INTRODUCTION

Let x be a number given by $\sum_{n=1}^{\infty} \frac{c_n}{3^n}$, where each of the numbers c_n has one of the values 0, 2. The set $\{x\}$ is a non-dense perfect set. Constructed by G. Cantor (1845-1918) this is the first example of a non-dense perfect set purposely constructed (Hobson, E. W., The theory of functions of a real variable and the theory of Fourier's series, Vol.I, Dover Publications, Inc., (1957), p.123). This set has been used by distinguished mathematicians and research workers with excellent results in various branches of mathematics, specially in the theory of sets as used in the theory of functions of real variables. We feel that the creation of this set is almost as happy an incident as Cantor's highly original creation of theory of sets.

The three distinguished men, whose influence on modern pure mathematics (and indirectly on modern logic and the philosophy which abut on it) is most marked, are Karl Weierstrass (1815-97), Richard Dedekind (1831-1916) and Georg Cantor (1845-1918). Undoubtedly, Cantor's works, such as construction of a non-dense perfect set, creation of theory of sets and discovery of the theory of infinite numbers, have brought about revolutionary progress in pure mathematics. Modern theories of infinity and continuity are based on

Cantor's discoveries and various branches of real variable theory and topology stem from them. The fruits of these discoveries have been rich - their promise even greater. The eminent mathematician Hilbert once said, 'From the Paradise created by Cantor no one can ever drive us' [Georg Cantor, Contributions to the founding of the theory of transfinite numbers (translated and provided with an introduction by Jourdain, 1915)].

A major part of this humble thesis is devoted to the study of some unique, as well as characteristic properties of the Cantor set or sets of similar type. We also discuss in this thesis about some sets with positive measure (in the Lebesgue sense) with a view to depicting similarity of their properties to those of the Cantor set whose Lebesgue measure is zero. Problems about cardinals of sets (introduced by Cantor) connected with the Cantor set or sets of similar type have been dealt with. Inquiries have been made whether some specially constructed classes of sets may be endowed with some properties of the Cantor set, in particular, those relating to distance set. These classes of sets in concern include both the classes which do not form metric spaces as well as those classes for each of which a metric is defined in a new sense of the term, as introduced in this thesis. We give below the summary of seven chapters with which this thesis is composed.

We shall be using the following notations. A set E whose distance set fills an interval about the origin is called an S-set and if the length of the interval is equal to the diameter of the set, then it is called an SD-set. The other notations used in this thesis will be explained as occasion arises.

Chapter I

It is a well-known property of the Cantor set [Randolph, J. F., Distances between points of the Cantor set, Amer. Math. Monthly, 47 (1940), p.549; Bose Majumder, N. C., On the distance set of the Cantor middle third set, III, Amer. Math. Monthly, 72 (1965), p.725] that each point of $(0,1)$ is the midpoint of at least one pair of Cantor points. We have shown in this chapter that each point of $(0,1)$ is a point of trisection of a segment contained in the interval $0 \leqslant x \leqslant 1$, the end points of which are Cantor points.

A linear set S is said to have the property (S_n), if there exists an η_n such that if

$$X_1 < X_2 < \dots \dots < X_n\ ,\quad X_n - X_1 < \eta_n$$

are any n real numbers, there exist n elements

$$Y_1\ ,\ Y_2\ ,\ \dots\dots,\ Y_n\ \varepsilon\ S$$

Congruent to $X_1, X_2, \ldots\ldots\ldots, X_n$.

E. Marczewski [Coll. Math. (1955), p.125] proposed the problem : does there exist a perfect set S of measure zero having the property (S_3) ? In this connection, it may be mentioned that the Cantor middle third set C, which is perfect and of Lebesgue measure zero, has the property (S_2) [Steinhaus, H., Nowa Vlasność mnogośći G. Cantora, Wektor (1917), p. 105; Randolph, J. F., Distances between points of the Cantor set, Amer. Math. Monthly, 47 (1940), p. 549; Utz, W. R., The distance set for the Cantor discontinuum, Amer. Math. Monthly, 58 (1951), p.407; Šalát, T., Cas. pro pest. Mat., 87 (1962), p.4; Bose Majumder, N. C., On the distance set of the Cantor middle third set, III, Amer. Math. Monthly, 72 (1965), p.725]. It is also known that Cantor set C does not possess the property (S_3) [Šalát, T., Čas. pro. pěst. Mat., 87 (1962), p.4] and in this chapter we have investigated the reasons as to why this property fails for the Cantor middle third set C.

