Chapter 8

MagnetoHydrodynamic Flow Between Two Oscillating Disks

§ 8.1 Introduction

Hide and Robert (1960) considered steady hydromagnetic boundary layer flows in an incompressible homogeneous rotating viscous fluid in the presence of a uniform magnetic field. This analysis revealed the structure of the steady velocity distribution and the induced magnetic field in a semi-infinite expanse of fluid bounded by an infinite horizontal rigid plate. The solution of the problem was obtained for weak and strong rotation of the fluid. In addition to velocity modes, the existence of magnetic modes in proved. Several hydrodynamic and hydromagnetic, rotating and non-rotating cases are recovered as particular limiting case of this study. It was indicated that the study has some applications for determination of the earth's liquid core motion.

The present analysis is introduced to make a generalisation of Hide and Robert's idealised model. This study is concerned with the steady hydromagnetic boundary layer flow in an electrically conducting viscous rotating fluid confined between two infinite horizontal rigid non-conducting disks in the presence of a uniform magnetic field. The velocity distribution and the induced magnetic field are calculated explicitly. The simultaneous influence of the coriolis force and the electromagnetic force on the flow is determined. The velocity and the
magnetic modes associated with the problem are analysed. Several limiting cases of interest are recovered as special cases. It is shown that the results of Hide and Robert can also be derived as particular cases of this analysis.

§ 8.2 Mathematical Formulation

We consider here the flow of fluid bounded by two plates parallel to z-axis. The whole system is in a state of rigid body rotation with uniform angular velocity \( \Omega \) about the z-axis normal to the plate. A uniform magnetic field \( \mathbf{B}_0 = (0, 0, B_0) \) is applied normal to the plate.

The motion of the fluid in the rotating co-ordinate system is governed by the equations

\[
\frac{\partial \mathbf{u}}{\partial t} + 2 \mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{j} \times \mathbf{B} + \nu \Delta \mathbf{u}
\]

\[
\text{div } \mathbf{u} = 0 \quad \ldots (8.1)
\]

\[
\mathbf{j} = (j_x, j_y, j_z) \quad \ldots (8.2)
\]

\( \mathbf{u} = (u, v, \omega) \) is the velocity field, \( p \) is the hydrostatic pressure, \( \rho \) is the density, \( \mathbf{j} \) the electric current density. \( \mathbf{B} \) is the total strength of the magnetic field and \( \nu \) is the coefficient of viscosity of the fluid.

Assuming no electric current in the basic state and neglecting the displacement currents, the Maxwell equations and generalised Ohm's law are
\[
\begin{align*}
\text{\textbf{96}} := 0, \quad & \text{curl} \vec{\beta} = \vec{\mu}^f, \quad \text{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \\
\dot{B} := \sigma \cdot (\vec{E} + \vec{u} \times \vec{B}) \quad & \text{..(8.3)}
\end{align*}
\]

Where \( \vec{E}, \mu \) and \( \sigma \) are respectively electric field, magnetic permeability and electrical conductivity.

We assume that \( \vec{B} = \vec{B}_0 + \vec{b} \) where \( \vec{b} \) is the impressed magnetic field.

Velocity field is taken to be of the form
\[
\vec{u} = (u(x,t), v(y,t), \omega(z,t)) \quad \text{..(8.5)}
\]

The motion is generated in the fluid from rest by moving the plates with prescribed time dependent velocity in their own planes, so that the boundary conditions on the velocity field are
\[
\begin{align*}
\vec{u} + i \omega & = f(t) \quad \text{on } z = 0, \\
\vec{u} + i \omega & = g(t) \quad \text{on } z = h_1
\end{align*}
\]

where \( f(t) \) and \( g(t) \) are arbitrary functions of time and they will be chosen later on. The boundary conditions to be satisfied by \( \vec{b} \) are
\[
\begin{align*}
\vec{b} \cdot (\hat{z}, t) & = 0 \quad \text{on } z = 0, \\
\vec{b} & = 0 \quad \text{on } z = h_1
\end{align*}
\]

Since both the plates are non-conducting.
By virtue of boundary conditions in this case it is found that $\omega(x, t)$ and $b_j(x, t)$ vanish everywhere in the flow field.

