CHAPTER II

STUDY OF THE SYSTEM OF AUTOMATIC CONTROL
WITH LIMITS IN THE OUTPUT AMPLITUDE
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2.1 Necessity of considering limits in the system under investigation.

The cutting process carried out in a vertical milling machine may be simple or complex. During simple cutting process the width and depth of the cut remains constant and the load on the instrument does not change abruptly. In the complex regime of operation, where the depth or width of cut changes abruptly, sudden change of load on the instrument takes place, and consequently, on the shaft of the driving motor. For example, when the instrument, for a certain time, moves along a relatively wider channel and then has to move along a relatively narrower channel a collision phenomenon between the metal and the instrument takes place. Collision phenomenon, may further take place under the following conditions: when each tooth of the cutting tool is cutting into the metal; due to sudden change of thickness of the casting or the semifinished products; during the movement of the stepper
motor just after the cutting process is over i.e., when the load from the instrument is suddenly withdrawn; when the table meets a rigid obstacle, during vibratory operation of the milling machine and so on. These types of processes are often met with in practice.

In all these cases there may be vibration of the table, stepper motor or the cutting tool. They are considered undesirable situations. But the presence of the undesirable situations depends on the values of such parameters as $C$, $J$, $B$. In case it is present, it may be either an elastic collision or an inelastic collision.

All this results in deterioration of quality of the finished product, specially in case of elastic collision, where presence of vibration is highly probable, consequently it necessitates verification of some of the parameters of the system under investigation. Parameters of the system must be chosen in such a way that there is a good damping to the vibration. With this object the performance of the system is investigated with limits in the output amplitude. A knowledge of the two extreme cases is very useful for predicting the behaviour of the system under actual operating conditions. The two extreme cases are: Inelastic collision and Elastic collision. Actual operating condition will always be between these two limiting cases.
Theory of operation of the system of automatic control for collision condition is not yet fully worked out. The question of obtaining the amplitude vs. phase characteristic of the kinematic system, containing elastic link, viscous friction, inertial element and limit for displacement is examined below. The block diagram for such a system is shown in figure 2.1.

Investigation of the system is carried out with viscous friction only, although in a real system both dry friction and viscous friction may be present. But it is true, that in a number of cases the dry friction or damping coefficient may be replaced by its equivalent viscous friction. Here it is assumed that equivalence remains unaffected by the change in operating speed of the mechanism. However, such assumption may not always be justified. For example, with increase in frequency of input impulses to a stepper motor the motion of the mechanism becomes smoother and the sign of torque due to dry friction does not change or changes rather rarely compared to the low frequency operation. As a result, coefficient of equivalent damping depends not only on the amplitude of dry friction but also on the friction of input impulses. This dependence has been investigated in Moscow Power Institute (34) by Molonov by analogue computer simulation and comparing the motions of stepper motor with dry friction to its motion with viscous friction - proportional to speed. Curves
FIG 2.1 BLOCK DIAGRAM OF A SYSTEM WITH ELASTIC LINK (EL), INERTIAL ELEMENT (IE), VISCOUS FRICTION (VF), AND LIMITING ELEMENTING (LE).

FIG 2.2 PLOT OF INPUT VS OUTPUT AT LOW FREQUENCY

FIG 2.3 PLOT OF FIRST HARMONIC AMPLITUDE AT LOW FREQUENCY
of equivalent viscous friction vs. frequency of input impulses for different values of dry friction have been constructed. The relative values of dry friction have been taken as 0.05; 0.1, 0.15, 0.2 \( T_{\text{max}} \). These values cover the range of dry friction which may be present on the stepper-motor shaft, acting as the drive for the vertical milling machine.

For obtaining the amplitude-phase characteristics of the combination mentioned above, the method of harmonic linearisation has been applied.

2.2 Investigation under inelastic collision.

The differential equation of the system may be written in the following form:

\[
J \ddot{\theta}_2 + B \dot{\theta}_2 = C(\theta_2 - \theta_1)
\]

or \( \ldots(2.1) \)

and \( \theta_2 = \text{Const.}, \quad \dot{\theta}_2 = 0 \quad \text{for} \ |\theta_2| = \theta_0 \)

Here, \( \theta_0 \) amplitude of the limit on the output angle,
\( J \) moment of inertia of the inertial element,
\( B \) coefficient of viscous damping,
\( C \) coefficient of elasticity of the elastic link.

Assuming a harmonic signal \( M \sin \omega t \) applied at the input to the system the mutual relationship between the input \( (\theta_1) \) and output \( (\theta_2) \) of the system at low operating frequencies may be illustrated through figure 2.2.
Assuming that harmonic linearisation technique can be applied to the system \( \theta_2 \) may be expressed by the following equation:

\[
\theta_2(t) = A_0 + A_1 \sin \omega t + B_1 \cos \omega t \quad \ldots (2.2)
\]

neglecting higher order terms of the Fourier Series.

Here,

\[
A_0 = \frac{1}{2\pi} \int_0^{2\pi} \theta_2(t) \sin \omega t \, d(\omega t)
\]

\[
A_1 = \frac{1}{\pi} \int_0^{2\pi} \theta_2(t) \sin \omega t \, d(\omega t)
\]

\[
B_1 = \frac{1}{\pi} \int_0^{2\pi} \theta_2(t) \cos \omega t \, d(\omega t)
\]

For the system under investigation

\[
A_0 = 0
\]

\[
A_1 = \frac{2}{\pi} \left( \text{arc} \sin k + k \sqrt{1-k^2} \right) \quad \text{for } k < 1
\]

\[
= 1 \quad \text{for } k \geq 1
\]

\[
B_1 = 0
\]

\[
k = \theta_0 / M = \sin \theta_a
\]
Variation of $A_4$ against $K$ is illustrated in figure 2.3.

