CHAPTER IV

ON A CERTAIN PROPERTY OF THE CANTOR SET UNDER THE STEINHAUS TRANSFORMATION OF PERMUTATION

Introduction and Notations : Steinhaus (1961) introduced the following transformation called Permutation of a linear set :

Let

$$f_0(t) = 0 \text{ for } 0 \leq t < 1, \qquad f_0(t+2) = f_0(t) \text{ for } t \geqslant 0,$$

$$f_0(t) = 1 \text{ for } 1 \leq t < 2.$$

Also let $f_n(t) = f_{n-1}(2t)$ for $0 \leq t < 1$, for every natural number n.

Let us now consider the interval $0 \leq t < 1$ and for any given t of this interval, we have in the scale 2,

$$t = 0 \cdot e_1 \, e_2 \ldots e_n \ldots \ldots (2)$$

where $e_n = f_n(t)$, $n = 1, 2, 3, \ldots$ and thus each e_n is either 0 or 1 (and $e_n = 0$ for infinitely many n).

<u>Transformation of Permutation</u> : Let N be the set of all natural numbers $(1, 2, 3, \ldots\ldots\ldots\ldots)$ and $P(n)$ be a permutation of these numbers, given by $P = P(n) = k_n$, $n = 1, 2, 3, \ldots\ldots\ldots;\ k_n \varepsilon N$ and the reciprocal (or inverse) permutation P^{-1} $[= P^{-1}(k)]$ which restores the original arrangement, is given by

$$P^{-1}(k) = n_k, \quad k = 1, 2, 3, \ldots\ldots;\ n_k \varepsilon N.$$

When we apply $P\ [= P(n)]$ to the right hand side of

$$t = \frac{e_1}{2} + \frac{e_2}{2^2} + \cdots\cdots\cdots + \frac{e_n}{2^n} + \cdots\cdots.$$

$$[= 0 \cdot e_1 e_2 \ldots\ldots e_n \ldots\ldots\ldots (2)$$

$$= \frac{f_1(t)}{2} + \frac{f_2(t)}{2^2} + \cdots + \frac{f_n(t)}{2^n} + \cdots]$$ we get another development (and thus a point t' of $[0, 1)$ which may or may not be t itself) :

$$t' = \frac{e_{k_1}}{2} + \frac{e_{k_2}}{2^2} + \cdots\cdots\cdots + \frac{e_{k_n}}{2^n} + \cdots\cdots\cdots$$

$$[= 0 \cdot e_{k_1} e_{k_2} \ldots\ldots\ldots e_{k_n} \ldots\ldots (2)$$

$$= \frac{f_{k_1}(t)}{2} + \frac{f_{k_2}(t)}{2^2} + \cdots\cdots\cdots + \frac{f_{k_n}(t)}{2^n} + \cdots\cdots$$

$$= P(t)\ ].$$

If we now apply the inverse transformation P^{-1} [or inverse permutation P^{-1}] to t' we get back t, given by

$$t = \frac{e_{n_1}}{2} + \frac{e_{n_2}}{2^2} + \cdots\cdots + \frac{e_{n_k}}{2^k} + \cdots\cdots$$

$$\Big[= 0\cdot e_{n_1} e_{n_2} \cdots\cdots e_{n_k} \cdots\cdots \quad (2)$$

$$= \frac{f_{n_1}(t')}{2} + \frac{f_{n_2}(t')}{2^2} + \cdots\cdots + \frac{f_{n_k}(t')}{2^k} + \cdots\cdots$$

$$= P^{-1}(t') \Big].$$

We have thus the relations:

$$f_1(t) = e_1, \quad f_2(t) = e_2, \ldots\ldots, f_s(t) = e_s, \ldots\ldots,$$

and also $f_{n_1}(t') = e_1$, $f_{n_2}(t') = e_2, \ldots\ldots, f_{n_s}(t') = e_s, \ldots$

If there is a point set $E \subset [0,1)$, each element of which is expressed in the scale 2, then by applying a transformation P of the above type, we get the transformed set $E' = P(E) \subset [0,1)$ and by applying the inverse permutation on E' we get back $E = P^{-1}(E')$. It has been shown by Steinhaus (1961) that $|E| = |P(E)|$ $(= |E'|)$, where $|X|$ is the Lebesgue measure of a linear measurable set X.

It has been shown by Mukhopadhyaya (1964b) that if E is a linear set of positive measure then there exists a permutation $P(n)$ under which $E \cap P(E)$ is non-empty.

In the followihg, we propose to show that the Cantor Middle third set C, though of measure zero, has the above property under any transformation of permutation P.

<u>Theorem 1.</u> The intersection $C \cap P(C)$ under any transformation of permutation P is non-empty and has the power c of the continuum.

