CHAPTER VII

CANTOR SETS IN THE CONSTRUCTION OF SPACE-FILLING CURVES

INTRODUCTION

In 1890 Peano [1] gave the first continuous curve which passes through every point of a square at least once. He further observed that there is an everywhere dense enumerable set of points through each of which the curve passes twice and another everywhere dense enumerable set of points through each of which it passes four times (Hobson [2]) (Boas [3] in this connection mentions that the construction of such curves may be further refined so that the curve has nothing worse than triple points, but further than this we cannot go).

In 1938 I. J. Schoenberg [4] constructed a space-filling curve which is continuous and passes through every point of the unit square $[0,1] \times [0,1]$ by using the Cantor set based on 3, i.e., the Cantor middle third set.

The aim of the present paper is to construct a class of space-filling curves, each continuous and each passing through every point of the unit square $[0,1] \times [0,1]$ by using subsets of the Cantor set based on $2\lambda+1$ where $\lambda = 1, 2, 3, \ldots$. Incidentally, the construction of these curves helps us to prove an important theorem on cardinal numbers, viz. $c^2 = c$.

<u>Construction of a space-filling curve</u> : Let λ be a positive integer and let us first define a function $f(t)$ in the interval $[0, 2]$ as follows :

$$
\begin{aligned}
f(t) &= 0, && \text{if } 0 \leq t \leq \frac{2\lambda-1}{2\lambda+1} \text{ or if } \frac{4\lambda+1}{2\lambda+1} \leq t \leq 2, \\
&= (2\lambda+1)t - (2\lambda-1), && \text{if } \frac{2\lambda-1}{2\lambda+1} \leq t \leq \frac{2\lambda}{2\lambda+1}, \\
&= 1, && \text{if } \frac{2\lambda}{2\lambda+1} \leq t \leq \frac{4\lambda}{2\lambda+1}, \\
&= -(2\lambda+1)t + (4\lambda+1), && \text{if } \frac{4\lambda}{2\lambda+1} \leq t \leq \frac{4\lambda+1}{2\lambda+1}.
\end{aligned}
$$

For other points on the real line $(-\infty, \infty)$ let $f(t)$ be defined by $f(t+2) = f(t)$. The function $f(t)$ is obviously continuous and positive on $(-\infty, \infty)$. Let us now define two functions $x(t)$, $y(t)$ on $-\infty < t < \infty$ as follows :

$$x(t) = \sum_{k=1}^{\infty} \frac{f[(2\lambda+1)^{2k-2} t]}{2^k}$$

and $$y(t) = \sum_{k=1}^{\infty} \frac{f[(2\lambda+1)^{2k-1} t]}{2^k}$$

Each term of the above two series is positive and each term in both the series is continuous everywhere and moreover each series converges uniformly (as $0 \leq f(t) \leq 1$ and Weierstrass's M-test is applicable with $M_n = \frac{1}{2^n}$). It follows that $x(t)$ and $y(t)$ are both continuous in $(-\infty, \infty)$.

Let $\varphi(t) = (x(t), y(t))$ and let F_λ denote the map of the unit interval $0 \leq t \leq 1$ under φ. We propose to show that F_λ fills the unit square $Q \equiv [0,1] \times [0,1]$, i.e., F_λ passes through every point of Q at least once.

Since $0 \leq f(t) \leq 1$ for every t and since $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, it is obvious that $0 \leq x(t) \leq 1$ and $0 \leq y(t) \leq 1$. Hence $F_\lambda \subset Q$.

Let $P(a, b)$ be any point in Q. We shall show that P lies on F_λ. Using dyadic scale, let

$$a = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \quad \text{and} \quad b = \sum_{k=1}^{\infty} \frac{b_k}{2^k},$$

$$a_k = 0, 1 \quad \text{and} \quad b_k = 0, 1$$

for $k = 1, 2, 3, \ldots\ldots$. Consider the Cantor point c (on the base $2\lambda+1$) given by $c = \sum_{k=1}^{\infty} \frac{2\lambda c_k}{(2\lambda+1)^k}$ where $c_k = 0, 1$ for all k and $c_{2k-1} = a_k$ and $c_{2k} = b_k$, $k = 1, 2, \ldots\ldots$.

