# CHAPTER III

## ON SOME PROPERTIES OF SETS WITH POSITIVE MEASURE

Kestelman [1947, Th.2] and Lahiri [1959, Th. 2] have given some theorems on sets with positive measure. In this paper we show that the results given by these authors can further be sharpened.

Theorem 1. Let $C$ be a closed bounded set in $R_N$ having positive measure and let $\{\lambda_n\}$ be a null sequence of vectors in $R_N$. Then there exists a set $\Omega = \{\xi \mid \xi \in C ; \xi + \lambda_n \in C$ for infinitely many $n\}$, which is closed and has a positive measure and moreover $\left| |C| - |\Omega| \right| < \epsilon$; where $\epsilon\ (>0)$ is arbitrary.

Proof. Let $\epsilon\ (>0)$ be chosen, small at pleasure (and therefore we can take, $0 < \epsilon < |C|$). We can now define a sequence $\{U_n\}$ of open sets satisfying $U_1 \supset U_2 \supset U_3 \supset \ldots$,

$$C = \prod_{r=1}^{\infty} U_r \qquad (1)$$

and

$$|C| \leq |U_r| < |C| + \frac{\epsilon}{2^r},$$

and hence

$$\sum_{r=1}^{\infty} | U_r - C | < \epsilon . \qquad (2)$$

Let $\mu_r$ be the distance between $C$ and $\tilde{U}_r$ ( = complement of $U_r$ ); then $\{\mu_r\}$ is a null sequence of positive numbers.

Since $\{\lambda_n\}$ is a null sequence of vectors in $R_N$, therefore $|\lambda_n| < \mu_1$, for all indices $n \geqslant m_1$ where $m_1$ is a positive integer depending on $\mu_1$. Let us choose one such index, say $n_1$, (preferably the least of all such indices); then $|\lambda_{n_1}| < \mu_1$. In the same manner, we have $|\lambda_n| < \mu_2$ for all indices $n \geqslant m_2$ where $m_2$ is a positive integer depending on $\mu_2$. Let us choose one such index, say $n_2$ satisfying $n_2 > n_1$; then $|\lambda_{n_2}| < \mu_2$. Proceeding in this way indefinitely, we have a null sequence of vectors $\{\lambda_{n_r}\} = \{\lambda_{r'}\}$ which is a subsequence of the given null sequence $\{\lambda_n\}$ and moreover $|\lambda_{r'}| = |\lambda_{n_r}| < \mu_r$, $r = 1, 2, \ldots$.

Let $C_r = T(C ; -\lambda_{r'})$ i.e. $C$ is translated by $-\lambda_{r'} = -\lambda_{n_r}$, $r = 1, 2, \ldots$, and let $\Omega = \prod_{r=1}^{\infty} C_r$.

Since $C$ is closed, therefore, $C_r$ is closed for all $r$, and hence $\Omega$ is closed and thus $\Omega$ is measurable. $\ldots (3)$

Since $U_r \supset C_r$ for all $r$ we have

$$\Omega = U_1 - \sum_{r=1}^{\infty} (U_1 - C_r) \quad \text{by De-Morgan's formula}$$

$$= U_1 - \{\sum_{r=1}^{\infty}(U_1 - U_r) + \sum_{r=1}^{\infty}(U_r - C_r)\}$$

$$= U_1 - \sum_{r=1}^{\infty}(U_1 - U_r) - \sum_{r=1}^{\infty}(U_r - C_r)$$

$$= \prod_{r=1}^{\infty} U_r - \sum_{r=1}^{\infty}(U_r - C_r) \qquad \text{by De-Morgan's formula}$$

$$= C - \sum_{r=1}^{\infty}(U_r - C_r) \qquad \text{by (1)}$$

Hence $\quad |\Omega| \geqslant |C| - \sum_{r=1}^{\infty}|U_r - C_r| = |C| - \sum_{r=1}^{\infty}|U_r - C|$

$$> |C| - \epsilon \quad \text{by (2)} \qquad (4)$$

Thus $\quad |\Omega| > 0 \quad$ and hence $\Omega$ is non-empty. (5)

If $\xi$ is any point of $\Omega$ then $\xi \in C_r$ and thence

$$\xi + \lambda_{r'} \in C \quad \text{i.e.} \quad \xi + \lambda_{n_r} \in C, \; r = 1,2,3,\ldots \qquad (6)$$

But since $\lim\limits_{r\to\infty} |\xi + \lambda_{r'}| = |\xi|$ as $\lim\limits_{r\to\infty} |\lambda_{r'}|$ $= \lim\limits_{r\to\infty} |\lambda_{n_r}| = 0$, $\xi$ is a limiting point of $C$ and hence $\xi \in C$ as $C$ is closed.

