CHAPTER II

ON SOME COUNTABLE SUBSETS OF CANTOR'S MIDDLE THIRD SET

In Bose Majumdar's papers [1959, 1960, 1965] it has been shown that for each $d \varepsilon [0, 1]$ the set of points $\{(x,y)\}$, $y-x=d$, $y \varepsilon C$, $x \varepsilon C$ where C is the Cantor set, is either finite or infinite with the power c of the continuum (but never infinite with the power a of the rational set). Each of the subsets of the Cantor product set $C \times C$ $(=C^2)$, corresponding to each separate $d \varepsilon [0,1]$, where $y=mx+d, (x,y) \varepsilon C^2$ may be considered to be a linear distribution of the Cantor points in the unit square $Q\,[(0,0);(1,0);(1,1);(0,1)]$.

Now the question arises, is there any linear distribution of the Cantor points in Q, corresponding to a given d which is infinite with the power a of the rational set? The present paper is an attempt in this direction. It has been shown by Utz [1951] from geometrical considerations that if m satisfies the relation $1/3 \leqslant |m| \leqslant 3$, then for each $d \varepsilon [0, 1]$ there exists at least one Cantor point $(x,y) \varepsilon C^2$ in the unit square Q satisfying $y = mx + d$. Hence $m = \frac{1}{2}$ is one of the admissible values and this particular result has been incidentally arrived at by set-theoretic method in this paper, in which we have mainly investigated the power of the set $\{(x,y)\}$ each point satisfying $y = \frac{x}{2} + d$, $y \varepsilon C$, $x \varepsilon C$ for each $d \varepsilon [o, 1]$.

Notations. The Lebesgue Measure of a set E is denoted by $|E|$. The symbol $C \ni x$ is used to mean that x is a point of the set C . The power of a set E is denoted by $\overline{\overline{E}}$.

Theorem 1. For a given $d\ [o \leq d \leq 1]$, the power of the set

$$\Delta_d = \{(x,y) \mid y = \tfrac{x}{2} + d,\ x \varepsilon C,\ y \varepsilon C\} \text{ is given by}$$

$$\overline{\overline{\Delta_d}} = 1 \text{ or } a .$$

Proof : Let us consider any $d \varepsilon [o, 1]$ and write it as

$$d = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i} , \qquad \delta_i = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Hence,

$$\frac{d}{2} = \sum_{i=1}^{\infty} \frac{\gamma_i}{3^i} , \qquad \gamma_i = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} .$$

Let us now write

$$\frac{d}{2} = \sum_{i=1}^{\infty} \frac{\beta_i - \frac{1}{2}\alpha_i}{3^i} \qquad \text{where} \quad \beta_i - \tfrac{1}{2}\alpha_i = \gamma_i \qquad \text{and}$$

$\beta_i = \binom{0}{1}$, $\alpha_i = \binom{0}{1}$, β_i (and α_i) may in particular cases be $1/2$, if all other β's (and α's) after this stage be zero. We observe that

$$\beta_i = 0, \ \alpha_i = 0, \quad \text{when} \quad \gamma_i = 0,$$

$$\beta_i = 1, \ \alpha_i = 1, \quad \text{when} \quad \gamma_i = 1/2$$

and $\beta_i = 1$, $\alpha_i = 0$, when $\gamma_i = 1$, satisfy the above equation. Thus, we have

$$d = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} - \frac{1}{2} \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i} = y - \frac{x}{2},$$

where

$$C \ni y = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} \quad \text{and} \quad C \ni x = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i},$$

which is Utz's theorem when $m = 1/2$ i.e., for every $d \varepsilon [0,1]$, the straight line $y = \frac{x}{2} + d$ passes through at least one Cantor point $(x, y) \varepsilon C^2$.

