CHAPTER VI

ON THIN SETS OF STRAIGHT LINES AND CIRCLES

INTRODUCTION

While dealing with some properties of the Cantor sets on various bases, Sengupta and Bose Majumder (1955) constructed in the unit square $Q\ [(0,0);(1,0);(1,1);(0,1)]$ an everywhere dense set A which is of first category having the power c, such that a straight line $y = mx + d$ (m and d rationals) while intersecting Q does not intersect A. Though they did not mention it, the set A that they constructed was of measure zero. In theorem 1 of this chapter we have sharpened this property.

In a recent paper Kinney (1968) has used Cantor set to construct a set K with plane Lebesgue measure zero, such that for every positive $\alpha \leqslant 1$, the set K contains a circle of diameter α. In theorem 2 of this chapter we have used the same Cantor set to construct a plane set K, having similar properties, but little more sharpened than those contained by Kinney's set.

<u>Theorem 1</u>. There exists a set $B \subset Q$ such that B is of plane measure 1 and is a residual set (therefore everywhere dense and of power c) such that it is possible

to draw countably infinite number of straight lines each intersecting Q but none meeting B.

Proof : Let R and S be the set of rationals and irrationals in $0 \leq x \leq 1$. We denote by $m_2(A)$ the plane Lebesgue measure of a plane set $A \subset Q$. Let us consider the plane sets A_1, A_2, A_3, A_4 (all subsets of Q) given by

$$A_1 \equiv \{(x,y): x \in R, y \in S\}$$

$$A_2 \equiv \{(x,y): x \in S, y \in R\}$$

$$A_3 \equiv \{(x,y): x \in R, y \in R\}$$

and

$$A_4 \equiv \{(x,y): x \in S, y \in S\}.$$

If A_x is the set of points $\{(x,y)\}$ in Q, where $y \in S$ on the line $X = x$, x being a fixed rational number of R, then by Fubini's theorem

$$m_2(A_1) = \int_R m_y(A_x)\, dm_x$$

where m_x and m_y are the Lebesgue linear measures on the axes of X and Y respectively. Hence,

$$m_2(A_1) = \int_R 1.\, dm_x = m_x(R) = 0.$$

Similarly,

$$m_2(A_2) = 0 \quad \text{and} \quad m_2(A_3) = 0 .$$

Also

$$m_2(A_4) = \int_S m_y(A_x)\, dm_x$$

where A_x is the set of points $\{(x,y)\}$ in Q where $y \varepsilon S$ on the line $X = x$, x being a fixed irrational number of S.

Hence

$$m_2(A_4) = \int_S 1.\, dm_x = m_x(S) = 1.$$

Let $y = mx + d$ be any straight line (m and d being rationals and $0 \leq d \leq 1$) . Let $x = i$ be any irrational on the X-axis ($0 < i < 1$) . Then the line $X = i$ will cut the line $y = mx + d$ at a point (i, j) where j is also irrational. Now if we keep d fixed and allow m to vary (taking only rational values) then the set of such straight lines $y = mx + d$ will be enumerable. Thus the set A_i of points $\{(i, j)\}$ on the line $X = i$ intersecting all possible lines $y = mx + d$ with d fixed and m variable (m and d both rationals) will be enumerable.

If A_5 is the set of points $\{(\xi, \eta)\} \subset Q$ with both ξ and η irrationals and lying on the straight lines $y = mx + d$ with m and d both rationals $(0 \leq d \leq 1)$

then

$$m_2(A_5) = \int_S m_y(A_x)\, dm_x$$

where x is any irrational on S (on the X-axis in the unit interval),

$$\text{or} \qquad m_2(A_5) = \int_S 0 . dm_x = 0 . m_x(S) = 0.1 = 0.$$

[This result could also be obtained otherwise, as the set $\{c\}$ is enumerable]. Hence, since $m_2(A_4) = 1$, the set A_5 cannot be identical with A_4.

This indicates that there are infinitely many points $(\xi, \eta) \in Q$ (with power greater than $\aleph_0$; we shall show below that the power is c), where ξ, η are both irrationals such that no straight line $y = mx + d$ can be drawn through them, with m and d both rationals $(0 \leq d \leq 1)$.

Let B be the set of points $\{(\xi, \eta)\} \subset Q$ when both the coordinates in each point are irrationals and no point lying on a straight line $y = mx + d$, m and d both rationals with $0 \leq d \leq 1$.

