short and critical study to characterise vibration and wave propagation, mathematically is made.

In next two chapters usual Laplace transform method is used to solve two problems of fluid mechanics. Firstly chapter eight deals with problems of hydromagnetic flow phenomenon. The analysis presented determines the simultaneous influence of the coriolis force and electromagnetic forces on the flow phenomenon.

Chapter nine is concerned with the unsteady analysis of the hydromagnetic boundary layer flows generated in a semi infinite expanse of an incompressible homogeneous viscous, electrically conducting fluid bounded by an infinite flat plate which performs small amplitude harmonic oscillations in its own plane.

In chapter ten concluding remarks are made and critical observation has been made on the work done.

Let us now discuss in short each chapter with the new results obtained therein together with the characteristic features of the method used.

As mentioned earlier in chapter one Cinelli's method is reported as this is the basis of calculations made in the subsequent five chapters.
In chapter two, the problems of vibration of thick elastic cylindrical and spherical shells are discussed when Young's modulus $E$ varies from point to point on the region according to certain laws. To solve the problem Cinelli's transform method is used.

In a paper Cinelle (1966) made an application of his method in problem of vibration of shells when stresses on the boundaries are prescribed. In that paper he considers only isotropic and homogeneous bodies. But discussions on non-homogeneous bodies are now gaining much importance and in consequence several papers appeared on the linear theory of elasticity where the elastic coefficients are assumed to vary with the location of the point on the region. The problem discussed in chapter two is essentially a mathematical problem of investigation of range of variability of the law of inhomogeneity in which Cinelli's method can profitably be used. In problems of vibration of isotropic and non-homogeneous bodies we assume variability of Young's modulus $E$, Poisson's ratio $\nu$ remains constant, validity of this assumption has been discussed by Golecki (1968). The law of variation of $E$ has been found out to be

\[
A_1 \left\{ r^{2\nu_1 - \alpha_1} \left( 1 - B_1 r^{\alpha_1} \right)^2 \right\} \cdot r^{2\nu_1} \left[ A_2 J_{\nu_1} \left( k_1 r \right) + B_2 Y_{\nu_1} \left( k_1 r \right) \right]^2
\]

Or

\[
A_2 \left\{ r^{2\nu_1 - \alpha_2} \left( 1 - B_2 r^{\alpha_2} \right)^2 \right\} \cdot r^{2\nu_1} \left[ A_2 J_{\nu_1} \left( k_2 r \right) + B_2 Y_{\nu_1} \left( k_2 r \right) \right]^2
\]
Vibration has been supposed to be in the radial direction \( r' \) only. Cinelli's method has been applied to obtain the response of thick elastic cylindrical and spherical shells. The method, however, is not applicable for arbitrarily varying \( E \) but we may see easily that due to presence of arbitrary constants in the expression of \( E(\gamma') \) there is considerable arbitrariness in \( E(\gamma) \) and these expressions embrace a good many forms of \( E(\gamma) \), generally found in some recent papers. One such example has been sited and solved in this chapter.

This chapter is based on one of my published papers 

In chapter III problems of vibration of inhomogeneous and anisotropic bodies have been solved. The displacements and stresses in thick cylindrical and spherical shells when the surfaces are subjected to certain dynamic loads have been found out by some method. In Cinelli's paper materials regardless of their composition have been considered to be homogeneous and isotropic. But these simplified assumption often leads to results, inadequate to explain physical relations properly, because in some bodies anisotropy of the material is too pronounced and hence cannot be neglected. A metallic single crystal is an extreme example of an anisotropic material, metals when severely deformed in a particular direction as in rolling also exhibit anisotropy (Dieter, 1960).

This chapter is based on one of my published papers
Merits of the method, used in above-mentioned two chapters lie in the fact that in using this method one can avoid the complicated inversion problems of other transforms specially of Laplace transform technique in which case only some approximate results can be obtained. Also it is noticeable that different authors Ghosh (1968), Sur (1967) and many others usually take only the power law of variation in treating non-homogeneous problems but in this chapter a broader law of variation of $E(\gamma)$ is found out for which non-homogeneous problems can be well-managed.

Several authors are found to close their discussion on vibration-problems simply with the frequency equation obtained from the basic equation of motion supplemented with certain boundary conditions. But in these two chapters it is found that Cinelli's method helps one to obtain the exact solutions of the problems. The kernels are so chosen that boundary conditions are automatically satisfied. The fineness of this transform is revealed in construction of the kernel, written, absorbing the boundary conditions from the very beginning.