In the dynamic regime of operation, e.g., at relatively higher frequencies the output signal may or may not attain the limiting value even though the input amplitude is higher than the limiting value, and depending on this, the portion of the system of automatic control under investigation may behave as a non-linear or linear system. The amplitude and phase of the system in absence of the limit depend not only on the amplitude of the input signal but on its frequency also. Naturally, it is expected that in presence of limit the amplitude and phase of the output signal will depend on the following three quantities: amplitude and frequency of the input signal, and the amplitude of the limit. Two modes of operation are obtained - Non-linear and linear. The first mode may be illustrated through figure 2.4. Here $\theta_2(t)/M$ attains unity at or before $\omega t = \pi$. When this condition is not satisfied the transient process becomes as shown in figure 2.5. Equation (2.1) is valid for the first mode and the solution of this equation is given in equation (2.5).

For the portion $a - b - c$:

$$\frac{\theta_2(t)}{M} = G \sin(\omega t - \Theta_1 + \psi) + e^{-\frac{\gamma}{\omega_1 t}} \left( P \cos \omega_2 t + Q \sin \omega_2 t \right)$$

...(2.5)

for the portion $c - d$ : $\frac{\theta_2(t)}{M} = 1.$
FIG 2.4 & 2.5 ILLUSTRATING DIFFERENT MODES OF NON-LINEAR OPERATION FOR INELASTIC COLLISION
Here, \( \omega_n = \sqrt{c/J} \), \( \gamma = B/2\sqrt{cJ} \)

\[
\omega_2 = \omega_n \sqrt{1 - \gamma^2}
\]

\[\phi = \arctan \left[ \frac{2 \gamma \omega_n}{(\omega_n^2 - \omega^2)} \right]\]

\[
\Delta = (\omega_n^2 - \omega^2)^2 + (2 \gamma \omega_n)^2
\]

\[
G = \frac{\omega_n^2}{\sqrt{\Delta}}
\]

\[
P = G \sin(\theta_a + \phi) - \phi_0/M
\]

\[
Q = \frac{g}{\sqrt{1 - \gamma^2}} P - G \frac{\omega}{\omega_n \sqrt{1 - \gamma^2}} \cos(\theta_a + \phi)
\]

\[
\theta_a = \arcsin(\phi_0/M)
\]

**Substituting (2.5) into (2.2)**

\[
A_4 = \frac{4}{T} (g_1 + g_2 + g_3 + g_4)
\]

\[
B_4 = \frac{4}{T} (b_1 + b_2 + b_3 + b_4)
\]

\[\ldots (2.7)\]
Where, \( T = \frac{2\pi}{\omega} \)

\[
\begin{align*}
\mathcal{G}_1 &= G \int_0^{t_c} \sin(\omega t - \theta_0 + \psi) \cdot \sin \omega t \, dt \\
&= G \left[ \frac{\cos \phi (t_c - \frac{\sin 2\theta_c}{2\omega}) + \frac{\sin \phi (\cos 2\theta_c - 1)}{4\omega}}{4} \right] \\
\mathcal{G}_2 &= P \int_0^{t_c} e^{-\gamma \omega t} \cdot \cos \omega t \cdot \sin \omega t \, dt \\
&= \frac{P}{2} \frac{1}{\omega_c^2 + (\gamma \omega)^2} \left[ \left( \omega_3 \cos \omega_3 t_c - \gamma \omega_3 \sin \omega_3 t_c \right) e^{-\gamma \omega t} - \omega_3 \right] \\
&+ \frac{P}{2} \frac{1}{\omega_4^2 + (\gamma \omega)^2} \left[ \left( \omega_4 \cos \omega_4 t_c - \gamma \omega_4 \sin \omega_4 t_c \right) e^{-\gamma \omega t} - \omega_4 \right] \\
\mathcal{G}_3 &= Q \int_0^{t_c} e^{-\gamma \omega t} \cdot \sin \omega t \cdot \sin \omega t \, dt \\
&= -\frac{Q}{2} \frac{1}{\omega_3^2 + (\gamma \omega)^2} \left[ \left( \omega_3 \sin \omega_3 t_c - \gamma \omega_3 \cos \omega_3 t_c \right) e^{-\gamma \omega t} + \omega_3 \right] \\
&+ \frac{Q}{2} \frac{1}{\omega_4^2 + (\gamma \omega)^2} \left[ \left( \omega_4 \sin \omega_4 t_c - \gamma \omega_4 \cos \omega_4 t_c \right) e^{-\gamma \omega t} + \omega_4 \right] \\
\mathcal{G}_4 &= \sin \theta_a \int_{t_c}^{T/2} \sin \omega t \, dt \\
&= \sin \theta_a \left( 1 + \cos \theta_c \right) / \omega
\end{align*}
\]
\[ b_1 = G \int_0^t \sin(\omega t - \phi) \cdot \cos \omega t \cdot dt \]
\[ = G \left[ \frac{\sin \phi}{2} \left( t - \frac{\sin 2\theta_e}{2\omega} \right) + \frac{\cos \phi}{4\omega} (\cos 2\theta_e - 1) \right] \]
\[ b_2 = \int_0^t e^{-\gamma \omega t} \cdot \cos \omega t \cdot \cos \omega t \cdot dt \]
\[ = \frac{p}{2} \frac{1}{\omega_3^2 + (\gamma \omega)^2} \left[ (\omega_3 \sin \omega_3 t - \gamma \omega \cos \omega_3 t) \cdot e^{-\gamma \omega t} \right] \]
\[ + \frac{p}{2} \frac{1}{\omega_4^2 + (\gamma \omega)^2} \left[ (\omega_4 \sin \omega_4 t - \gamma \omega \cos \omega_4 t) \cdot e^{-\gamma \omega t} \right] \]
\[ b_3 = \int_0^t e^{-\gamma \omega t} \cdot \sin \omega t \cdot \cos \omega t \cdot dt \]
\[ = \frac{Q}{2} \frac{1}{\omega_3^2 + (\gamma \omega)^2} \left[ (\omega_3 \cos \omega_3 t - \gamma \omega \sin \omega_3 t) \cdot e^{-\gamma \omega t} \right] \]
\[ - \frac{Q}{2} \frac{1}{\omega_4^2 + (\gamma \omega)^2} \left[ (\omega_4 \cos \omega_4 t - \gamma \omega \sin \omega_4 t) \cdot e^{-\gamma \omega t} \right] \]
\[ b_4 = \sin \theta_a \int_{t_e}^{T/2} \cos \omega t \cdot dt \]
\[ = -\sin \theta_a \sin \theta_e / \omega \]
where, \( \Phi = \theta_a + \psi \)
\[
\begin{align*}
\omega_3 &= \omega_n \sqrt{1 - q^2} + \omega = \omega_2 + \omega \\
\omega_4 &= \omega_2 - \omega
\end{align*}
\] ...
(2.10)