The fundamental field equations for $\mathbf{u}$ and $\mathbf{b}$ can then be obtained as

$$(\partial_t - \eta \frac{\partial^2}{\partial z^2}) b^y = -B_0 \frac{\partial u^y}{\partial j} \quad \ldots (8.10)$$

$$(\partial_t - \frac{\gamma \omega^2}{3 \omega^2}) u^y + 2 j^y \times u^z = \frac{B_0}{\mu} \frac{\partial b^y}{\partial j} \quad \ldots (8.11)$$

where $\eta = \frac{1}{\sigma' \mu}$ may be called the electromagnetic viscosity.

We introduce the non-dimensional variables $t'$ and $z'$ defined by $t' = \omega t$ and $z' = \sqrt{\frac{\omega}{B_0}} z$. The flow is governed by the following non-dimensional parameters $\alpha, \beta$ and $\delta$ defined by

$$\alpha = \frac{VA}{\omega}, \quad \beta = \frac{\gamma}{\omega}, \quad \delta = \frac{2 \sigma / \omega}{\mu}$$

and $\delta' = 2 \mu / \omega$ where $V_A^2 = B_0^2 / \mu$ is the Alfvén velocity.

In terms of non-dimensional variables and parameters, equations (8.10) and (8.11) and the boundary conditions assume the following form on omitting the primes:

$$(\partial_t - \eta \frac{\partial^2}{\partial z'^2}) b^y = -B_0 \frac{\partial u^y}{\partial j} \quad \ldots (8.12)$$

$$(\partial_t - \frac{\gamma \omega^2}{3 \omega^2}) u^y + \delta k^y \times u^z = \frac{B_0}{\mu \sqrt{\omega} \sigma} \frac{\partial b^y}{\partial j} \quad \ldots (8.13)$$
The boundary conditions are
\[ u + iv = f(t) \quad \text{on} \quad \bar{z} = 0 \]  
\[ = g(t) \quad \text{on} \quad \bar{z} = \ell \quad \text{where} \quad \ell = \sqrt{\frac{\omega}{v}} \]  
and  
\[ \bar{b}(\bar{z}, t) = 0 \quad \text{on} \quad \bar{z} = 0 \]  
\[ = 0 \quad \text{on} \quad \bar{z} = \ell \]  

Hence we are to solve the coupled equations (8.12) and (8.13) subject to above boundary conditions as stated in (8.14)-(8.17).

§ 8.3 The Solution of the Problem

From (8.12) and (8.13) eliminating \( \bar{b} \) we get
\[ \left[ \frac{\partial}{\partial t} - \frac{\partial^3}{\partial \bar{z}^2} \right] \left\{ \frac{\partial}{\partial t} - \frac{\partial^3}{\partial \bar{z}^2} \right\} \bar{u} + \delta \bar{K}_1 \times \bar{u} = \lambda \frac{\partial^2}{\partial \bar{z}^2} \bar{u} \]
\[ \text{On} \quad \ell \quad \frac{\partial^4}{\partial \bar{z}^4} \bar{u} = \lambda \frac{\partial^2}{\partial \bar{z}^2} \bar{u} - \delta \bar{K}_1 \times \frac{\partial^2}{\partial \bar{z}^2} \bar{u} - (1 + \lambda) \frac{\partial^2 \bar{u}}{\partial \bar{t} \partial \bar{z}^2} \]
\[ \frac{\partial^{2^3}}{\partial \bar{t} \partial \bar{z}^2} - \delta \bar{K}_1 \times \frac{\partial^2}{\partial \bar{t} \partial \bar{z}^2} \bar{u} \]

Let  
\[ \bar{u}(\bar{z}, t) = e^{it} \bar{u}_1(\bar{z}) \]

Then we have from (8.18)
Let \( P = u_1 + i u_1 \), where \( u_1 = \vec{U}_1 \cdot \vec{U}_1 + \vec{V}_1 \cdot \vec{V}_1 \).
\( \vec{U}_1, \vec{V}_1, \vec{k}_1 \) are unit vectors along the co-ordinate axes.