Since $\sum_{k=1}^{\infty} \frac{2\lambda}{(2\lambda+1)^k} = 1$, it follows that $0 \leq c \leq 1$.

It will now be shown that $x(c)=a$ and $y(c)=b$.

We have

$$(2\lambda+1)^k c = \{2\lambda c_1 (2\lambda+1)^{k-1} + \ldots \ldots + 2\lambda c_{k-1}(2\lambda+1) + 2\lambda c_k\}$$

$$+ \left\{ \frac{2\lambda c_{k+1}}{2\lambda+1} + \frac{2\lambda c_{k+2}}{(2\lambda+1)^2} + \frac{2\lambda c_{k+3}}{(2\lambda+1)^3} + \ldots\ldots\ldots \text{ to } \infty \right\}$$

$= 2\mu + d_k$, where μ is a non-negative integer

and $d_k = \frac{2\lambda c_{k+1}}{2\lambda+1} + \frac{2\lambda c_{k+2}}{(2\lambda+1)^2} + \ldots\ldots\ldots$

Hence $f[(2\lambda+1)^k c] = f(2\mu + d_k) = f(d_k)$ as f is positive with period 2.

If $c_{k+1} = 0$ then obviously $0 \leqslant d_k \leqslant \frac{1}{2\lambda+1}$ as

$$\frac{2\lambda}{(2\lambda+1)^2} + \frac{2\lambda}{(2\lambda+1)^3} + \ldots\ldots\ldots\ldots = \frac{2\lambda}{(2\lambda+1)^2} \times \frac{2\lambda+1}{2\lambda} = \frac{1}{2\lambda+1}.$$

Hence $f(d_k) = 0 = c_{k+1}$ and thus $f[(2\lambda+1)^k c] = c_{k+1}$ (i)

Again, if $c_{k+1} = 1$, $\frac{2\lambda}{2\lambda+1} \leqslant d_k \leqslant 1$ as $\frac{2\lambda}{2\lambda+1} + \frac{2\lambda}{(2\lambda+1)^2} + \ldots\ldots = 1.$

Hence $f(d_k) = 1$ (as $\lambda > 1/2$) . Therefore, $f[(2\lambda+1)^k c] = c_{k+1}$ (ii)

From (i) and (ii) it follows that $f[(2\lambda+1)^k c] = c_{k+1}$, always.

Hence $f[(2\lambda+1)^{2k-2} c] = c_{2k-1} = a_k$, $k = 1, 2, 3, \ldots\ldots$

and $f[(2\lambda+1)^{2k-1} c] = c_{2k} = b_k$, $k = 1, 2, 3, \ldots\ldots\ldots$

Thus $x(c) = \frac{f(c)}{2} + \frac{f[(2\lambda+1)^2 c]}{2^2} + \cdots\cdots\cdots$

$$= \frac{c_1}{2} + \frac{c_3}{2^2} + \cdots\cdots\cdots$$

$$= \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots\cdots\cdots = a$$

and

$$y(c) = \frac{f[(2\lambda+1)c]}{2} + \frac{f[(2\lambda+1)^3 c]}{2^2} + \cdots\cdots$$

$$= \frac{c_2}{2} + \frac{c_4}{2^2} + \cdots\cdots\cdots$$

$$= \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots\cdots = b\,.$$

Hence at the point $t = c$, $(x(c), y(c)) = P(a, b)$. Therefore, P lies on F_λ . Hence F_λ completely fills Q .

Note 1 : If $\lambda = 1$ we get Schoenberg's curve Γ filling Q.

Note 2 : But taking $\lambda = 1, 2, 3, \ldots\ldots$ we get a sequence $\{F_\lambda\}_{\lambda=1}^{\infty}$ of continuous space-filling curves.

Note 3 : It is obvious that each of F_λ though continuous is non-differentiable (e.g. F_1 is non-differentiable as f is so at $t = 1/3$, $2/3$ etc.).

Note 4 : It is obvious that instead of taking the entire interval $0 \leq t \leq 1$ it would be enough if t is restricted only on the Cantor set C (based on 3) to fill Q by F_1 (similar considerations apply for the other F's) i.e. Cantor set may be mapped onto Q.