In this chapter we have constructed a class Λ of sets, where Λ behaves like the Cantor set C which is demonstrated in the theorem, viz.,

There exists a class Λ of sets, where Λ consists of continuum number of pairwise disjoint non-empty linear sets such that the distance set $\{\rho(A,B)\}$ of Λ, which is a set of real numbers, fills an interval of length $\delta(\Lambda)$, which is the diameter of Λ and $A \varepsilon \Lambda$, $B \varepsilon \Lambda$ (i.e. Λ is an SD-class).

Chapter II

It has been shown [Bose Majumder, N. C., On the distance set of the Cantor middle third set, III, Amer. Math. Monthly, 72 (1965), p. 725] that for each $d \varepsilon [0,1]$ the set of points $\{(x,y)\}$, $y-x=d$, $y \varepsilon C$, $x \varepsilon C$ where C is the Cantor set, is either finite or infinite with the power c of the continuum (but never infinite with the power $\aleph_0$ of the rational set).

Each of the subsets of the Cantor product set $C \times C$ $(=C^2)$ corresponding to each separate $d \varepsilon [0,1]$ where $y = mx + d$, $(x,y) \varepsilon C^2$, may be considered to be a linear distribution of the Cantor points in the unit square

$$Q\ [(0,0)\ ;\ (1,0)\ ;\ (1,1)\ ;\ (0,1)].$$

If m satisfies the relation $1/3 \leq |m| \leq 3$, then Utz [Utz, W. R., The distance set for the Cantor discontinuum, Amer. Math. Monthly, 58 (1951), p.407] has shown from geometrical considerations, that for each $d \varepsilon [0,1]$ there exists at least one Cantor point $(x,y) \varepsilon C^2$ in the unit square Q

satisfying $y = mx + d$. Bose Majumder [Bose Majumder, N. C., Properties of the Cantor set and sets of similar type, Amer. Math. Monthly, 68 (1961), p.444] gave the demonstration of Utz's theorem in the cases $m = \frac{1}{3^n}$, $n = 1, 2, 3, \ldots\ldots$, when $d = 0$. Now, $m = \frac{1}{2}$ is also one of the admissible values and Utz's result in this particularly case has incidentally been arrived at in this chapter by set-theoretic method, in which we have mainly investigated the power of the set $\{(x,y)\}$, each point $(x,y) \varepsilon C^2$ satisfying $y = \frac{x}{2} + d$, for each $d \varepsilon [0, 1]$. Among the results obtained in this chapter, the main results are :

<u>Theorem 1</u> For a given $d\ [0 \leqslant d \leqslant 1]$, the power of the set

$$\Delta_d = \left\{ (x,y) \,\middle|\, y = \tfrac{x}{2} + d ,\ x \varepsilon C,\ y \varepsilon C \right\}$$

is given by

$$\overline{\overline{\Delta_d}} = 1 \text{ or } \aleph_0 .$$

<u>Theorem 3</u> For almost all points $d \varepsilon [0, 1]$, the set

$$\Delta_d = \left\{ (x,y) \,\middle|\, d = y - \frac{x}{2} ,\ (x,y) \varepsilon C^2 \right\}$$

has the power

$$\overline{\overline{\Delta_d}} = \aleph_0 .$$

Chapter III

In this chapter we have discussed some properties of sets (in the N -dimensional Euclidean space) with positive measure. The main results are embodied in the following theorems :

<u>Theorem 1</u> Let C be a closed bounded set in R_N having positive measure and let $\{\lambda_n\}$ be a null sequence of vectors in R_N. Then there exists a set

$$\Omega = \{\xi \mid \xi \in C \; ; \; \xi + \lambda_n \in C \quad \text{for infinitely many } n\}$$

which is closed and has a positive measure and moreover

$$\big| |C| - |\Omega| \big| < \epsilon ,$$

where $\epsilon \; (>0)$ is arbitrary, and Ω depends on ϵ.