Hence, the amplitude and phase of the first harmonic are:
\[
\begin{align*}
A &= M \sqrt{A_1^2 + B_1^2} \\
\varphi &= \arctan \frac{A_1}{B_1} - \theta_a
\end{align*}
\] ...
(2.11)

When the system operates in the other non-linear region e.g., when (2.5) does not attain infinity before and the condition \( G > 1 \) is satisfied, the nature of variation of output may be portrayed as in figure 2.5.

Here, for the portion \( a-b-c \):
\[
\theta_2(t)/M = G \sin(\omega t - \theta'_a - \psi) + \left( P' \cos \omega_2 t + Q' \sin \omega_2 t \right) e^{-\frac{\omega_0 t}{2}}
\] ...
(2.12)

This equation must satisfy the following equality:
\[
\theta_2(t)/M = 1 \text{ at } \omega t = \pi
\] ...
(2.13)

As distinct from the previous case, now \( \theta'_a \) changes with frequency. This quantity may be positive or negative so as to satisfy equation (2.13).

Replacing \( \theta_a \) by \( \theta'_a \) in equation (2.6),
\[
\begin{align*}
P' &= G \sin(\theta'_a + \psi) - \theta_a / M \\
Q' &= \frac{y}{\sqrt{1 - q^2}} P' - G \frac{\omega}{\omega_2} \cos(\theta_a + \psi)
\end{align*}
\] ...
(2.14)
From equations (2.3) and (2.12):

\[ A' = \frac{4}{T} (q'_1 + q'_2 + q'_3) \]

\[ B'_1 = \frac{4}{T} (b'_1 + b'_2 + b'_3) \]

\[ g'_1 = G \int_0^{T/2} \sin(\omega t - \theta'_a - \psi) \cdot \sin \omega t \cdot dt \]

\[ = G \cos(\theta'_a + \psi) \cdot \frac{T}{4} \]

\[ g'_2 = P' \int_0^{T/2} e^{-\eta \omega t} \cdot \cos \omega t \cdot \sin \omega t \cdot dt \]

\[ = \frac{P'}{2} \frac{1}{\omega^3 + (\eta \omega)^2} \left[ \left( \omega_3 \cos \omega_3 T/2 - \eta \omega_3 \sin \omega_3 T/2 \right) e^{-\eta \omega_3 T/2} - \omega_3 \right] \]

\[ + \frac{P'}{2} \frac{1}{\omega_4^2 + (\eta \omega)^2} \left[ \left( \omega_4 \cos \omega_4 T/2 - \eta \omega_4 \sin \omega_4 T/2 \right) e^{-\eta \omega_4 T/2} - \omega_4 \right] \]

\[ g'_3 = Q' \int_0^{T/2} e^{-\eta \omega t} \cdot \sin \omega t \cdot \sin \omega t \cdot dt \]

\[ = -\frac{Q'}{2} \frac{1}{\omega_3^2 + (\eta \omega)^2} \left[ \left( \omega_3 \sin \omega_3 T/2 - \eta \omega_3 \cos \omega_3 T/2 \right) e^{-\eta \omega_3 T/2} + \omega_3 \right] \]

\[ + \frac{Q'}{2} \frac{1}{\omega_4^2 + (\eta \omega)^2} \left[ \left( \omega_4 \sin \omega_4 T/2 - \eta \omega_4 \cos \omega_4 T/2 \right) e^{-\eta \omega_4 T/2} + \omega_4 \right] \]
\[ b_1' = G \int_0^{T/2} \sin(\omega t - \theta_0 - \psi) \cos \omega t \, dt \]
\[ = -6 \sin(\theta_0 + \psi) \cdot T/4 \]

\[ b_2' = p' \int_0^{T/2} e^{-\gamma \omega n t} \cos \omega_2 t \cos \omega t \, dt \]
\[ = \frac{p'}{2} \frac{1}{\omega_3^2 + (\gamma \omega n)^2} \left[ (\omega_3 \sin \omega_3 T/2 - \gamma \omega n \cos \omega_3 T/2) e^{-\gamma \omega n T/2} + \gamma \omega n \right] \]
\[ + \frac{p'}{2} \frac{1}{\omega_4^2 + (\gamma \omega n)^2} \left[ (\omega_4 \sin \omega_4 T/2 - \gamma \omega n \cos \omega_4 T/2) e^{-\gamma \omega n T/2} + \gamma \omega n \right] \]
\[ = B \quad \text{(2.17)} \]

\[ b_3' = Q' \int_0^{T/2} e^{-\gamma \omega n t} \sin \omega_2 t \cos \omega t \, dt \]
\[ = \frac{Q'}{2} \frac{1}{\omega_3^2 + (\gamma \omega n)^2} \left[ (\omega_3 \cos \omega_3 T/2 - \gamma \omega n \sin \omega_3 T/2) e^{-\gamma \omega n T/2} - \omega_3 \right] \]
\[ - \frac{Q'}{2} \frac{1}{\omega_4^2 + (\gamma \omega n)^2} \left[ (\omega_4 \cos \omega_4 T/2 - \gamma \omega n \sin \omega_4 T/2) e^{-\gamma \omega n T/2} - \omega_4 \right] \]
In the non-linear region of operation, knowing to 
vide figure 2.4 , the values of terms on the left hand 
side of equations (2.8) and (2.9), and similarly, knowing 
$\theta_0$ (vide figure 2.5), - values of terms on the left hand 
side of equations (2.16) and (2.17) may be evaluated. 
Consequently, with the help of (2.7) and (2.11) or (2.15) 
and (2.11), as the case may be, the amplitude and phase 
of the output signal may be determined.