<u>Proof</u> : Let us consider the series

$$\frac{2}{3} + \frac{2}{3^2} + \cdots\cdots\cdots + \frac{2}{3^n} + \cdots\cdots$$

If we take any point $x \varepsilon [o, 1)$ given by $x = 0 \cdot \epsilon_1 \epsilon_2 \ldots\ldots \epsilon_n \ldots (2)$ where $\epsilon_n = o, 1$ ($= o$ for infinitely many n) for all n , then the corresponding point

$$(x) = \frac{2\epsilon_1}{3} + \frac{2\epsilon_2}{3^2} + \cdots\cdots + \frac{2\epsilon_n}{3^n} + \cdots\cdots \ \varepsilon \ [o, 1).$$

It is obvious that $(x) \varepsilon C$, where C is the Cantor Middle third set.

Let $P(n)$ be any transformation of permutation of the type mentioned above. Then obviously $P[o, 1) = [o, 1)$. Let C_1 be the subset of the Cantor set C , the elements of which are each the left hand end points of the contiguous intervals of the Cantor set C in $[o, 1)$. Then Σ $(= C - C_1)$ is the subset of the Cantor set C , which consists of the external points of C together with the right hand end points of the contiguous intervals of C in $[o, 1)$. It is obvious that the power of Σ i.e. $\overline{\overline{\Sigma}}$ is that of the continuum $(= c)$.

Now let α be any point of $[o,1)$ expressed in the dyadic scale as above; then $P(\alpha) = \alpha' \varepsilon [o,1)$ is the corresponding transformed point. Obviously $(\alpha) \varepsilon \Sigma$ and $(\alpha') \varepsilon P(\Sigma)$ where $P(\Sigma)$ is the set obtained by applying the transformation P on Σ. It is clear that $P(\Sigma) = \Sigma$. Hence $P(\Sigma) \cap \Sigma = \Sigma$ and therefore $\overline{\overline{P(\Sigma) \cap \Sigma}} = \overline{\overline{\Sigma}}$. Hence the theorem.

Hardy and Wright (1938) have introduced the following definitions :

Suppose that $x \varepsilon [o,1]$ is expressed in the scale r, and that the digit b occurs n_b times in the first n places.

If $\frac{n_b}{n} \to \beta$ when $n \to \infty$ then we say that b has the frequency β. We also say that x is simply normal in the scale r if $\frac{n_b}{n} \to \frac{1}{r}$ when $n \to \infty$ for each of the r possible values of b.

Let $\sum_{n=1}^{\infty} d_n$ be an infinite series, then $\sum_{n=1}^{\infty} d_{k_n}$ is called a subseries of this, where $\{k_n\}_{n=1}^{\infty}$ is an increasing sequence of natural numbers. Let a number $x \varepsilon [o,1)$ be expressed in the dyadic scale as $x = 0 \cdot \epsilon_1 \epsilon_2 \ldots \epsilon_n \ldots\ldots$ (2), where each $\epsilon_n = 0$ or 1 (with $\epsilon_n = 0$ for infinitely many n). It follows that $(x) = \sum_{n=1}^{\infty} \epsilon_n d_n$ is a subseries of $\sum_{n=1}^{\infty} d_n$, corresponding to the above x. Conversely any given subseries of $\sum_{n=1}^{\infty} d_n$ corresponds to a unique x of $[o,1)$.

Definition : A property P is said to be true for almost all subseries of $\sum_{n=1}^{\infty} d_n$ if the set of all $x \in [0, 1)$ corresponding to all the subseries having the property P, has the Lebesgue measure 1.

We propose to prove the following theorem.

Theorem 2. Almost all points of the Cantor set (expressed in the ternary scale) have each the frequency $1/2$ (in each of the digits 0 and 2) and this property is invariant under any Steinhaus' transformation of permutation.

We require the following lemma.

Lemma. If A and B are two linear measurable sets both contained in the interval $[a, b]$, with $|A| = b - a = |B|$, then $|A \cap B| = b - a$.

Proof : We have $A \cap B = C(CA \cup CB)$, by De-Morgan's formula, where CE represents the set complementary to $E \subset [a, b]$.

$$\therefore |A \cap B| = b - a - |CA \cup CB| \geqslant (b-a) - \{|CA| + |CB|\} = (b-a) - \{0+0\} = b-a.$$

But since $A \cap B \subset [a, b]$, hence $|A \cap B| \leqslant b - a$.
It follows that $|A \cap B| = b - a$.

Proof of Theorem 2. Let N_2 be the set of simply normal numbers of the interval $[0, 1)$ each expressed in the dyadic scale. Then we know that $|N_2| = 1$ (Hardy and Wright, 1938).