<u>Case I</u>. Now, take a $d_1 \varepsilon [0,1]$ which is such that

$$d_1 = y_1 - \frac{1}{2} x_1, \quad y_1 = \sum_{i=1}^{k} \frac{2\beta_i^{(1)}}{3^i} \quad \text{and} \quad x_1 = \sum_{i=1}^{k} \frac{2\alpha_i^{(1)}}{3^i},$$

with $\beta_i^{(1)} = \alpha_i^{(1)} = 0$ (or $\beta_i^{(1)} = \alpha_i^{(1)} = 1$), $i \geqslant k+1$; k

being a finite positive integer, and this happens when

$$d_1 = \sum_{i=1}^{k} \frac{\delta_i}{3^i}, \quad \delta_i = 0 \quad \text{for} \quad i \geqslant k+1,$$

$$\left(\text{or} \quad d_1 = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i}, \quad \delta_i = 1, \quad \text{for} \quad i \geqslant k+1\right).$$

If in this representation of d_1 in the above form we put

$$\beta_{k+1}^{(1)} = \frac{1}{2} \text{ and } \alpha_{k+1}^{(1)} = 1 \text{ with } \beta_i^{(1)} = \alpha_i^{(1)} = 0, \ i \geqslant k+2,$$

then we have

$$y_1' - \frac{1}{2} x_1' = \left(\frac{2\beta_1^{(1)}}{3} + \cdots\cdots + \frac{2\beta_k^{(1)}}{3^k} + \frac{1}{3^{k+1}}\right) - \frac{1}{2}\left(\frac{2\alpha_1^{(1)}}{3} + \cdots\cdots + \frac{2\alpha_k^{(1)}}{3^k} + \frac{2}{3^{k+1}}\right)$$

$$= \left(\frac{2\beta_1^{(1)}}{3} + \cdots\cdots + \frac{2\beta_k^{(1)}}{3^k}\right) - \frac{1}{2}\left(\frac{2\alpha_1^{(1)}}{3} + \cdots\cdots + \frac{2\alpha_k^{(1)}}{3^k}\right)$$

$$= y_1 - \frac{x_1}{2} = d_1,$$

where

$$y_1' = \frac{2\beta_1^{(1)}}{3} + \cdots\cdots + \frac{2\beta_k^{(1)}}{3^k} + \frac{1}{3^{k+1}} \in C,$$

and

$$x_1' = \frac{2\alpha_1^{(1)}}{3} + \cdots\cdots + \frac{2\alpha_k^{(1)}}{3^k} + \frac{2}{3^{k+1}} \in C.$$

In general, we put

$$\beta_{k+1}^{(1)} = \beta_{k+2}^{(1)} = \cdots\cdots = \beta_{k+m}^{(1)} = 0 = \alpha_{k+1}^{(1)} = \alpha_{k+2}^{(1)} = \cdots\cdots = \alpha_{k+m}^{(1)}$$