However, $t_c$ or $\theta_0$ are not known beforehand. They 
may be determined by trial and error method. This gives 
rise to difficulty in the theoretical determination of 
amplitude-phase characteristics of the system. Qualitative 
results and amplitude-phase characteristics of the given 
system were obtained by analogue computer simulation. 
More accurate results were obtained from digital computer 
MINSK-22. These results tally (within the permissible 
range of variation) with those obtained from the analogue 
computer MN-7. The flow diagram for the digital computer 
programme is given in figure 2.6.

When the system operates in the linear region, 
e.g., when the output quantity does not attain the 
limiting value, then the amplitude and phase of the output 
signal is easily obtained from the solution of the linear 
differential equation (2.1).
FIG 2.6 SEQUENCE OF CALCULATION BY THE COMPUTER TO GET THE FIRST HARMONIC IN CASE OF INELASTIC COLLISION
FIG 27 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK WITH VISCOUS FRICTION AND INERTIA WITH LIMIT ON THE OUTPUT AMPLITUDE, FOR INELASTIC COLLISION ($\xi = 1.0$)
FIG 2.8 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK WITH VISCOUS FRICTION AND INERTIA WITH LIMIT ON THE OUTPUT AMPLITUDE, FOR INELASTIC COLLISION ($\xi = 0.8$).
FIG 2.9 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK WITH VISCOSUS FRICTION AND INERTIA WITH LIMIT ON THE OUTPUT AMPLITUDE, FOR INELASTIC COLLISION ($\xi = 0.6$)
FIG. 2.10 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK WITH VISCOUS FRICTION AND INERTIA WITH LIMIT ON THE OUTPUT AMPLITUDE FOR INELASTIC COLLISION (\(\xi = 0.4\))
\[ A = M \frac{\omega_h^2}{\sqrt{(\omega_h^2 - \omega^2) + (2 \omega \omega_n)^2}} \]
\[ \varphi = \arctan \frac{2 \omega \omega_n}{(\omega_h^2 - \omega^2)} \]  \( \ldots \) (2.18)

From the results obtained from digital computer, the amplitude vs. phase plots are made and illustrated in figures 2.7 to 2.12.

2.3. Interpretation of computer results for the case of inelastic collision.

Amplitude-phase characteristics, obtained from computer results for the combination "elastic link plus viscous friction plus inertia plus element with limit" under inelastic collision are shown in figures 2.7 - 2.12. Here, inphase and quadrature components of first harmonic of the output signal are plotted along X and Y axes respectively. Each family of characteristics corresponds to a given value of relative damping, and the individual characteristic of a particular family corresponds to a particular value of the limiting amplitude. The amplitude-phase characteristics are represented by continuous lines, whereas, the dotted lines are the curves for constant frequency.

The above mentioned characteristics reveal the following features.
(1) Presence of inelastic stop, which reduces the output velocity to zero immediately after collision modifies the amplitude and phase of the first harmonic of the output signal if the amplitude of the output signal in absence of the inelastic stop had exceeded the limiting value, that is, the presence of the stop is perceived under such condition only.

(2) The system may operate in the non-linear region even though the amplitude of the input signal is less than the limiting value, provided the first condition is satisfied.

(3) Even at frequencies higher than the resonance frequency, the phase of the first harmonic of output signal may remain less than $90^\circ$, if the input amplitude is higher than the limiting amplitude. This is explained by the fact, that if the limiting had not existed, then the output would have reached its maximum value as determined from the linear relationship. However, in presence of the limit, upto a particular frequency, the output amplitude is limited, retains the limiting value for a certain period and then starts decreasing only when the input amplitude becomes less than the limiting value. This takes place at such a time when the input amplitude has not yet attained its zero value. Consequently, the peak of the input signal and peak of the first harmonic of output signal can not be separated by more than $90^\circ$. Therefore, under such
conditions the phase lag of the first harmonic of the output signal can not be more than 90°. Here, equation (2.5) describes the operation of the system.

(4) For $M > \theta_0$ as soon as the phase of first harmonic of the output signal exceeds 90° it changes abruptly. This phenomenon can be explained by the fact that if the disagreement of the output amplitude from the limiting amplitude starts after the point $d$ (vide figure 2.4), then the maximum amplitude of the output signal will not afterwards reach the limiting amplitude and hence violates the initial assumption. For frequencies at which the output amplitude (considering the limit stop to be non-existent) is greater than the limiting amplitude, this, however, does not apply. Therefore, the interval between the points of disagreement of the input and output amplitude from the limiting amplitude starts becoming wider and wider as a result of which, the abrupt change in the phase of first harmonic of output signal becomes a probability.

(5) In presence of limit stop, if $M > \theta_0$ the amplitude of first harmonic of the output signal does not change appreciably so long its phase does not reach 90°. However, at lower frequencies the amplitude of first harmonic is somewhat greater than the limiting amplitude, because under such conditions the output remains constant for the portion $c - d$ (figure 2.4). The presence of this
portion explains the higher value of the first harmonic of the output signal.

(6) For the reasons mentioned in paragraphs (3) and (4) the curves of constant frequency (shown by dotted lines) show a stiff curvature to the left. In other words, the amplitude-phase characteristics approach the characteristics obtained in absence of limit stop as the effectiveness of the limit stop becomes less and less.