As A_1, A_2, A_3, A_5 and B are disjoint and

$$Q = A_1 + A_2 + A_3 + A_5 + B$$

$$\therefore\ m_2(Q) = m_2(A_1) + m_2(A_2) + m_2(A_3) + m_2(A_5) + m_2(B)$$

$$\text{or} \quad 1 = 0 + 0 + 0 + 0 + m_2(B)$$

$$\therefore\ m_2(B) = 1.$$

Now with rational d fixed on $0 \leqslant y \leqslant 1$ the set of points $\ell \cap Q$ is non-dense in Q, where ℓ is the set of points on $y = mx + d$ (m any rational number).

Hence

$$L_d = \bigcup_m (\ell \cap Q)$$

is a set of first category, when the point set union is taken over all rational $d\,(0 \leqslant d \leqslant 1)$. Since A_1, A_2, L are each of first category it follows that $L \cup A_1 \cup A_2$ is also of first category.

Hence

$$B = Q - (L \cup A_1 \cup A_2)$$

is a residual set. This set B satisfies the requirements of the theorem.

Theorem 2. There exists a plane set K with plane Lebesgue measure zero, such that, for every positive α_i belonging to a set of countably infinite number of α_n's

$$\{\alpha_n\}_{n=1}^{\infty}, \quad 0 < \alpha_n < 1,$$

the set K contains continuum number of circles each of diameter α_i.

<u>Proof</u> : As mentioned in the introduction, Kinney's (1968) set K is constructed as follows : on $\{0 \leq x \leq 1, y=0\}$ Cantor middle third set is considered. Now by the well-known Steinhaus theorem (1917), we know that the distance set of the Cantor set fills the unit interval $0 \leq \alpha \leq 1$. For each $\alpha \varepsilon [0, 1]$ the left most pair $(x_1(\alpha), x_2(\alpha))$ of the set of Cantor points for which $x_2(\alpha) - x_1(\alpha) = \alpha$, is selected. Then a circle $C(\alpha)$ with $\{(x,y) \mid x \varepsilon [x_1(\alpha), x_2(\alpha)], y=0\}$ as diameter is drawn. Finally the plane set

$$K = \bigcup_{0 \leq \alpha \leq 1} C(\alpha)$$

is considered. It has been shown by Kinney that the two-dimensional Lebesgue measure $m_2(K)$ of K is zero. Since $C(\alpha) \subset K$ it follows that, for every positive $\alpha \leq 1$ the set K contains a circle of diameter α , even though $m_2(K) = 0$.

Now Kinney has taken only one pair of Cantor points corresponding to each $\alpha \varepsilon [0, 1]$ on which the circle $C(\alpha)$ is drawn. It follows, Kinney's set K contains continuum number of circles.

We know that (Boas, 1962; Bose Majumder, 1965) for each of almost all $\alpha \varepsilon [0, 1]$ the power of the set

$$E_\alpha = \{(x_1(\alpha), x_2(\alpha))\}$$

of pairs of Cantor points, is c that of the continuum, where

$$x_2(\alpha) - x_1(\alpha) = \alpha.$$

Now we take such an α (which equals the difference of continuum number of pairs of Cantor points), and consider each pair $(x_1(\alpha), x_2(\alpha)) \in E_\alpha$ [where $x_2(\alpha) - x_1(\alpha) = \alpha$]. Then we draw circles $C(\alpha)$ with $\{(x,y) \mid x \in [x_1(\alpha), x_2(\alpha)], y = 0\}$ as diameter for each such pairs of points of E_α and then define the set K consisting of all these continuum number of circles given by

$$K_\alpha = \acute{U} \; C(\alpha) . \qquad (1)$$

Then we prove a lemma.

<u>Lemma</u> : $m_2(K_\alpha) = 0$, where $m_2(\;)$ is the two-dimensional measure in the Lebesgue sense.

<u>Proof</u> : Let us take any circle of $K(\alpha)$, say the circle $C(\alpha) \in K(\alpha)$, on AB as diameter, where $AB = \alpha$ and $A(x_1(\alpha), 0)$ and $B(x_2(\alpha), 0)$ are points of C satisfying $x_2(\alpha) - x_1(\alpha) = \alpha$ [i.e. $(x_1(\alpha), x_2(\alpha)) \in E_\alpha$].