The mathematical modification and extension of Cinelli's method is presented in chapter four. Problems so long solved by Cinelli's method involve second order differential operator of the form

$$L \equiv \frac{d^2}{d\gamma^2} + \frac{1}{\gamma} \frac{d}{d\gamma}$$

only.
Now as regards to a operator of the higher order say $L^n$, we can write for any positive integer
\[ L^n f = H L L^{n-1} f \]
To evaluate this we need end-point conditions of $L^{n-1} f$ if those can be supplied from the problems given, we can write
\[ L L^{n-1} f = \beta_i - \xi_i^2 H L^{n-1} f, \]
where $\beta_i$ containing boundary conditions of $L^{n-1} f$.
$\xi_i's$ are positive roots of an equation of the form
\[ J_\mu (\xi a) Y_\mu (\xi b) - J_\mu (\xi b) Y_\mu (\xi a) = 0, \]
form of this equation depends on the end-point conditions given.

With the use of boundary conditions and the basic equations of the problems, the end-point conditions necessary for finding truncated Hankel transform of $f(L)$, where $f(L)$ is a quadratic in $L$, have been found. The calculation and procedure ensure the fact that the modified transform technique is also applicable where $f(L)$ is any polynomial or even an integral function in $L$.

The content of this chapter is based on a part of one of my papers. [Sen, 1972].

The problem solved by this method is a boundary value problem for longitudinal vibration of a thick, isotropic homogeneous circular cylindrical shell of infinite length coupled with a thermal field in a case when lateral surface of the cylinder is kept at the ambient temperature.
The fundamental equations for a thermoelastic medium are

\[(\lambda + 2\mu) \nabla \cdot \nabla \cdot \mathbf{u} - \mu \nabla \cdot \nabla \times \mathbf{u} - \nabla \cdot \mathbf{Q} = \mathbf{f} \]

and

\[K \nabla \cdot \nabla \cdot \mathbf{u} - \frac{\partial \mathbf{u}}{\partial t} \cdot \nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} \]

\(\mathbf{u}\) is the displacement vector, \(T\) the temperature and \(\mathbf{Q}\) is the temperature change from the equilibrium temperature \(T\), rest of the quantities are constants.

From these two equations eliminating \(\mathbf{Q}\) we get a fourth order equation of the form,

\[\left( L + \lambda_1 \right) \left( L + \lambda_2 \right) \tilde{\mathbf{D}} = 0\]

where \(L = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\) and \(\tilde{\mathbf{D}} = \nabla \cdot \mathbf{u}\).

This is solved for \(\tilde{\mathbf{D}}\) by the use of the above modified method, boundary condition is that the lateral surface of the cylinder is kept at the ambient temperature.

\(\mathbf{u}\) and temperature field are found and correspondingly thermal stresses also.

Suhebi (1964) has considered the same problem and found out only the frequency relation, whereas the newly extended method enables us to find out the exact solution directly. The transform parameter \(\tilde{\mathbf{D}}\), stated earlier is found to be related with the eigen frequency of vibration. Here only one problem has been solved but now following the above mentioned method one can treat various kinds of coupled boundary value problem.
encountered in mathematical physics.

This chapter is based on one my papers to appear in the "Proceeding of the first Annual CASAMCU Symposium (1972)."

Chapter six presents the solution of the problem of deformation of a consolidating spherical shell of poroelastic material, by the use of the method described above.

Following Biot (1955) a physical assumption is made which is that the skeleton is purely elastic and contains a compressible viscous fluid.

For solving the problem of deformation assume polar symmetry about the origin and take the solid and fluid displacements.

\[ u_r = U(r, t) \quad , \quad u_\theta = 0 \quad , \quad u_\phi = 0 \]
\[ v_r = V(r, t) \quad , \quad v_\theta = 0 \quad , \quad v_\phi = 0 \]

Equations for an isotropic poroelastic medium (Biot, 1955) are

\[ \sigma_{rr} = 2Ne_{rr} + Ae + Q\varepsilon \]
\[ \sigma_{\theta\theta} = \sigma_{\phi\phi} = 2Ne_{\theta\theta} + Ae + Q\varepsilon \]
\[ \sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\theta\phi} = 0 \quad , \quad \sigma = Q\varepsilon + R\varepsilon . \]