On the basis of the above mentioned characteristics the describing functions of the system are obtained. They are helpful in determining the equivalent scheme (linearised) for the whole of the given system of automatic control. The method of applying these characteristics is discussed in chapter III for the purpose of obtaining control algorithms.

2.4. Investigation with elastic limit stop.

As mentioned earlier, a practical limit stop will be somewhat in between the perfectly elastic and perfectly inelastic stop. The nature of collision with the metal to be cut, as well as the subsequent motion of the system may depend on several factors, such as speed of the table before the tool meets the stop, hardness of the metal, elasticity of the cutting tool and the elastic link, and so on. Investigation of operation with elastic stop is useful from the point of view of understanding and explaining the effect
FIG 2.13 ILLUSTRATING DIFFERENT MODES OF NONLINEAR OPERATION FOR ELASTIC COLLISION
of a practical stop and the operation is examined with this object.

As distinct from inelastic stop, here the energy of the system is not dissipated immediately after collision. Assuming that the tool holder is immovable and both the objects taking part in the collision are elastic, the table will start moving backwards after the collision. With such assumptions the equations of motion may be written down in the following way:

\[ J \frac{d^2 \theta}{dt^2} + \beta \frac{d\theta}{dt} = C(\theta_1 - \theta_2) \text{ for } |\theta_2| < \theta_0 \]  

Immediately after the collision, the initial conditions for the motion following the collision are,

\[ \theta_{20} = \pm \theta_0 \]
\[ \dot{\theta}_{20} = -\dot{\theta}_{20} \]

These initial conditions will be valid for collision at any instant. The nature of variation of \( \dot{\theta}_2 \) for a sinusoidal input signal is shown in figures 2.13a and 2.13b. The difficulty of obtaining a theoretical solution to this problem lies in the fact that the value of \( \dot{\theta}_{20} \) is not known beforehand and that during any particular half cycle collisions may take place several times as illustrated in figure 2.13a, especially at lower frequencies.

For the above reason qualitative investigation was
FIG 2.14 SEQUENCE OF CALCULATION BY THE COMPUTER 
TO GET THE FIRST HARMONIC IN CASE OF 
ELASTIC COLLISION
FIG-2.15 AMPLITUDE - PHASE CHARACTERISTIC OF ELASTIC LINK, VISCOUS FRICTION AND INERTIA WITH LIMIT ON OUTPUT AMPLITUDE, FOR ELASTIC COLLISION ($\xi=0.9$)
FIG-2.16 AMPLITUDE - PHASE CHARACTERISTIC OF ELASTIC LINK, VISCOUS FRICTION AND INERTIA WITH LIMIT ON OUTPUT AMPLITUDE, FOR ELASTIC COLLISION ($\zeta=0.8$)
FIG. 2.17 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK, VISCOUS FRICTION AND INERTIA WITH LIMIT ON OUTPUT AMPLITUDE, FOR ELASTIC COLLISION ($\xi = 0.6$)
FIG. 2.18 AMPLITUDE-PHASE CHARACTERISTIC OF ELASTIC LINK, VISCOUS FRICTION AND INERTIA WITH LIMIT ON OUTPUT AMPLITUDE, FOR ELASTIC COLLISION ($\xi=0.4$)
of the stepper motor is slowly and gradually loaded by torque of gradually varying magnitude and direction so that the dynamical properties of the linkages do not appear and if angle of rotation of the shaft and corresponding moments are noted down then the static characteristic is obtained. This characteristic will pertain to the mechanism lying in between the stepper motor and the point of fixation. Hence, fixation of the table w.r.t. the body of the machine permits determination of the static characteristic pertaining to the portion including the motor shaft and the "lead-screw-Half-nuts", whereas fixation of the table by means of the cutting tool gives the static characteristic pertaining to the whole of mechanical linkage from motor shaft to the instrument (cutting tool). Curves for the first portion for transverse axis are shown in figure 1.3, for longitudinal axis - in figure 4, curves for the whole of mechanical linkage for transverse axis and longitudinal axes are shown in figs.1.5 and 1.6 respectively. They bear some resemblance to the hysteresis loop.

1.3(a). To obtain the block diagram through the method of identification for the programme controlled milling machine 6N12P.

The kinematic chain of the vertical milling machine consists of the following elements (vide figure 1.2):

1. Power stepper motor, SM,
2. Mechanical linkage "lead-Screw-half-nut",
3. Table,

The problem of optimum control in the sense of minimising the error/deviation can be examined only after obtaining the equations of motion of the system as a whole. To this end, right at the beginning it is necessary to identify the given system and the elements contained in it. Identification of the present system consists of two stages:

1. Obtaining the parameters influencing on the static as well as on the dynamics of the system,
2. Obtaining the parameters influencing only on the dynamics of the system.

Parameters of the first group may be obtained from the static characteristics of the system. Here, the term static characteristic means the functional relationship between the angular displacement of the motor shaft and the torque applied to it, under static conditions. The essence of the method lies in experimental determination of the static characteristics and comparing these with the characteristics of typical elements like dry friction, elastic link, air gap, etc. which appear equally under static and dynamic conditions. Afterwards, adding these elements to the elements appearing only under dynamic
FIG 1.7 GENERAL NATURE OF STATIC CHARACTERISTICS
condition the equivalent block diagram of the whole of mechanical linkage may be obtained.

The static characteristics of the system are more or less similar in form. Consequently, they may be replaced by a typical curve as shown in figure 1.7. This curve may be considered to consist of two parts. In figure 1.7 they are shown as unshaded and shaded portions. The former may be considered due to the combination of two elements - dry friction and linear elastic link whereas the latter portion appears due to the presence of airgap and non-linear friction i.e., "jamming". Experimental curves show that the effect of the last two elements is not high. But they make the solution of the problem quite involved. Therefore, as a first approximation they may not be taken into consideration.

