CHAPTER V

CANTOR PROPERTY OF A CLASS OF SETS

I

INTRODUCTION

In 1962 Bose Majumder (Bose Majumder, 1962b) defined the terms, an SD- set and a set having properties σ, μ and δ. In the following we shall in general consider subsets of E $(0 \leq x \leq 1)$. We call a set $S \subset E$ symmetrical (i.e., S possesses the property σ) if for every $x \in S$, $1-x$ also belongs to S (taking 1 to be the diameter of S). If every point in $0 < x < 1$ is the mid point of at least one pair of points of S, then we say that S has the property μ. If every number x of $0 < x < 1$ is equal to the difference of at least one pair of points of S, then we say that S is an SD- set (i.e. S possesses the property δ). If a set S has all the three properties σ, μ and δ then S is said to possess the Cantor property π.

It has been shown (Bose Majumder, March, 1962a) that if a linear set $S \subset E$ has the properties σ and μ then it possesses the property δ also (i.e., S possesses the Cantor property π); if S has the properties σ and δ then it must possess the property μ also. But it has been shown that if S possesses the properties μ and δ then S may or may not possess the property σ .

Now the set E has obviously all the properties σ , μ and δ . The set R of rationals in E has the property σ but it cannot obviously have the property μ (and hence δ), i.e., R does not possess the Cantor property π . Also the set Q of irrationals in E has obviously the properties σ and μ (and hence δ), i.e., Q possesses the Cantor property π . Thus we see that if we withdraw from E the set R (which is symmetrical, of measure zero and of first category) then the set Q $(\equiv E - R)$ still possesses the Cantor property π (which E originally possessed). We know that the Cantor middle third set C has the property π . If $\tilde{C}$ is the set complementary to the middle third set C , then $E - \tilde{C}$ $(= C)$ still possesses the Cantor property (where $\tilde{C}$ is symmetrical, of measure 1 and is a set of second category - a residual set).

It is thus seen that if from E we withdraw a symmetrical set F , does not matter what its measure is (it may even be a non-measurable set), what its category is, the set $E-F$ is still an SD-set, provided it can retain the property μ.

We now propose to prove the following theorems.

<u>Theorem 1</u>. The system Σ of sets, all subsets of E each of which has the property σ, is a ring.

<u>Proof</u> : If $A \varepsilon \Sigma$ and $B \varepsilon \Sigma$ then $A+B \varepsilon \Sigma$, since if $x \varepsilon A+B$, then assuming that $x \varepsilon A$ it follows that $1-x \varepsilon A$ as $A \varepsilon \Sigma$ and hence $1-x \varepsilon A+B$.

Then we show that $A-B \varepsilon \Sigma$

Let $x \varepsilon A-B$ then $x \varepsilon A$ and $x \bar{\varepsilon} B$. Hence $1-x \varepsilon A$ as A has the property σ. Also $1-x \bar{\varepsilon} B$ (since if $1-x \varepsilon B$ then $x \varepsilon B$ as B has the property σ). It follows that $1-x \varepsilon A-B$. Hence $A-B \varepsilon \Sigma$.

Thus $A \triangle B \; (\equiv (A-B) \cup (B-A)) \; \varepsilon \Sigma$ (1)

Again if $x \varepsilon A \cap B$ then $x \varepsilon A$ and $x \varepsilon B$.

Hence $1-x \varepsilon A$ and $1-x \varepsilon B$ and thus $1-x \varepsilon A \cap B$.

Therefore $A \cap B \varepsilon \Sigma$. (2)

From (1) and (2) it follows that Σ is a ring.

Corollary As $E\ (\varepsilon\ \Sigma)$ is obviously a unit of Σ, it follows that Σ is a Borel algebra.

Theorem 2. The system M of sets, all subsets of Σ E each of which has the property μ, is not a ring.

Proof : If possible, let M be a ring. Then $\Sigma \cap M$ is also a ring. Let $A \varepsilon \Sigma \cap M$, then A has the properties σ and μ and hence δ.

Thus all the symmetric SD- sets of E form a ring $S\ (= \Sigma \cap M)$. Now as $Q \varepsilon S$, therefore $E - Q = R\ \varepsilon\ S$ i.e., the rational set R (in E) forms an SD- set, which is absurd. Hence M cannot form a ring.

Theorem 3. The system D of sets, all subsets of E each of which has the property δ, is not a ring.

Proof is similar to one given in Theorem 2.