Note 5. It is obvious that above is not a $(1,1)$ correspondence between the unit interval $0 \leq t \leq 1$ and Q [for we have the theorem (Hobson [2]) : no continuous $(1,1)$ correspondence can exist between all the points in a square and all the points in a linear interval (also Boas [3] and Randolph [5])].

Note 6. If at least one of the functions $x(t)$, $y(t)$ be continuous and of bounded variation then they cannot be semi-independent (see Note 7 below) since either $x(t)$ or $y(t)$ or both fail to assume each of its values non-enumerably infinity of times (for, the Banach indicatrix $N(y)$= Number of roots of the equation $f(x)=y$ for a given y is finite almost everywhere, Natanson [6]) and hence such functions cannot be used to fill the square.

Note 7. H. M. Sengupta [7] has shown (1948 and 1954) that the continuous functions $x(t)$, $y(t)$ which map the linear interval $0 \leq t \leq 1$ onto the square $R\ [l \leq x \leq L;\ m \leq y \leq M]$, l, L and m, M being the exact bounds of $x(t)$ and $y(t)$ on $[0, 1]$ are semi-independent (i.e., given $\alpha, \beta\ ;\ \alpha', \beta'$ satisfying $l \leq \alpha \leq \beta \leq L$ and $m \leq \alpha' \leq \beta' \leq M$ then the sets of points on $[0, 1]$, $E_1 \equiv E_x\ (\alpha \leq x(t) \leq \beta)$

and $E_2 \equiv E_y (\alpha' \leq y(t) \leq \beta')$ are such that $|E_1 E_2| > 0$ for every choice of pairs of α, β and α', β'. He has proved the following theorem in his first paper.

Theorem : If $f(x)$ and $g(x)$ be continuous in $0 \leq x \leq 1$ and be not constant in it, then a necessary and sufficient condition for semi-independence i.e., for $|E_1 E_2| > 0$ is that the set of values assumed by $f(x)$ over the set of points where $g(x)$ has the value a $(m \leq a \leq M)$ must be identical with the entire interval $l \leq x \leq L$ and also that the set of values assumed by $g(x)$ over the set of points where $f(x) = a'$ $(l \leq a' \leq L)$ must be identical with the entire interval $m \leq y \leq M$ where l, L and m, M are the exact bounds of $f(x)$ and $g(x)$ in $[0, 1]$.

In the latter paper Sengupta proved the following two theorems.

Theorem 1. Two functions $f(x)$ and $g(x)$ continuous and semi-independent in $[0, 1]$ and not constant in it, furnish a continuous representation on a rectangular area of the unit linear interval.

Theorem 2. Any continuous representation of a rectangle on a linear interval defines a pair of continuous semi-independent functions on the interval.

on the t-axis where $\overline{\overline{A_1}} = c$, $\overline{\overline{A_2}} = c$ and $A_1 \cap A_2 = \emptyset$. Also as 'a' moves on $0 \leq a \leq 1$ on the $y^{(t)}$ -axis we get disjoint sets $\{A\}$'s on the t-axis where

$$U_a A_{a \varepsilon [0,1]} = [0,1] \quad (3)$$

$$\therefore \overline{\overline{U_a A_a}} = c \quad (4)$$

We can now use the theorem (Kuratowski [8] , p. 69-70) : let $\overline{\overline{T}} = n$ and let F be a function whose arguments run over the set T and whose values are disjoint sets of power m, that is $\overline{\overline{F_t}} = m$, $F_t \cap F_{t'} = \emptyset$ for $t \neq t'$,

then $\overline{\overline{U_t F_t}} = m.n.$ (5)

Taking $T \equiv [0,1]$ on the $y^{(t)}$-axis, we get $\overline{\overline{T}} = n = c$ also taking $F_t = \bar{y}^1$ we get $\overline{\overline{F_t}} = m = c$ (by the above lemma) and also $F_t \cap F_{t'} = \emptyset$ for $t \neq t'$ (t, t' are points on $y^{(t)}$ -axis in $[0,1]$) . Hence by (5) $\overline{\overline{U_t F_t}} = m.n = c.c = c^2$ or $c = c^2$ by (4). Hence the theorem.

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