Here, it is necessary to note that the values of dry friction and coefficient of elasticity are determined from figures 1.3 and 1.4 for the portion upto the table (but excluding it). In a similar way, dry friction and coefficient of elasticity for the whole system were obtained. Subtraction of the value of dry friction for the first portion from that of the second portion gives the dry friction in the table. Coefficient of elasticity of the cutting tool may be determined by the following relationship
FIG 18 BLOCK DIAGRAM OF THE AUTOMATIC CONTROL SYSTEM SM - STEPPER MOTOR, DF - DRY FRICTION, EL - ELASTIC LINK, C, C' - TABLE AND SLIDING BLOCK.

FIG 19 NORMALIZED LOGARITHMIC CHARACTERISTIC OF BACK LASH TYPE NON-LINEARITY.
where, 

\[ C_{eq} = \frac{C_\Phi \cdot C_{dc}}{C_\Phi + C_{dc}} \]  

...(1.1)

where,  \( C_{eq} \) - Coefficient of elasticity for the whole system of mechanical linkages,

\( C_{dc} \) - Coefficient of elasticity of the "lead screw - half nut",

\( C_\Phi \) - Coefficient of elasticity of the cutting tool.

Experimental results together with the foregoing discussion permits construction of the block diagram as shown in figure 1.8.

1.3(b). Frequency response method of identification.

The problem of identification may be solved by the frequency response approach. The general shape of the static characteristics ( \( \theta_d \) or \( \theta_c \) vs \( T_{syn} \)) has the appearance of that of a "back-lash". The parameters appearing only under dynamic condition do not have any effect on the static characteristics. Such parameters are inertia, viscous friction etc. Therefore, the static characteristics may be replaced by an inertialess non-linear element having no viscous friction. For such an element it is not difficult to determine the amplitude-phase
characteristics or logarithmic characteristics. Non-linear characteristic of the "back lash" is shown in figure 1.9. The describing function of the back lash type of nonlinearity for \( u = \frac{A}{b} \geq 1 \) is determined by the following relationship:

\[
q(u) = \frac{1}{K} \left[ \frac{\alpha}{2} + \arcsin \frac{\mu^2 - 2}{\mu} \right] + \frac{2(\mu - 2)}{\mu^2} \sqrt{\frac{1}{\mu}(1 - \frac{1}{\mu})} \]
\[
q'(u) = -\frac{4}{\mu} \frac{\mu - 1}{\mu^2}
\]

Here,

\( q(u), q'(u) \) - Real and imaginary components of the relative amplitude of first harmonic of the output signal.

The non-dimensional logarithmic characteristics may be expressed through the following equation:

\[
L_0(u) = 20 \log \sqrt{q^2(u) + [q'(u)]^2}
\]
\[
\varphi(u) = \arctan \frac{q'(u)}{q(u)}
\]

For small values of the amplitude of the input the effect of back lash on the system performance is quite appreciable whereas its effect decreases for higher values of input. In other words, for higher values of turning moment the effect of "back lash" (e.g., dry friction in the present case) reduces.
$w_1, w_3, \ldots, w_n$ - ratio of 1st, 3rd, ..., $n$th harmonic of output to the input.

**Figure I.10** Block diagram representation for the control system by separating out the nonlinearities.

**Figure I.11** Four phase stepper motor windings.
However, for lower values of input amplitude the effect of higher harmonics becomes pronounced. In such cases, it becomes necessary to consider the higher harmonics in the Fourier Series, so as to obtain more accurate results. Consequently, the block diagram will be of the form shown in figure 1.10.

Knowing \( w_1, w_3, \ldots, w_n \) the amplitude-phase characteristics of the whole system may be determined for each harmonic. The characteristics thus obtained may be utilised in the manner outlined in Chapter III for obtaining the control algorithm.

1.3(c). **Parameters of the System.**

1. For transverse axis.

(a) When table fixed w.r.t. machine body:

Mean value of coefficient of elasticity (of the elastic link)

\[ C_{dc} = 12.02 \text{ Nm}\text{/rad}, \]

Dry friction \( T_d \) 3.0 Nm.

(b) When tool is fixed w.r.t. to the job:

1) For tool-dia. of 28 mm: -

Equivalent coefficient of elasticity,

\[ C_{eq} = 9.05 \text{ to } 9.01 \text{ Nm}\text{/rad}. \]

Total dry friction = 2.3 to 3 Nm.
Coefficient of elasticity of the tool, transferred to the motor shaft, \( C_\Phi = 37 \text{ Nm/rad.} \)

(ii) For tool dia. of 20 mm.:
Equivalent coefficient of elasticity, \( C_{eq} = 4.9 - 5.5 \text{ N-m/rad.} \)
Total dry friction = 3 - 4.5 N-m.
Coefficient of elasticity of the tool \( C_\Phi = 9.2 \text{ N-m/rad.} \)

2. For longitudinal axis.

a) When table is fixed w.r.t. machine body:
Mean value of coefficient of elasticity,
\( C_{dc} = 25.6 \text{ N-m/rad.} \)
Dry friction = 1 - 2 N-m.

b) When tool is fixed w.r.t. the job:
For tool dia. of 28 mm.:
Equivalent coefficient of elasticity,
\( C_{eq} = 9.65 \text{ to } 18.1 \text{ N-m/rad.} \)
Total dry friction = 7.5 to 9.0 N-m.

Coefficient of elasticity referred to motor shaft of the tool, \( C_\Phi = 38.85 \text{ N-m/rad.} \)

For the stepper motor, coefficient of elasticity (when motor is considered to be equivalent to an elastic link)
\( C_d = 418 \text{ N-m/rad.} \)
Besides the above-mentioned elements, the block-diagram must contain elements, parameters of which entry into the dynamics of the system. Consequently, moment of inertia of the stepper motor, \( J_d \), coefficient of its internal damping, \( B \), moment of inertia of the table, \( J_c \), and coefficient of viscous damping have to be considered. All the parameters are referred to motor-shaft.