Let $P(n)$ be any Steinhaus' transformation of permutation. It follows that $|P(N_2)| = 1$ (Steinhaus, 1961). It follows from the above lemma that $|N_2 \cap P(N_2)| = 1$. If $x \varepsilon N_2 \cap P(N_2) \subset [0,1)$ then, $x \varepsilon N_2$ and also $x \varepsilon P(N_2)$.

Let $x = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \ldots\ldots + \frac{\epsilon_n}{2^n} + \ldots\ldots$, ($\epsilon_n = 0$ or 1 for all n and $\epsilon_n = 0$ for infinitely many n).

Hence $(x) = \sum_{n=1}^{\infty} \frac{2\epsilon_n}{3^n}$ (εC) is a Cantor point with frequency $1/2$ (in each of the digits 0 and 2). Thus all points of the Cantor set C corresponding to the common points of N_2 and $P(N_2)$ have this property. Also since almost all points of $P(N_2)$ are common to N_2, the required theorem follows.

<u>Definition</u> : If σ be a point of $[0,1)$ which is such that under a certain Steinhaus' transformation of permutation P, we have $P(\sigma) = \sigma$ then we say σ is an invariant point of P. Mukhopadhyaya (1964a) has shown that if E is a set of positive measure then there exists a permutation $P(n)$ under which there exists a non-empty subset of E (with positive measure) each point of which is an invariant point of P. We propose to show in the following that the Cantor Middle third set C, though of measure zero, possesses the same property.

<u>Theorem 3.</u> There exists a Steinhaus' transfommation of permutation P under which a non-empty subset of the

Cantor set C with a power greater than a can be found such that each of its points is an invariant point of P and at the same time each of its points (expressed in ternary scale) has the frequency $1/2$ (in each of the digits 0 and 2).

Proof : Let N_2 be the set of simply normal numbers of the interval $[0,1)$ each expressed in the scale 2. Hence $|N_2| = 1$ (Hardy and Wright, 1938). It follows by Mukhopadhyaya's Th. 3 (1964a) that there exists a permutation P under which the set of invariant points of N_2 is of positive measure. It follows that the set $\{x\} \subset [0,1)$ of points satisfying the relations $x \varepsilon N_2$, $x \varepsilon P(N_2)$ and $P(x) = x$ is of power greater than a ($a =$ the power of rational numbers).

Now if x be any such invariant point of P then

$x = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots\cdots + \frac{\epsilon_n}{2^n} + \cdots\cdots$, $\epsilon_n = 0$ or 1 for all n ($= 0$ for infinitely many n) has the property that its frequency (in each of the digits 0 and 2) is $1/2$.

It follows that the corresponding point

$$(x) = \frac{2\epsilon_1}{3} + \frac{2\epsilon_2}{3^2} + \cdots\cdots + \frac{2\epsilon_n}{3^n} + \cdots\cdots$$

belongs to C and $P(C)$ and moreover $P((x)) = (x)$ with the additional property that it has the frequency $1/2$ (in each of the digits 0 and 2). Obviously the power of the set of such (x)'s is greater than a .

Theorem 4. Every Steinhaus transformation of permutation P considered as a function $P(t)$ defined in $0 \leq t \leq 1$ is R-integrable over $[0, 1]$.

Proof. Let $\delta\ (>0)$ be arbitrarily chosen and let s be a positive integer so chosen that $\frac{1}{2^s} < \delta$. Let us divide the interval $[0, 1)$ into 2^s equal parts and let these dyadic intervals be

$$\left[\frac{r}{2^s}, \frac{r+1}{2^s}\right),\quad r = 0, 1, 2, \cdots\cdots\cdots, 2^s - 1.$$

Let P be any permutation. For any given r, the dyadic interval $\left[\frac{r}{2^s}, \frac{r+1}{2^s}\right)$ on the t-axis, will be transformed into $2^{n_s - s}$ dyadic intervals on the $P(t)$-axis, each of length $\frac{1}{2^{n_s}}$ (with the notations given in the introduction, assuming that n_s is the greatest of the numbers $n_1, n_2, \cdots\cdots\cdots\cdots, n_s$). Let E_i be the subsets of $\left[\frac{r}{2^s}, \frac{r+1}{2^s}\right)$, $i = 1, 2, \cdots\cdots, n_s - s$ which correspond to those $2^{n_s - s}$ dyadic intervals on $P(t)$- axis. These sets are measurable and pairwise disjoint (they are measurable, since the dyadic intervals of the $P(t)$- axis are measurable, by Steinhaus' theorem (Steinhaus, 1961).