Therefore $\Omega \subset C$ and hence $|\Omega| \leqslant |C| < |C| + \epsilon \qquad (7)$

Therefore from (4) and (7) we get

$$|C| - \epsilon < |\Omega| < |C| + \epsilon$$

$$\text{or} \quad \big|\,|C| - |\Omega|\,\big| < \epsilon \qquad (8)$$

In view of (3), (5) and (8) the theorem is completely proved.

<u>Theorem 2.</u> Let $\{\alpha_n\}$ be a sequence of real numbers such that $\lim \alpha_n = 1$, $\alpha_n > 1$ for all $n$ and let $C$ be a

closed bounded set of positive measure contained in the open unit sphere $E_N$. Then there exists a set $X = \{\xi \mid \xi \in C ; \frac{\xi}{\alpha_n} \in C$ for infinitely many $n\}$ which is closed and has a positive measure and moreover $\big| |C| - |X| \big| < \epsilon$, where $\epsilon\ (>0)$ is arbitrary.

Proof. Let $\epsilon\ (>0)$ be chosen small at pleasure (and therefore we can take $0 < \epsilon < |C|$ ). We can then define a sequence $\{U_n\}$ of open sets satisfying $U_1 \supset U_2 \supset \ldots ; C = \prod_{r=1}^{\infty} U_r$

and $$|C| \leqslant |U_r| < |C| + \frac{\epsilon}{2^r}$$

and hence $$\sum_{r=1}^{\infty} |U_r - C| < \epsilon \qquad (1)$$

Let $\delta_n\ (>0)$ be the distance between the closed set $C$ and $\widetilde{U}_n$ (complement of $U_n$ ). Then $\{\delta_n\}$ is a positive null sequence.

Now $\lim\limits_{n \to \infty} \alpha_n = 1$ and hence corresponding to $\delta_1$, $1 < \alpha_n < 1 + \delta_1$ for $n \geqslant m_1$ where $m_1$ is a positive integer depending on $\delta_1$. Let us choose one such index, say $n_1$ (preferably the least of all such indices), then $\alpha_{n_1} < 1 + \delta_1$. In the same manner we have $\alpha_n < 1 + \delta_2$ for all $n \geqslant m_2$ where $m_2$ is a positive integer depending on $\delta_2$. Let us choose one such index, say $n_2$ satisfying $n_2 > n_1$ ; then $\alpha_{n_2} < 1 + \delta_2$. Proceeding in this way indefinitely we get the sequence of numbers $\alpha_{n_1}, \alpha_{n_2}, \ldots\ldots$ which is a subsequence of $\{\alpha_n\}$ and $\lim_{r\to\infty} \alpha_{n_r} = 1$, $\alpha_{n_r} > 1$.

Then we consider the sets $C_r = \alpha_{n_r} C$, $r = 1, 2, 3, \ldots, \infty$, and let $X = \prod_{r=1}^{\infty} C_r$.

Now $C_r$ is closed for all $r$ and hence $X$ is closed and therefore measurable.

If $\xi$ be any point of $C$ then $\xi \alpha_{n_r} \varepsilon C_r$ and $|\xi - \xi\alpha_{n_r}| = |\xi| . |1 - \alpha_{n_r}| < |\xi| . \delta_r < \delta_r$ since $\xi \varepsilon E_N$, which shows that $C_r \subset U_r$.

Now
$$X = \prod_{r=1}^{\infty} C_r \quad [\subset \prod_{r=1}^{\infty} U_r = C]$$

$$\therefore \quad |X| \leqslant |C| < |C| + \epsilon \qquad (3)$$

Also $X = U_1 - \sum_{r=1}^{\infty} (U_1 - C_r)$ by De-Morgan's formula

$$= U_1 - \left\{ \sum_{r=1}^{\infty} (U_1 - U_r) + \sum_{r=1}^{\infty} (U_r - C_r) \right\}$$

$$= \left\{ U_1 - \sum_{r=1}^{\infty} (U_1 - U_r) \right\} - \sum_{r=1}^{\infty} (U_r - C_r)$$

$$= C - \sum_{r=1}^{\infty} (U_r - C_r)$$ by De-Morgan's formula

Hence $|X| \geqslant |C| - \sum_{r=1}^{\infty} |U_r - C_r|$

Now $C_r = \alpha_{n_r} C$.