<u>Theorem 2</u> Let $\{\alpha_n\}$ be a sequence of real numbers such that $\lim \alpha_n = 1$, $\alpha_n > 1$ for all n and let C be a closed bounded set of positive measure contained in the open unit sphere E_N. Then, there exists a set

$$X = \{\xi \mid \xi \in C \; ; \; \frac{\xi}{\alpha_n} \in C \quad \text{for infinitely many } n\}$$

which is closed and has a positive measure and moreover

$$\big| |C| - |X| \big| < \epsilon ,$$

where $\epsilon \; (>0)$ is arbitrary and X depends on ϵ.

<u>Corollary</u> The above theorem is true even when the closed bounded set does not lie within the unit sphere E_N.

It is to be noted in this connection that a theorem on the Cantor set viz., 'Let C be the Cantor set and let E be the product set $C \times C \times \cdots \times C$ $(= C^N)$ in R_N. Let $n \leq N$ be any positive integer. Then if λ is a vector in R_N whose length is less than $\sqrt{n}$, the set $\{\xi\}$ such that $\xi \varepsilon E, \xi + \lambda \varepsilon E$ is non-empty for each of an infinite number (of power c) of directions of λ and the set $\{\xi\}$ of such ξ is closed for each particular direction of λ' [Bose Majumder, N. C., Properties of Cantor set and sets of similar type, Amer. Math. Monthly, 68 (1961), p.444] shows that the Cantor set C though of measure zero behaves almost similarly as a set with positive measure, as given in the theorem 1 of this chapter.

In the next chapter also, we exhibit some properties of the Cantor set which are very much similar to those of a set with positive measure.

Chapter IV

Steinhaus [Steinhaus, H., Sur une transformation des ensembles linéaires, Mathematica, 3 (26), 1 (1961), p.179-182] introduced the following transformation called Permutation of a linear set :

Let

$$f_0(t) = 0 \ , \qquad \text{for} \quad 0 \leq t < 1,$$

$$f_0(t+2) = f_0(t), \quad \text{for} \quad t \geq 0,$$

$$f_0(t) = 1 \ , \qquad \text{for} \quad 1 \leq t < 2.$$

Also, $f_n(t) = f_{n-1}(2t)$, for $0 \leq t < 1$, for every positive integer n.

For every $t \varepsilon [0, 1)$, we have in the scale 2

$$t = 0 \cdot e_1 e_2 \ldots . e_n \ldots \ldots \text{(scale 2)}$$

where $e_n = f_n(t)$, $n = 1, 2, 3, \ldots .$ and thus each e_n is either 0 or 1 (and $e_n = 0$ for infinitely many n).

Let $P(n)$ be a permutation of the set N of natural numbers 1,2,3, ... given by

$$P = P(n) = k_n \ , \quad n = 1, 2, 3, \ldots \ldots ,$$

$k_n \varepsilon N$ and the inverse permutation $P^{-1} [= P^{-1}(k)]$ which restores the original arrangement is given by

$$P^{-1}(k) = n_k \ , \quad k = 1, 2, 3, \ldots \ldots ; \ n_k \varepsilon N .$$

When we apply $P [= P(n)]$ to the right hand side of

$$t = \frac{e_1}{2} + \frac{e_2}{2^2} + \ldots \ldots .$$

$$\left[= 0 \cdot e_1 e_2 \ldots \ldots \ldots \text{(scale 2)} = \frac{f_1(t)}{2} + \ldots \ldots + \frac{f_n(t)}{2^n} + \ldots \ldots \right]$$

we get another development (and thus a point t' of $[0,1]$ which may or may not be t itself) ;

$$t' = \frac{e_{k_1}}{2} + \cdots\cdots + \frac{e_{k_n}}{2^n} + \cdots\cdots$$

$$\Big[= 0 \cdot e_{k_1} e_{k_2} \ldots\ldots e_{k_n} \ldots\ldots \quad (\text{scale } 2)$$

$$= \frac{f_{k_1}}{2} + \cdots\cdots + \frac{f_{k_n}}{2^n} + \cdots\cdots$$

$$= P(t) \Big].$$

If we now apply the inverse transformation P^{-1} (or inverse permutation P^{-1}) to t' , we get back t , given by