Following values are obtained from Molonov\(^3\)\(^4\) and name-plate ratings:

\[
J_d = 1.57 \times 10^{-3} \text{ Kg-m}^2; \\
B = 0.204 \text{ N-m/rad/sec}; \\
J_c = 0.242 \times 10^{-3} \text{ Kg-m}^2; \\
J_c' = 0.495 \times 10^{-3} \text{ Kg-m}^2.
\]

1.4. Mathematical description of the dynamics of the system.

The system under investigation consists of elements shown in block diagram of figure 1.8. The first element, the motor stepper, is an electromechanical energy conversion device.

In general, an \( m \)-phase stepper motor with constant excitation consists of \( (m + 1) \) electrical contours. To obtain the simplified model of a stepper motor the permanent magnets are replaced by an equivalent fictitious excitation winding\(^{11}\). Saturation of the magnetic circuit
is neglected. This simplifies understanding the operation of a stepper motor and explaining the electromagnetic coupling inside the motor. Saturation is quantitatively considered during magnetic circuit calculations. Coefficient of saturation, values of inductances and time constants thus obtained are used for further calculations and are considered to be independent of instantaneous values of currents in the windings of the motor. This assumption permits writing down the following equations for flux linkages in the windings as linear functions of currents with periodically varying coefficients, which depend only on the angular displacement of the rotor.

\[ \Psi_k(\theta, i) = L_{k1}(\theta)i_1 + L_{k2}(\theta)i_2 + \cdots + L_{kK}(\theta)i_K + L_{k,m+1}(\theta)i_{m+1} \]

\[ K = 1, 2, \ldots, m+1. \]

\( (1.4) \)

Torque developed is expressed through instantaneous values of currents, inductances of phase windings and excitation winding (real or fictitious).

\[ T_e = \frac{dW}{d\theta_{mech}} = b \frac{dW}{d\theta} = \frac{b}{2} \sum_{k=1}^{m+1} i_k \frac{d\Psi_k}{d\theta} \]

\[ = \frac{b}{2} \sum_{j=1}^{m+1} \sum_{k=1}^{m+1} i_j i_k \frac{dL_{jk}}{d\theta} \]

\( (1.5) \)

Winding currents and angular displacement appear
as the variables of the electromechanical system under investigation. For mathematical description of the system \((m + 1)\) equations of electrical equilibrium are necessary. These circuit equations are to be derived from the circuits containing the windings, commutating elements and the voltage sources combined with one equation for mechanical equilibrium of the drive. They can be written down in the following way:

\[
\begin{align*}
\dot{U}_K &= R_K i_K + \frac{d\psi_K}{dt} \\
J \frac{d^2\theta_{\text{mech}}}{dt^2} + T_L &= T_e
\end{align*}
\]

\((1.6)\)

where, \(K = 1, 2, \ldots, m\) - no. of electrical contours containing the phase windings of the S.M.;

\(K = m + 1\) - no. of the fictitious electrical contour, replacing the permanent magnet;

\(U_K, i_K, \psi_K\) - instantaneous values of voltage, current and flux linkage in the \(K\)th electrical contour;

\(R_K\) - electrical resistance of the \(K\)th contour.

\(J\) - resultant moment of inertia, referred to the stepper motor shaft;

\(T_L\) - resultant load torque, including the no-load losses;

\(T_e\) - turning moment of the motor;
\[ \theta_{\text{mech}} = \theta/p \] - mechanical angular displacement of the rotor.

Equation (1.6) may be rewritten in the following form:

\[ R_{ij} + \sum_{k=1}^{m+1} L_{jk} \frac{di_k}{dt} + \frac{d\theta}{dt} \sum_{k=1}^{m+1} i_k \frac{dL_{jk}}{d\theta} = u_j \]

\[ j = 1, 2, 3 \ldots \ldots m \]

\[ \frac{J}{p} \frac{d^2\theta}{dt^2} + T_L = \frac{p}{2} \sum_{j=1}^{m+1} \sum_{k=1}^{m+1} i_j i_k \frac{dL_{jk}}{d\theta} \]

\text{...(1.7)}

where \( R \) - resistance of each phase winding and the excitation winding.

For an \( m \)-phase stepper motor, without any excitation winding or permanent magnet, equation (1.7) remains unchanged, but the indices \( j \) and \( k \) through which summation is carried out take values from 1 - \( m \).

In presence of permanent magnet \((m + 1)\)th. coordinate (i.e. current) is replaced by a constant.

\[ i_{m+1} = I_{m+1} = I'_c \]

\text{...(1.8)}

and the equation for the drive may be written down in the following form:

\[ R_{ij} + \sum_{k=1}^{m} L_{jk} \frac{di_k}{dt} + \frac{d\theta}{dt} \sum_{k=1}^{m} i_k \frac{dL_{jk}}{d\theta} = u_j \]

where, \( j = 1, 2, \ldots \ldots n \)

\text{...(1.9)}
Excitation voltage, applied to the (m \ l)th. winding is considered to be external input when it changes during the working process. However, this change is at a comparatively slow rate. Therefore, its influences on the electromechanical energy conversion process is very little. Considering only the external inputs the equations of motion for a 4-phase stepper motor in

\[U = 0, d, q - 0 \ (\text{vide figure 1.11})\] may be written down in the following form:

\[
\begin{align*}
U_{t_0} &= \frac{1}{4}(u_1 + u_2 + u_3 + u_4) \\
U_d &= \frac{1}{2}(u_1 - u_3)\cos(\theta + \phi_0) + \frac{1}{2}(u_2 - u_4)\sin(\theta + \phi_0) \\
U_q &= -\frac{1}{2}(u_1 - u_3)\sin(\theta + \phi_0) + \frac{1}{2}(u_2 - u_4)\cos(\theta + \phi_0) \\
U_{t_0} &= \frac{1}{4}(u_1 - u_2 + u_3 - u_4)
\end{align*}
\]