and

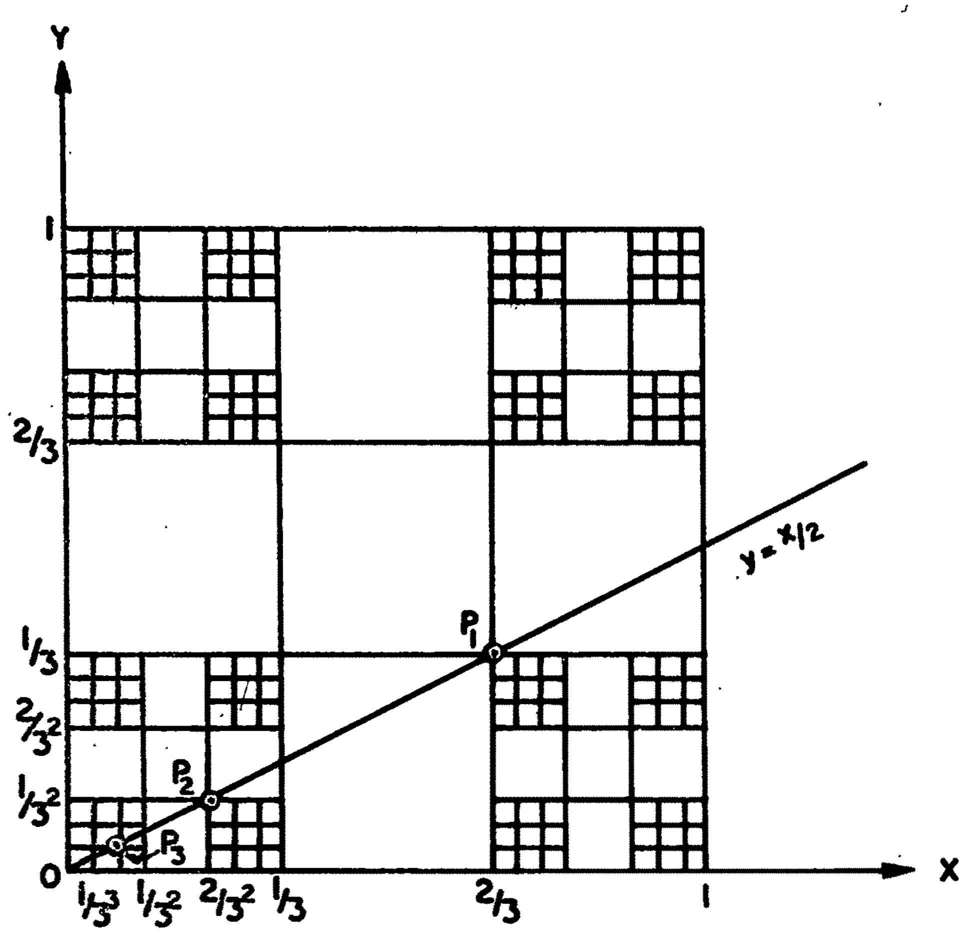
$$\beta_{k+m+1}^{(1)} = \frac{1}{2} \text{ with } \alpha_{k+m+1}^{(1)} = 1 \text{ and } \beta_i^{(1)} = \alpha_i^{(1)} = 0, \ i \geqslant k+m+2.$$

Then we get

$$y_1^{(k+m+1)} - \frac{1}{2} x_1^{(k+m+1)} = y_1 - \frac{1}{2} x_1 = d_1,$$

where

$$C \ni y_1^{(k+m+1)} = \frac{2\beta_1^{(1)}}{3} + \cdots\cdots + \frac{2\beta_k^{(1)}}{3^k} + \frac{0}{3^{k+1}} + \cdots\cdots + \frac{0}{3^{k+m}} + \frac{1}{3^{k+m+1}},$$



The points P_1, P_2, P_3 ······ belong to the set $C \times C$ $(= C^2)$.

and

$$C \ni x_1^{(k+m+1)} = \frac{2\alpha_1^{(1)}}{3} + \cdots + \frac{2\alpha_k^{(1)}}{3^k} + \frac{0}{3^{k+1}} + \cdots\cdots + \frac{0}{3^{k+m}} + \frac{2}{3^{k+m+1}} \cdot$$

Thus we notice that if $d \in [0,1]$ is a number of the form $d = \cdot\delta_1\delta_2\ldots\ldots\delta_k$ (i.e. of the form .011201 .... 20000 ...) or of the form $d = \cdot\delta_1\delta_2\ldots\delta_k 111\ldots\ldots$ (i.e. of the form $\cdot 0112012111111\ldots$), then there lie an enumerably infinity of points

$$(x,y) \in C^2 \text{ on } y = \frac{x}{2} + d$$

$$\text{i.e. } \overline{\overline{\Delta}}_d = a \,.$$

<u>Note</u> : This result that we have found by set-theoretic method can be seen also with reference to a diagram in which we construct the set $C \times C$ in the unit square by successive deletion of crosses. We take a particular case

$$y = \frac{x}{2} = \frac{x}{2} + 0 = \frac{x}{2} + (\cdot 000\ldots)$$

The diagram vividly demonstrates the existence of an enumerably infinity of points of C^2 lying on the line $y = \frac{x}{2}$.