$$t = \frac{e_{n_1}}{2} + \cdots\cdots + \frac{e_{n_k}}{2^k} + \cdots\cdots$$

$$\Big[= \frac{f_{n_1}(t')}{2} + \cdots\cdots + \frac{f_{n_k}(t')}{2^k} + \cdots\cdots$$

$$= P^{-1}(t') \Big]$$

Thus

$f_1(t) = e_1, \ldots\ldots, f_s(t) = e_s, \ldots\ldots$ and also

$f_{n_1}(t') = e_1, \ldots\ldots, f_{n_s}(t') = e_s, \ldots\ldots.$

If t be a point of $[0,1)$ which is such that under a certain Steinhaus transformation of permutation P , we have $P(t) = t$, then we say that t is an invariant point of P.

Thus a point set $E \subset [0,1)$ is transformed to a point set $E' = P(E) \subset [0,1)$ by this transformation of permutation and we also get $E = P^{-1}(E')$ by applying the inverse permutation. It has been shown by Steinhaus that $|E| = |P(E)|$, where $|X|$ is the Lebesgue measure of the set X.

In 1964 Mukhopadhyaya [Mukhopadhyaya, A. K., On sets of positive measure under certain transformations, Matematichki Vesnik, 1 (16) (1964), C_B, p.4] has proved that if E is a linear set of positive measure then there exists a permutation $P(n)$ under which $E \cap P(E)$ is non-empty. In this chapter we have shown that the Cantor set C though of measure zero, has the above property under any transformation of permutation P. This is given in Theorem 1.

<u>Theorem 1</u> The intersection $C \cap P(C)$ under any transformation of P is non-empty and has the power c of the continuum.

Let $x \in [0,1]$ be expressed in the scale h and let a digit b occur n_b times in the first n places. If $\lim_{n\to\infty} \frac{n_b}{n} = \beta$, then we say that b has the frequency β [Hardy and Wright, Theory of numbers, Oxford (1938), p.124]. We have proved the following theorems in this chapter.

<u>Theorem 2</u> Almost all points of the Cantor set expressed in the ternary scale, have each the frequency $1/2$ (in each of the digits 0 and 2) and this property is invariant under any Steinhaus transformation of permutation.

Mukhopadhyaya [Mukhopadhyaya, A. K., On certain transformations of sets of points, Mathematica 6(29), 2 (1964), p.273] has also shown that if E is a set of positive measure, then there exists a permutation $P(n)$ under which there exists a non-empty subset of E (with positive measure) each point of which is an invariant point of P. We have shown in Theorem 3 that though $|C| = 0$, C possesses the same property.

<u>Theorem 3</u> There exists a Steinhaus transformation of permutation P under which a non-empty subset of the Cantor set C with a power greater than $\aleph_0$ can be found such that each of its points is an invariant point of P and at the same time each of its points, expressed in ternary scale, has the frequency $1/2$ (in each of the digits 0 and 2).

We have also proved the following theorem in this chapter.

<u>Theorem 4</u> Each Steinhaus transformation of permutation P considered as a function $P(t)$ defined in $0 < t < 1$ is R-integrable over $[0, 1]$.

<u>Corollary</u> The function $\mathcal{P}(t)$ is continuous almost everywhere in $[0,1]$ for every Steinhaus transformation of permutation $\mathcal{P}$.

Finally, we have investigated in this chapter the existence of a transformation of permutation $\mathcal{P}$ such that the corresponding function $\mathcal{P}(t)$ may be linear.

Chapter V

Let E be the unit interval $0 \leq x \leq 1$. We call a set $S \subset E$ symmetrical (i.e. S possesses the property σ), if for every $x \in S$, $1-x$ also belongs to S. If every point in $0 < x < 1$ is the mid-point of at least one pair of points of S , then we say that S has the property μ . If every number x of $0 < x < 1$ is equal to the difference of at least one pair of points of S , then we say that S possesses the property δ (or S is an SD-set, when we assume that 1 is the diameter of the set S). If a set S has all the three properties σ , μ and δ , then S is said to possess the Cantor property π [Bose Majumder, N. C., On the construction of some SD -sets, Bull. Cal. Math. Soc., 54 (3), September (1962), p.163]. It has been shown by Bose Majumder [Bose Majumder, N. C., A study of certain properties of the Cantor set and of an SD -set, Bull. Cal. Math. Soc., 54 (1), March (1962), p.8] that if $S (\subset E)$

has the properties σ and μ then it possesses the property δ also; if S has the properties σ and δ then it possesses the property μ also. But it has been shown by him [Bose Majumder, N.C., On the construction of some SD-sets, Bull. Cal. Math. Soc., 54 (3), September (1962), p.163] that if S possesses μ and δ, S may or may not possess σ.