\( \sigma_{rr}, \sigma_{\theta\theta} \) etc. are stress components and \( \sigma \) is the fluid stress.
Expressing the stresses in terms of the displacements \( u \) and \( v \) the dynamic equations are
\[ N \nabla^2 \mathbf{u} + \text{grad} \left[ (A+N) \mathbf{e} + \mathbf{aE} \right] = \frac{\partial^2}{\partial t^2} \left( \int_{s_1} \mathbf{u} + \int_{s_2} \mathbf{v} \right) + C \frac{\partial}{\partial t} (y - \nu). \]

and \[ \text{grad} \left[ \mathbf{aE} + \mathbf{RE} \right] = \frac{\partial}{\partial t} \left( \int_{s_2} \mathbf{u} - \int_{s_2} \mathbf{v} \right) - C \frac{\partial}{\partial t} (u - v) \]

\( A, N, Q, R \) are elastic constants, \( C \) is related to Darcy's coefficient of permeability \( K \) by \( C = \mu \beta^2 / K \) where \( \mu \) is the fluid viscosity and \( \beta \) the porosity.

From these two equations eliminating \( v \) we get a fourth order equation in \( u \) and with the help of boundary conditions prescribed, Cinelli's method in extended form has been successfully utilised to find out \( u \) and similarly \( v \) also.

Jana and Sanayal (1971) have treated the same problem, using Laplace transform technique, they have obtained approximate solution for \( u \). Inversion of Laplace transform presents some difficulties, to overcome this short-time and long-time approximations are to be made. At this point our method has an advantage which is that, exact solutions have been obtained.

In this context, we like to refer that the boundary conditions are
\[ \sigma = f_1(t), \quad \sigma_{rr} = f_2(t) \quad \text{on} \quad r = a, \quad t > 0 \]
\[ \sigma = f_3(t), \quad \sigma_{rr} = f_4(t) \quad \text{on} \quad r = b, \quad t > 0. \]

It is however, not realizable how one can prescribe different stresses on the fluid and solid parts of the same boundary of
This discussion has been reported in one of my papers communicated in 1972.

Seventh chapter contains a critical study on mathematical characterization of vibration and wave propagation.

In the eighth chapter, a problem of hydromagnetic flow has been discussed. The analysis presented there is concerned with a generalization of Hide and Robert's hydromagnetic boundary layer flow phenomenon, in an idealized model consisting of a semi-infinite expanse of electrically conducting rotating fluid bounded by an infinite horizontal rigid disk. The hydromagnetic boundary layer flow is examined in an electrically conducting rotating viscous fluid confined between two infinite horizontal rigid non-conducting disks in presence of a uniform magnetic field. The velocity distribution and the induced magnetic field are explicitly calculated for the investigation of the structure of the flow. The simultaneous influence of the Coriolis force and electromagnetic forces on the flow phenomenon is determined. The "velocity mode" and "magnetic modes" associated with the problem are analysed. Several limiting cases of interest are recovered as special cases.

This chapter is based on one of my joint papers with Professor Debnath to appear in "Revue Roumaine des Sciences Techniques, Pure and Applied Mech," 1973.
The chapter nine deals with an unsteady problem on hydromagnetic boundary layer flow generated in a semi-infinite expanse of an incompressible, homogeneous, viscous, electrically conducting fluid bounded by an infinite flat plate which performs small amplitude harmonic oscillations in its own plane. The unsteady velocity field and the induced magnetic field are calculated explicitly by using the Laplace transform method with suitable approximations. The solutions for the velocity and the induced magnetic field are obtained for small and large time with physical implications. The structures of the boundary and current layers including the flow field outside these layers are examined. It is shown that the solution consists of the hydrodynamic boundary and current layers and two different hydromagnetic waves which propagate with velocities \( \frac{V_A}{\sqrt{\nu}} \) and \( 2\frac{V_A}{\sqrt{\nu}} \) for small time.

The Alfven wave decays over a distance \( \frac{\sqrt{3}}{\omega A (\nu + \nu)} \) and the other is damped out in the Hartmann layer of thickness of the order \( \left( \frac{\nu}{V_A} \right)^{1/5} \). Several limiting cases of physical interests are discussed.

This chapter is based on one of my joint papers with Professor Debnath already communicated.