<u>Case II</u>. If d is such that the equation $d = y - \frac{x}{2}$ holds good for $(x,y) \in C^2$, where

$$C \ni y = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} \text{ and } C \ni x = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i} \text{ with } \alpha_i = \beta_i \ (= 0 \text{ or } 1) \text{ not being}$$

<u>satisfied</u> for all i from and after a certain stage $i = k$, then we cannot change any pair (α_i, β_i) in the above representation of d (as has been done in Case I) without

throwing (x,y) out of C^2 . Hence it follows that in such a case $\overline{\overline{\Delta_d}} = 1.$

E.g. Take $d = \cdot 01222101211101 2112 \ldots (3)$ (which does not terminate either with a string of 0's or with a string of 1's).

Then, by applying the method mentioned in the above theorem, we get a solution $(x,y) \varepsilon C^2$, satisfying $y - \frac{x}{2} = d$, given by

$y = \cdot 022222022222 0 \ldots (\varepsilon C)$ and

$x = \cdot 020002020222 0 \ldots (\varepsilon C)$, where x and y do not end with a string of 0's , nor do they end with a string of 2's from a certain stage onward. Hence we cannot generate a countable number of (other) Cantor points satisfying this equation as we have done above. Hence for this d , we have $\overline{\overline{\Delta_d}} = 1.$

<u>Note</u> : Just as in Case I, here also we can make a geometrical construction to demonstrate the truth of the above conclusion in some simple particular cases.

<u>Theorem 2.</u> The measure of the set Γ , where

$$\Gamma = \Big\{ d \;\Big|\; d = y - \frac{x}{2},\; C \ni x = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i},\; C \ni y = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} \qquad (1)$$

$$\alpha_i, \beta_i = \binom{0}{1} \text{ and } \alpha_i \neq \beta_i, \text{ for all } i \geqslant k+1,$$

where k is a finite integer depending on $d \varepsilon [0,1] \Big\}$, is given by

$$|\Gamma| = 0.$$

Proof : We have observed in the introduction that given a $d \in [0,1]$ the equation $d = y - \frac{x}{2}$ has a solution (x,y) on the Cantor product set C^2.

If moreover there exists a $d \in [0,1]$ of the type mentioned in the above Theorem 2, i.e. if Γ is non-empty, then we have a point $(x,y) \in C^2$ satisfying $d = y - \frac{x}{2} = y + x - \frac{3x}{2}$.

We also see that the same Cantor point (x,y) satisfies

$$y + x = \frac{2(\alpha_1+\beta_1)}{3} + \cdots + \frac{2(\alpha_k+\beta_k)}{3^k} + \frac{2}{3^{k+1}} + \frac{2}{3^{k+2}} + \cdots \cdot$$

Hence

$$d = \frac{2(\alpha_1+\beta_1)}{3} + \cdots + \frac{2(\alpha_k+\beta_k)+1}{3^k} - \frac{3x}{2} .$$

Hence, the set Γ , given in (1), will be contained in a set which is the union of a finite number of sets, each obtained by dilating the Cantor set C by $3/2$, then reflecting it in the origin and then translating it by

$$\frac{2(\alpha_1+\beta_1)}{3} + \cdots + \frac{2(\alpha_{k-1}+\beta_{k-1})}{3^{k-1}} + \frac{2(\alpha_k+\beta_k)+1}{3^k}$$

where,

$$\alpha_m , \beta_m = \binom{0}{1} , \quad m = 1, 2, \ldots, k .$$

Since $|C| = 0$, it follows that $|\Gamma| = 0$ [see Boas].

1376

Theorem 3. Fornalmost all points $d \varepsilon [o,1]$, the set $\{(x,y)\} = \Delta_d = \{(x,y) | d = y - \frac{x}{2}, x \varepsilon C, y \varepsilon C\}$

has the power $\overline{\overline{\Delta}}_d = a$.

Proof : The result follows by combining the results of Theorem 1 and Theorem 2.

REFERENCES

Utz, W. R. (1951) - Amer. Math. Monthly, 58, 407.

Boas, Jr., R. P. (1962) - Bull Cal. Math. Soc., 54, 103.

Bose Majumdar, N. C. (1959) - Bull Cal. Math. Soc., 51, 93.

Bose Majumdar, N. C. (1960) - Bull Cal. Math. Soc., 52, 1.

Bose Majumdar, N. C. (1965) - Amer. Math. Monthly, 72, 725.