In this chapter we have studied these properties with reference to a class of sets and have found the following theorems.

<u>Theorem 1</u> The class Σ of sets, all subsets of E $(0 \leqslant x \leqslant 1)$ each of which has the property σ, is a ring.

<u>Theorem 2</u> The class M of sets, all subsets of E $(0 \leqslant x \leqslant 1)$ each of which has the property μ, is not a ring.

<u>Theorem 3</u> The class D of sets, all subsets of E $(0 \leqslant x \leqslant 1)$ each of which has the property δ, is not a ring.

It is known that the usual definition of the distance (a real number) between two sets [Randolph, Basic Real and Abstrac Analysis, Academic Press, N.Y. p.101] does not make a class of sets a metric space. Of course, the distance (a real number) has been defined in a different way by Kolmogorov and Fomin [Kolmogorov and Fomin, Elements of theory of functions and functional analysis, Vol. II, Graylock Press, (1961), p.47] to make a ring a metric space.

In this chapter we have defined the distance $\rho(A,B)$ which is not a real number but a set, between two sets A and B in a manner, such that a class S of sets of points becomes a metric space. Then taking the class S of sets with a unit X, we have defined (just as we have done above in the case of a point set S) SD-class and also the class S possessing the properties (σ, μ, δ) i.e. Π.

We have then proved the following theorems.

<u>Theorem 4</u> If X is any set of points and S is the class of all subsets of X, then S possesses the properties σ, μ and δ and is thus an SD-class.

<u>Theorem 5</u> A class S of sets with unit X, having the properties σ and μ must necessarily possess the property δ.

<u>Theorem 6</u> A class S of sets with unit X, having the properties σ and δ must possess the property μ.

<u>Theorem 7</u> A necessary and sufficient condition that an algebra of sets with unit X, is an SD-class is that it

should coincide with the class of all subsets of X.

<u>Corollary</u> If an algebra of sets is an SD-class, then it is a B-algebra and also a monotone class. Also, a monotone ring with a unit is an SD-class.

<u>Theorem 8</u> If L is a lattice of sets, then the set D of its distances contains a semi-ring.

<u>Theorem 9</u> If the lattice L of sets has the property σ, then $P(L)$ is an algebra [$P(L)$ is the class of all sets of L of the form $F - E$, where F and E are in L].

Chapter VI

While dealing with some properties of the Cantor sets on various bases, Sengupta and Bose Majumder [Sengupta, H. M. and Bose Majumder, N. C., A note on certain plane sets of points, Bull. Cal. Math. Soc., 47 (4), December (1955), p. 199.] constructed in the unit square Q [$(0,0)$; $(1,0)$; $(1,1)$; $(0,1)$] an everywhere dense set A which is of first category having the power c, such that a straight line $y = mx + d$ (m and d rationals), while intersecting Q does not intersect A. Though they did not mention it, the set A that they constructed is of measure zero. In this chapter we have sharpened this property in theorem 1.

Theorem 1 There exists a plane set $\mathcal{B} \subset Q$ such that $\mathcal{B}$ is of plane measure 1 and is a residual set (therefore everywhere dense and of power c) such that it is possible to draw countably infinite number of straight lines each intersecting Q but none meeting $\mathcal{B}$.

In a recent paper J. R. Kinney [Kinney, J. R., A thin set of circles, Amer. Math. Monthly, 75 (1968), p.1077] of Michigan State University has used the Cantor set to construct a set K with plane Lebesgue measure zero such that for every positive $\alpha \leqslant 1$, the set K contains a circle of diameter α .