Note : It is obvious that M or D do not even form a semiring.

II

The distance (a real number) between two sets has been defined in various books of real analysis (Copson, 1968). The distance (a real number) between two sets of a ring has been defined in a different way (Kolmogorov and Fomin, 1961) to make the ring a metric space. In the following we define the distance (which is not a real number, but is a set) between two sets in a manner such that a class of sets (of points) becomes a metric space in a new sense of the term.

Metrizing a Class of Sets : Let us take a class S of sets. If A and B are two elements of S, then we know that the symmetric difference between A and B represented symbolically as $A \triangle B$ is given by the relation $A \triangle B = (A-B) \cup (B-A)$.

Definition : We define the distance $\rho(A,B)$ between two elements A and B of the class S by the relation $\rho(A,B) = A \triangle B$.

Properties of ρ with Reference to the Class S

We note that ρ has the following properties :

(i) $\rho(A,B)$ is a set of points (may be an empty set $\emptyset$) for every pair of elements of S and

$$\rho(A,B) = \rho(B,A).$$

(ii) $\rho(A,B) \supseteq \emptyset$ for $A \in S$, $B \in S$ and $\rho(A,B) = \emptyset$ if and only if $A = B$ (Proof is obvious).

(iii) $\rho(A, B) \subseteq \rho(A, C) \cup \rho(C, B)$ where A, B, C are any three elements of S.

Proof : Let $x \in \rho(A, B)$. To be definite let $x \in A$ and $x \bar{\in} B$. Then there are two possibilities, either $x \in C$ or $x \bar{\in} C$.

If $x \in C$ then noting that $x \bar{\in} B$ it follows that $x \in \rho(C, B)$ and thus $x \in \rho(A, C) \cup \rho(C, B)$ and the inequality in this case is proved.

Again, if we take $x \bar{\in} C$ then noting that $x \in A$ it follows that $x \in \rho(A, C)$ and hence $x \in \rho(A, C) \cup \rho(C, B)$. The inequality in (iii) is thus completely proved.

Definition : In view of (i), (ii) and (iii) we shall call the set-valued function $\rho(A, B)$, a metric on S.

Note : If X is any non-empty set of points and S is the class of all subsets of X then $\rho(A, B)$ is a mapping of $S \times S$ into S. It will be shown later (Theorem 4) that this mapping is 'onto'.

As in the case of a bounded set of real numbers (where we defined an SD-set), we shall define an SD-class of sets as follows :

Definition: Let S be a class of sets with a unit X. If D be any subset of X and if it is possible to find two disjoint elements A and B of S such that $\rho(A, B) = D$ then we shall call the class S an SD-class.

If A be any member of S and if $\rho(X, A)$ $(= X - A)$ is also an element of S then we shall say that the class S has the property σ.

If D is any subset of X and if it is possible to find two disjoint elements A and B of S such that $(D \cup D =) D = A \cup B$ then we say that the class S has the property μ.

If D be any subset of X and if it is possible to find two disjoint elements A and B of S such that $D = \rho(A, B)$ then we say that the class S has the property δ. In this case S is an SD-set i.e. set of sets.

The class S will be said to possess the Cantor property if it possesses all the three properties σ, μ and δ.

Note : If E is a countable class of sets containing $\emptyset$ then the set $D = \{\rho(A, B)\}$ of distances between sets A and B of E is also a countable subclass of the countable class $\mathcal{R}(E)$ generated by E (Halmos, 1964).

Theorem 4. If X is any set of points and S is the class of all subsets of X then S possesses Cantor property and is thus an SD- class.

Proof : If $A \varepsilon S$ then $X - A \varepsilon S$. Hence S has the property σ.

If D is any subset of X then D is an element of S . We can obviously decompose D into two disjoint subsets A and B . As $A \varepsilon S$, $B \varepsilon S$ $A \cap B = \emptyset$ it is obvious that S has the property μ.

Let D be any subset of X (and thus $D \varepsilon S$) and let $D = A_1 \cup B_1$ where $A_1 \cap B_1 = \emptyset$ and hence $D = f(A_1, B_1)$. Hence S has the property δ and the theorem is completely proved.

Note : The similarity of the facts that the entire space (i.e., the segment $[0,1]$) is an SD- set and the class S of all subsets of a set X is an SD- class, is to be noted.

Theorem 5. A class S of sets (with unit X) having the properties σ , μ must necessarily possess the property δ .