From (8.19) then we get
\[
\beta \frac{d^3 P}{d z^3} \left[ \frac{-1}{1 - \frac{z}{l}} \right] \frac{d^2 P}{d z^2} - (1 + \delta) P = 0.
\]

...(8.20)

then
\[
P = A \sinh m_1 z + \beta \sinh m_2 (l - z) + C \sinh m_3 z + D \sinh m_4 (l - z).
\]

...(8.21)

where \( m_1, m_2, m_3, m_4 \) are roots of the equation
\[
\beta m^4 - [x + i \left\{ 1 - T^2 (1 + \delta) \right\}] m^2 - (1 + \delta) = 0.
\]

...(8.22)

A, B, C and D are constants.

In a similar manner we get
\[
Q = A_1 \sinh m_1 z + \beta_1 \sinh m_2 (l - z) + C_1 \sinh m_3 z - 1 + D_1 \sinh m_4 (l - z).
\]

...(8.23)

\( Q = bx + i by \),
\( b^2 = (bx, by, b^2). \)
Boundary conditions to be satisfied by $P$ and $q$ are respectively

\[ P = f(t) \quad \text{on} \quad y = 0 \quad \ldots \quad (8.24) \]
\[ = g(t) \quad \text{on} \quad y = L \quad \ldots \quad (8.25) \]

and

\[ q = 0 \quad \text{on} \quad y = 0 \quad \ldots \quad (8.26) \]
\[ = 0 \quad \text{on} \quad y = L \quad \ldots \quad (8.27) \]

By virtue of (8.24) - (8.27)

\[
\begin{align*}
P &= A \left( \sinh m_1 y - \frac{\sinh m_1 L \sinh m_3 y}{\sinh m_3 L} \right) + \\
&\quad + B \left\{ \sinh m_2 (L-y) - \frac{\sinh m_2 L \sinh m_4 (L-y)}{\sinh m_4 L} \right\} + \\
&\quad + g(t) \frac{\sinh m_3 y}{\sinh m_3 L} + \frac{f(t) \sinh m_4 (L-y)}{\sinh m_4 L} \\
&\quad \ldots \quad (8.28)
\end{align*}
\]

and

\[
\begin{align*}
q &= A_1 \left\{ \sinh m_1 y - \frac{\sinh m_1 L \sinh m_3 y}{\sinh m_3 L} \right\} + \\
&\quad + B_1 \left\{ \sinh m_2 (L-y) - \frac{\sinh m_2 L \sinh m_4 (L-y)}{\sinh m_4 L} \right\} \ldots \quad (8.29)
\end{align*}
\]
From (8.22) we notice that

\[
\begin{align*}
  m_2 &= -m_1, \\  m_4 &= -m_3
\end{align*}
\]  \hspace{1cm} \text{(8.30)}

From equations (8.12) and (8.13) we get

\[
\left( i - \beta \right) \frac{d^2}{dj^2} \Phi = \frac{\mathcal{B}_0}{V \omega \nu} \frac{d\rho}{dj} \hspace{1cm} \text{(8.31)}
\]

\[
\left\{ i (\gamma + \delta) - \frac{d^2}{dj^2} \right\} \rho = -\frac{\mathcal{B}_0}{\mu V \omega \nu} \frac{d\rho}{dj} \hspace{1cm} \text{(8.32)}
\]

From (8.13) and by (8.14) and (8.15) we get

\[
A_1 \left( \beta m^2 - i \right) + B_1 \left( -i \cosh m_1 \rho - \beta m_1^2 \cosh m_1 \tau - \sinh m_1 \rho + \beta m_1^2 \right)
+ A \left( -\frac{\mathcal{B}_0}{V \omega \nu} : m_1 \right) + B \left( -\frac{\mathcal{B}_0}{V \omega \nu} : m_1 \cosh m_1 \tau - \frac{\mathcal{B}_0}{V \omega \nu} : m_1 \sinh m_1 \rho \right) = 0 \hspace{1cm} \text{(8.33)}
\]