Hence $\sum E_i = \left[\frac{r}{2^s}, \frac{r+1}{2^s}\right)$ and therefore

$$m\left(\sum E_i\right) = \sum m E_i$$

$$= m\left[\frac{r}{2^s}, \frac{r+1}{2^s}\right) = \frac{1}{2^s}.$$ Now the oscillation ω_i of $P(t)$ (which is clearly bounded) on each of the sets E_i obviously satisfies the condition $\omega_i \leq \frac{1}{2^{n_s}}$ and hence

$$\omega_i \, mE_i \leqslant \frac{mE_i}{2^{n_s}}$$

$$\therefore \sum_{i=1}^{2^{n_s-s}} \omega_i \, mE_i \leqslant \sum \frac{mE_i}{2^{n_s}} = \frac{1}{2^{n_s}} \sum mE_i = \frac{1}{2^{n_s+s}} \qquad (1)$$

Let e_i be one of the $(n_s - s)$ dyadic intervals on $P(t)$-axis and let $e_{i_1}, e_{i_2}, \ldots\ldots, e_{i_p}$ (p finite) be the correspondiing disjoint dyadic intervals, all subintervals of $\left[\frac{r}{2^s}, \frac{r+1}{2^s}\right)$ and which correspond to e_i under the transformation P^{-1} (and $E_i = e_{i_1} + e_{i_2} + \cdots + e_{i_p}$). If ω_{i_p} be the oscillation of $P(t)$ on e_{i_p}, $p = 1, 2, \ldots\ldots$, then

$$m(e_{i_1})\,\omega_{i_1} + m(e_{i_2})\,\omega_{i_2} + \cdots\cdots + m(e_{i_p})\,\omega_{i_p}$$

$$\leqslant \omega_i \left[m(e_{i_1}) + m(e_{i_2}) + \cdots\cdots + m(e_{i_p}) \right]$$

$$= \omega_i \, mE_i .$$

Hence the Riemann oscillatory sum corresponding to the partition

$$\left\{ \left[\frac{r}{2^s}, \frac{r+1}{2^s} \right) \right\}, \quad r = 0, 1, 2, \ldots\ldots, 2^s - 1$$

of the interval $0 \leqslant t < 1$ is

$$\sum \sum m(e_{i_p})\,\omega_{i_p} \leqslant \sum_{i=1}^{2^{n_s-s}} \omega_i \, mE_i \leqslant \frac{2^s}{2^{n_s+s}} \quad \text{by (1)}$$

$$= \frac{1}{2^{n_s}} \leqslant \frac{1}{2^s} \quad (\text{as } n_s \geqslant s) \; < \delta$$

Hence $(R)\int_0^1 P(t)\,dt$ exists.

Corollary. The function $P(t)$ is continuous almost everywhere in $[0, 1]$ for every Steinhaus' transformation of permutation P.

Definition : A function f whose domain is R_1 is said to be linear if $f(x) + f(y) = f(x+y)$ for all real x and y (Boas, 1960).

Note. It may be asked, whether there exists any transformation of permutation such that the corresponding function $P(t)$ may be linear.

As it is a bounded function, it cannot be discontinuous (wildly discontinuous) (Boas, 1960). It has points of continuity (Cor. ofTh. 4), hence it must be continuous everywhere in $[0, 1)$ and its form is $P(t) = at$ where a is a constant, for all t in $[0, 1)$ (Boas, 1960). Hence we conclude that if a transformation of permutation P is to be linear, then its form must necessarily be $P(t) = at$ where a is a constant. We construct below an illustration of this type, when the transformation of permutation is not strictly of Steinhaus' type, but P is a permutation of the set $N = (0, 1, 2, \ldots)$.

Example : Let us take any point $x \in [0, 1)$ given by

$$x = 0 \cdot \epsilon_1 \epsilon_2 \ldots\ldots \epsilon_n \ldots\ldots (2)$$

$$= \epsilon_0 \cdot \epsilon_1 \epsilon_2 \ldots\ldots \epsilon_n \ldots\ldots (2),$$

where $\epsilon_0 = 0$ for all x and $\epsilon_n = 0$ or 1 ($= 0$ for infinitely many n) for other points. We define a transformation T as follows :

$$T(n) = n-1 \quad \text{for all} \quad n \geq 1 .$$

Hence the above x is transformed into $x' \in [0, \frac{1}{2})$ given by $x' = 0 \cdot \epsilon_0 \epsilon_1 \epsilon_2 \ldots\ldots \epsilon_{n-1} \ldots\ldots (2)$. Thus the interval $[0, 1)$ is mapped onto $[0, \frac{1}{2})$ by this transformation T . In this case, the measure is not invariant, every measurable set in $[0, 1)$ has its measure reduced to half its original measure by T . It is obvious that $T(t)$ is linear and $T(t) = \frac{1}{2} t$ for all t in $[0, 1)$.

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