<u>Proof</u> : As S has the property μ, if $D(\neq X, \emptyset)$ is any subset of X, then $D = A \cup B$ where $A \in S$, $B \in S$, $A \cap B = \emptyset$. $\therefore D = f(A, B)$. If $D = X$ then if $A \in S$ then the complementary set to A viz., $\complement A \in S$.

Hence $D = X = A \cup \complement A = f(A, \complement A)$. If $D = \emptyset$ then $\emptyset = f(\emptyset, \emptyset)$ as $\emptyset \cap \emptyset = \emptyset$. Hence S has the property δ.

<u>Corollary</u> If S has the properties σ and μ, S coincides with the class of all subsets of X.

<u>Theorem 6.</u> A class S of sets (with unit X) having the properties σ and δ must possess the property μ.

Proof is similar to one given in Theorem 5.

<u>Theorem 7.</u> A necessary and sufficient condition that an algebra $\mathcal{R}$ of sets (with unit X) is an SD-class is that it should coincide with the class of all subsets of X.

<u>Proof.</u> Sufficiency is obvious.

The necessity follows from the fact that if D is any subset of X and $D = A \cup B$, $A \cap B = \emptyset$ or $D = f(A, B)$ where $A \in \mathcal{R}$, $B \in \mathcal{R}$ (where $\mathcal{R}$ is the given algebra of sets), then $D \in \mathcal{R}$ (by the definition of a ring).

<u>Note</u> : It is now evident that if an algebra of sets is an SD- class, then it is a B-algebra and also a monotone

class (a non-empty class M of sets is monotone if for every monotone sequence $\{E_n\}$ of sets in M we have $\lim_n E_n \in M$) (Halmos, 1964). Also a monotone ring with a unit is an SD-class as a monotone ring is a σ-ring.

<u>Theorem 8.</u> If L is a lattice of sets, then the set D of its distances [i.e., $D = \{\rho(A, B)\}$, $A \in L, B \in L$] contains a semi-ring.

<u>Proof</u> : We know that (Halmos, 1964, p.25) $\emptyset \in L$ and if $E \in L$, $F \in L$ then $E \cup F \subset L$, $E \cap F \subset L$ and if $P = P(L)$ be the class of all sets of the form $F - E$ where $E \in L$, $F \in L$ and $E \subset F$ then P is a semiring.

Now $\rho(E, F) = F \triangle E = F - E$ therefore $(F - E) \in D$.

As $F - E$ is any element of P, hence $P \subset D$.

Thus the theorem is proved.

<u>Theorem 9.</u> If the lattice L of sets has the property σ, then $P(L)$ is an algebra.

<u>Proof</u> : As L has the property σ and $\emptyset \in L$ therefore $X \in L$ where X is the unit of L. Also if $E \subset L$ then $X - E$ belongs to L i.e., $\complement E \in L$. Thus if $E \in L$, $F \in L$ and $E \subset F$ then by definition $F - E \in P(L)$ i.e. $F \cap \complement E \in P(L)$. But $F \in L$ and $\complement E \in L$ therefore $F \cap \complement E \in L$. Therefore,

$\mathcal{P}(L) \subseteq L$. Also if $A \varepsilon L$ then $A - \emptyset \varepsilon L$. But $A = A - \emptyset \varepsilon \mathcal{P}(L)$. Therefore $L \subseteq \mathcal{P}(L)$.

Also $\complement E$, $\complement F$ belong to L (as L has the property σ). Thus

$$E - F = E \cap \complement F \varepsilon L \quad \text{and} \quad F - E = F \cap \complement E \varepsilon L$$

$$\therefore E \cap \complement F \cup F \cap \complement E \varepsilon L.$$

Thus $E \Delta F \varepsilon L$. Hence L is a ring with a unit X. Therefore $L (= \mathcal{P}(L))$ is an algebra.

REFERENCES

Bose Majumder, N. C. - A study of certain properties of the Cantor set and of an (SD) set, Bull. Cal. Math. Soc., 54, 1 (1962a), p.8.

Bose Majumder, N. C. - On the construction of some SD-sets, Bull. Cal. Math. Soc., 54, 3 (1962b), p.163.

Copson, E. T. - Metric spaces, Cambridge University Press, (1968), p.23.

Halmos, P. R. - Measure theory, D. Van Nostrand Co. Inc., (East-West Edition), (1964), p.23 and 25.

Kolmogorov and Fomin - Elements of theory of functions and Functional analysis, Vol.II, Graylock Press, (1961), p.47.