Let us take any straight line

$$L(d) : \; y = d$$

intersecting $C(\alpha)$ at A', B' (A', B' may coincide). The coordinates of A' and B' are respectively

$$\left(x_1(\alpha) + \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4d^2}}{2}, d\right) \text{ and } \left(x_2(\alpha) - \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4d^2}}{2}, d\right).$$

with α and d fixed, A' and B' are obtained by translating the Cantor points [situated on $L(d)$ in $(0, d)$ and $(1, d)$] $(x_1(\alpha), d)$ and $(x_2(\alpha), d)$ respectively by $\frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4d^2}}{2}$ and $-\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4d^2}}{2}$ along $L(d)$. Hence, when we consider all members of $K(\alpha)$, the linear measures of the sets $\{A'\}$ and $\{B'\}$ are each zero, for these sets are obtained by translating subsets of the Cantor set along $L(d)$.

It follows that

$$m_x\{L(d) \cap K_\alpha\} = 0$$

where $m_x(\ )$ is the one-dimensional measure in the Lebesgue sense in the direction of x-axis.

Hence by Fubini's theorem

$$m_2(K_\alpha) = 2\ (L)\int_0^{\alpha/2} m_x\{L(d) \cap K_\alpha\}\, dm_y ,$$

where $m_y(\ )$ is the one-dimensional Lebesgue measure in the direction of y-axis.

Thus

$$m_2(K_\alpha) = 2 \times 0 \times \frac{\alpha}{2} = 0.$$

Hence the lemma.

Proof of the Theorem. We now choose a sequence

$$\{\alpha_n\}_{n=1}^{\infty}, \qquad 0 < \alpha_n < 1,$$

such that each α_n of this sequence is the difference of continuum number of Cantor points $(x_1(\alpha_n), x_2(\alpha_n))$ and by the previous lemma we get

$$m_2(K_{\alpha_n}) = 0, \qquad n = 1, 2, 3, \cdots$$

where

$$K_{\alpha_n} = \bigcup C(\alpha_n) \qquad \text{as in } (1).$$

Let us consider the set

$$K = \bigcup_{n=1}^{\infty} K_{\alpha_n}.$$

Hence

$$m_2(K) \leqslant \sum_{n=1}^{\infty} m_2(K_{\alpha_n}) = 0$$

and thus

$$m_2(K) = 0.$$

We have thus shown the existence of a plane set K with plane Lebesgue measure zero, such that for every positive α_i

belonging to a set of countably infinite number of α_n's : $\{\alpha_n\}_{n=1}^{\infty}$, $0 < \alpha_n < 1$, the set K contains continuum number of circles each of diameter α_i .

Note 1. The set K constructed by Kinney, consists of continuum number c of circles; the set K that we have constructed also contains $\aleph_0 c$ $(= c)$ number of circles (Kuratowski, 1961).

Note 2. For every $\alpha \varepsilon [0,1]$ there exists either one pair of Cantor points $(x_1(\alpha), x_2(\alpha))$ or exist a finite number n pairs $(x_1^{(1)}(\alpha), x_2^{(1)}(\alpha))$, $(x_1^{(2)}(\alpha), x_2^{(2)}(\alpha)), \cdots \cdots, (x_1^{(n)}(\alpha), x_2^{(n)}(\alpha))$ of Cantor points or there exist continuum number of pairs $\{(x_1(\alpha), x_2(\alpha))\}$ of Cantor points such that $x_2(\alpha) - x_1(\alpha) = \alpha$ for each such pair. Kinney has considered all $\alpha \varepsilon [0,1]$ and has taken in each case (i.e. for each α) the left most pair on which he has drawn a circle. He is thus assuming that each such set of pairs (corresponding to a chosen α) is well-ordered (Natanson, Vol. II, 1955).

We have considered above only those α's , which are the difference of continuum number of pairs of Cantor points. If we assume the continuum hypothesis (Natanson, Vol.II, 1955) there exists continuum number of such α's . For each such α , we can construct a set of continuum number of circles as shown in the theorem 2 of this chapter.

Now, corresponding to a given α of the aforesaid type, instead of choosing the left-most circle (corresponding to the left-most pair of Cantor points, as indicated by Kinney), if we choose any circle [we now apply axiom of choice - Natanson, Vol. II, 1955] we can construct c^c [$= 2^c > c$, Kuratowski, 1961] number of sets of circles each of power c and each of the type of the set K constructed by Kinney (with the property: there exists a set K with plane measure zero such that for almost every positive $\alpha \leqslant 1$, the set K contains a circle of diameter α). [In Kinney's original theorem, he had here 'for every positive $\alpha \leqslant 1$'.]

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