Therefore $|C_r| = \alpha_{n_r}^N |C| > |C|$ as $\alpha_{n_r} > 1$.

Hence $|U_r - C_r| < |U_r - C|$ and thus

$$|X| \geq |C| - \sum_{r=1}^{\infty} |U_r - C| > |C| - \epsilon > 0 \quad \text{by (1)} \qquad (4)$$

Combining (3) and (4) we get

$$|C| - \epsilon < |X| < |C| + \epsilon \quad \text{or} \quad \big| |C| - |X| \big| < \epsilon \qquad (5)$$

That $X$ is non-empty follows from (4) and if $\xi$ is any point of $X$ then $\xi \varepsilon C_r$ and hence $\frac{\xi}{\alpha_{n_r}} \varepsilon C$, $r = 1, 2, 3, \ldots$.

But $\lim_{r\to\infty} \frac{\xi}{\alpha_{n_r}} = \xi$ since $\lim_{r\to\infty} \alpha_{n_r} = 1$, therefore $\xi$ is a limiting point of $C$ and hence $\xi \varepsilon C$ as $C$ is closed.

Hence $X$ is the set $\left\{ \xi \,\middle|\, \xi \varepsilon C ; \frac{\xi}{\alpha_n} \varepsilon C \right.$ for infinitely many $\left. n \right\}$ and in view of (2) and (5) the theorem is completely proved.

Corollary. The above theorem can also be shown to be true even when the closed bounded set $C$ does not lie in the unit sphere $E_N$.

We can choose a positive constant $k (> 1)$ such that the set $C$ lies wholly in the sphere (with centre at the origin) of radius $k$.

We now apply the transformation of dilation on $C$ and consider the set $E = \frac{1}{k} C$.

Hence if $\xi$ be any point of $C$, then the point $\frac{\xi}{k}$ belongs to $E$ and $|\frac{\xi}{k}| < 1$ (since $|\xi| < k$ ), that is if $\xi'$ be any point of $E$ then $|\xi'| < 1$.

Hence applying the argument used in this theorem, we can find a subsequence $\{\alpha_{n_r}\}$ of the sequence $\{\alpha_n\}$, $\alpha_n > 1$ with $\lim_{n\to\infty} \alpha_n = 1$ and a closed set $Y \subset E$ such that $|E| - \frac{\epsilon}{k^N} < |Y| < |E| + \frac{\epsilon}{k^N}$, where $\epsilon\,(>0)$ is arbitrarily chosen (6)

This shows that $Y$ is non-empty (as $|Y| > 0$ ) and hence if $\xi' \varepsilon Y$ then $\frac{\xi'}{\alpha_{n_1}}, \frac{\xi'}{\alpha_{n_2}}, \ldots\ldots$ will also each belong to $Y$.

We now consider the set $\Gamma = kY$ which is closed and $|\Gamma| = k^N |Y|$ which shows that $|\Gamma| > 0$ and if $\xi' \varepsilon Y$ then $\xi\,(= k\xi') \varepsilon \Gamma$ and moreover

$$\frac{k\xi'}{\alpha_{n_r}} \varepsilon \Gamma \quad \text{i.e.} \quad \frac{\xi}{\alpha_{n_r}} \varepsilon \Gamma, \quad r = 1, 2, 3, \ldots\ldots$$

Also from (6)

$$\frac{|C|}{k^N} - \frac{\epsilon}{k^N} < \frac{|\Gamma|}{k^N} < \frac{|C|}{k^N} + \frac{\epsilon}{k^N} \quad \text{or} \quad \big||C| - |\Gamma|\big| < \epsilon.$$

Hence the corollary is proved.

# REFERENCES


Kestelman, H. (1947) - The convergent sequences belonging to a set, Journal, Lond. Math. Soc., 22, 130.

Lahiri, B. K. (1959) - A property of sets of positive measure, Bull. Cal. Math. Soc., 51, 79.