In this chapter, we have used the same Cantor set to construct a plane set K having properties a little more sharpened than those contained by Kinney's set. This is considered in theorem 2.

Theorem 2 There exists a plane set K with plane Lebesgue measure zero, such that for every positive α_i belonging to a set of countably infinite number of α_n's:

$$\{\alpha_n\}_{n=1}^{\infty} \quad , \quad 0 < \alpha_n < 1 ,$$

the set K contains continuum number of circles each of diameter α_i . We have also indicated that there exist c^c $(= 2^c > c)$ number of sets each of Kinney's type.

Chapter VII

In 1890 Peano /Peano, Sur une Courbe, qui remplit toute une aire plane, Math. Ann., XXXVI (1890), p.157/ gave the first continuous curve which passes through every point of a square at least once. He further observed that there is an everywhere dense enumerable set of points through each of which the curve passes twice and another everywhere dense enumerable set of points through each of which it passes four times /Hobson, E. W., The theory of functions of a real variable and the theory of Fourier's series, Vol. I, Dover Publications, Inc., (1957), p.453/. Boas /Boas, A primer of Real functions, John Wiley and Sons, Inc., (1960), p.92/ in this connection mentions that the construction of such curves may be further refined so that the curve has nothing worse than triple points, but further than this we cannot go.

In 1938, Schoenberg /Schoenberg, I. J., On the Peano curve of Lebesgue, Bull. Amer. Math. Soc., 44 (1938), p.519/ constructed a space-filling curve which is continuous and passes through every pointof the unit square $[0,1]\times[0,1]$ by using the Cantor set based on 3 (i.e. the Cantor middle third set).

In this chapter we have constructed a class of space-filling curves, each continuous and each passing through every point of the unit square $[0,1]\times[0,1]$ by using subsets of the Cantor set based on $2\lambda+1$, $\lambda=1,2,3,\ldots$.

Schoenberg's curve is a particular member of the above class.

Sengupta [Sengupta, H. M., On continuous independent functions, Quart. J. Math., 19 (1948), p.129; On continuous semi-independent functions, Quart. J. Math. (2), 5 (1954), p.172] gave some properties of semi-independent functions. [Two functions $x(t)$ and $y(t)$ continuous in $0 \leq x \leq 1$ and not constant in it are said to be semi-independent in $0 \leq x \leq 1$, if given α, β and α', β' satisfying $l \leq \alpha \leq \beta \leq L$ and $m \leq \alpha' \leq \beta' \leq M$ where l, L and m, M are lower and upper bounds of $x(t)$ and $y(t)$ respectively, the sets of points E_1 and E_2 on $[0,1]$ given by

$$E_1 \equiv E_x\,(\alpha \leq x(t) \leq \beta) \quad and \quad E_2 \equiv E_y\,(\alpha' \leq y(t) \leq \beta')$$

are such that

$$|E_1 E_2| > 0$$

for every choice of pairs of α, β and α', β']. With the help of Sengupta's theorems we have constructed the following two semi-independent functions $x(t)$ and $y(t)$ on $[0,1]$.

Let $a\,(0 \leq a \leq 1)$ be one of the values of $y(t)$ in $0 \leq t \leq 1$ and $E_{y(t)=a}$ be the set of points t on the t-axis in $0 \leq t \leq 1$, for which $y(t)$ has the value a. Let $F_{a=x}\,[E_{y(t)=a}]$ be the set of values assumed by $x(t)$ at the points $E_{y(t)=a}$ in $0 \leq t \leq 1$. Sengupta has shown that F_a is identical with $[0,1]$ and that $E_{y(t)=a}$

is not enumerable. We have in this chapter, given a more definite result in the lemma viz.

The power $\overline{\overline{E}}_{y(t)=a}$ of the set $E_{y(t)=a}$ is c, where c is the cardinal number of the continuum.

Incidentally, the space-filling curves constructed above and this lemma help us to give an alternative proof of the following important theorem on cardinals [Kuratowski, K., Introduction to set theory and topology, Pergamon Press, (1961), p.79].

Theorem If c be the cardinal number of the continuum then

$$c^2 = c .$$

In conclusion, the author acknowledges her indebtedness to authors of different papers and text books which have been consulted frequently during the preparation of this thesis.