\[
A_1 \left( \beta m^2 - i \right) + B_1 \left( -i \cosh m_1 \rho - \beta m_1^2 \cosh m_1 \tau - \sinh m_1 \rho + \beta m_1^2 \right)
+ A \left( -\frac{\mathcal{B}_0}{V \omega \nu} : m_1 \right) + B \left( -\frac{\mathcal{B}_0}{V \omega \nu} : m_1 \cosh m_1 \tau - \frac{\mathcal{B}_0}{V \omega \nu} : m_1 \sinh m_1 \rho \right) = 0 \hspace{1cm} \text{(8.34)}
\]

\[
A_1 \left[ \sinh m_1 \rho \left( \beta m_3^2 - i \right) \right] + B_1 \left[ \left( \beta m_3^2 - i \right) \sinh m_3 \rho \left( \cosh m_3 \tau - \sinh m_3 \rho \right) \right]
+ A \left( -\frac{\mathcal{B}_0}{V \omega \nu} : m_1 \sinh m_3 \rho \right) + B \left( \frac{\mathcal{B}_0}{V \omega \nu} : m_1 \cosh m_3 \tau \sinh m_3 \rho \right) = 0 \hspace{1cm} \text{(8.35)}
\]
From (8.33), (8.34), (8.35) and (8.36)

\[
A = \left[ \frac{-g(t) \cosh m_1 l}{\sinh m_1 l} - m_3 \left\{ \frac{f(t) \sinh m_3 l}{m_1} \left( \beta m_3^2 - 1 \right) \right\} \right] \left/ \sinh m_1 l \left( a_1, b_1 - b_2, a_2 \right) \right.
\]

\[
A_1 \left\{ \sinh m_1 l \left( i - \beta m_3^2 \right) \right\} + A_1 \left\{ \cosh m_1 l \left( i - \beta m_3^2 \right) \right\} + A_3 \left\{ \cosh m_3 l \right\}
\]

\[
\text{steady state: } m_3/\sinh m_3 l = 0 \quad \text{..(8.36)}
\]
and
\[
\beta = (i - \beta m_1^2) m_3 \left[ m_1 g(t) \left( \beta m_3^2 - i \right) \left( \cosh m_3 \ell - \cosh m_1 \ell \right) \right]
- f(t) \sinh m_3 \ell \left[ m_3 \left( \beta m_1^2 - \sinh m_1 \ell \right) + m_1 \sinh m_3 \ell \left( i - \beta m_3^2 \right) \right] \\
\sinh m_1 \ell \left[ \left\{ m_1 \left( \beta m_1^2 - i \right) \sinh m_1 \ell + m_3 \cosh m_3 \ell \sinh m_3 \ell \right. \right. \\
x \left. \left( i - \beta m_3^2 \right) \right\} - \left\{ m_3 \left( \beta m_1^2 - \sinh m_1 \ell \right) + m_1 \sinh m_3 \ell \left( i - \beta m_3^2 \right) \right\} \\
- \left\{ (i - \beta m_1^2) \left( \beta m_3^2 - i \right) \left( \cosh m_3 \ell - \cosh m_1 \ell \right) \right\} m_1 m_3 \right]
\]

\[\ldots (8.38)\]

When \( \lambda = 0 \) \& \( \delta = 0 \) the roots of the equations (8.5) are
\[
m_1 = \frac{1 + i}{2 \beta} \sqrt{2} \quad m_2 = \frac{1 + i}{2 \beta} \sqrt{2} \quad m_3 = \frac{1 + i}{2 \beta} \sqrt{2}
\]

\( m_1 \) corresponds to the magnetic mode.

In that case \( \beta \) comes out to be zero and if we make \( g(t) = 0 \) as \( \ell \to \infty \) and take \( f(t) = u_0 \cos \omega t \) the result is same as that deduced by Hide and Robert (1960).
§ 8.4 Concluding Remarks

In this chapter, the problem has been solved as a boundary value problem, it would be interesting to consider the problem as an initial value problem for an understanding of the unsteady motion.

This chapter is based on a paper by [Sen and Debnath 1973] accepted for publication in "Revue Roumaine des Sciences techniques".