\[
\begin{align*}
RI_{t_0} + \frac{d\psi_{t_0}}{dt} &= U_{t_0} \\
RI_d + \frac{d\psi_d}{dt} - \psi_q \frac{d\theta}{dt} &= U_d \\
RI_q + \frac{d\psi_q}{dt} + \psi_d \frac{d\theta}{dt} &= U_q \\
RI_{t_0} + \frac{d\psi_{t_0}}{dt} &= U_{t_0}
\end{align*}
\]

and

\[
\frac{J}{2r} \frac{d^2\theta}{dt^2} + T_L = T_e
\]
Flux linkage may be expressed as:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Psi_{t_0} & L_0 & L_{1/2} & 0 & 0 \\
\hline
\Psi_d & L_1 & L_0 & 0 & L_1 \cos 2\theta \\
\hline
\Psi_q & 0 & 0 & L_0 & -L_1 \sin 2\theta \\
\hline
\Psi_{-t_0} & 0 & L_{1/2} \cos 2\theta & -L_{1/2} \sin 2\theta & L_0 \\
\hline
\end{array}
\]

\[
...(1.12)
\]

Electromagnetic energy of the system is expressed through flux linkage and current in the following way:

\[
\mathcal{W}_E = \frac{i}{2} \sum_{k=1}^{m} \Psi^*_k i_k
\]

\[
= \frac{i}{2} \left[ \frac{m}{2} (\Psi_d I_d + \Psi_q I_q) + m (\Psi_{t_0} I_{t_0} + \Psi_{-t_0} I_{-t_0}) \right]
\]

\[
...(1.13)
\]

Electromagnetic torque may be obtained by differentiating this equation with respect to angular displacement of the rotor:

\[
T_e = Z r \frac{d\mathcal{W}_E}{d\theta}
\]

\[
= 2 Z_r L_1 [I_{t_0} I_q - I_{-t_0} (I_d \sin 2\theta + I_q \cos 2\theta)]
\]

\[
...(1.14)
\]

Substituting flux linkage (equation 1.12) and electromagnetic torque (equation 1.14) into equation (1.11):

\[
\frac{J}{Z_r} \frac{d^2\theta}{dt^2} + T_L = T_e
\]

\[
= 2 Z_r L_1 [I_{t_0} I_q - I_{-t_0} (I_d \sin 2\theta + I_q \cos 2\theta)]
\]

\[
...(1.15)
\]
Equations are transformed into \( \alpha, \beta, q, \phi \) co-ordinates only for four phase stepper motor, since this type of motor is installed in the vertical milling machine 6N12P.

Demands of programme control, from the point of view of fastness, stability of moment, and the degree of discreteness in combination with reliability of commutation are satisfied more fully by multiphase (m \( \geq 3 \)) stepper motors, windings of which are excited sequentially or in groups by rectangular voltage pulses of only one polarity.

In comparison with 3-phase stepper motor, the four phase motor has better characteristics and gives lesser value of absolute error in the system and operates at lower speed at a given frequency of input impulses.
Equations of electrical and mechanical equilibrium (1.10 and 1.15) for the stepper motor express the mutual relationships of parameters in a more compact form and simplify the computation of electromechanical torque.

The control system under consideration contain elements other than the stepper motor. Effect of these elements also are to be considered in determining the dynamical behaviour of the system. They are shown in (figures 8(a) and 1.8(b) ) the block diagram of the kinematic links.

Angular displacement of the stepper motor shaft may be expressed in the following form:

\[ J_d \ddot{\theta}_d + B \dot{\theta}_d = T_e - C_{dc} (\theta_d - \theta_c) - \text{sign}(\dot{\theta}_d) T_d \]  

...(1.16)

Here,

- \( J_d \) - moment of inertia of the stepper motor,
- \( B \) - coefficient of internal damping of the stepper motor,
- \( C_{dc} \) - coefficient of elasticity of the lead screw,
- \( T_d \) - dry friction on the stepper motor shaft,
- \( \theta_d, \theta_c \) - angular displacement of the motor shaft and linear displacement of the table referred to stepper motor shaft.
\( \theta_c \) may be expressed by the following equation:

\[
J_c \dot{\theta}_c = C_{dc}(\theta_d - \theta_c) - \text{Sign}(\dot{\theta}_c)T_c - T_L \quad \text{...(1.17)}
\]

Here, 
- \( J_c \) - moment of inertia of the table, referred to the motor shaft,
- \( T_c \) - Dry friction in the table, referred to motor shaft,
- \( T_L \) - Load torque, referred to motor shaft.

Equation (1.16), (1.15), (1.16) and (1.17) fully characterise the given system of automatic control.

Resume

1. Static characteristics of the kinematic chain were obtained by plotting angular displacement of the stepper motor shaft against the turning moment by loading the motor shaft with load of gradually changing magnitude and sign. When the table is fixed relative to the machine body, the characteristics obtained pertain to the kinematic chain from stepper motor to the table, whereas fixation of the table through the cutting tool gives the characteristics pertaining to the whole of kinematic chain.

2. The parameters which influence equally on the
static and dynamic conditions were obtained from the two families of curves mentioned above, by comparing the obtained characteristics with the well-known characteristics of simple elements. The schematic diagram contain linear element - elastic link and non-linearities of the following types: dry friction, back lash and jamming.

3. Addition of such elements like inertia and viscous friction, which appear only during the dynamic condition with the above mentioned elements give the complete block diagram of the system for each coordinate axis.

4. Mathematical description of the drive is given by equations (1.4) and (1.5).

5. The complete system of equations is non-linear and sufficiently complex. Table and rotor of the stepper motor are mutually coupled together through the elastic link. Elastic link, through its action increases the order of differential equation of double-mass system upto fourth order.

6. The system of automatic control contain non-linearities of the following types: dry friction on the motor shaft and the table, back lash and non-linear friction - jamming of the lead screw.
7. Solution of the problem incorporates some simplifying assumptions, discussed in the third chapter.

8. For obtaining the algorithm for optimum control in the sense of minimising the error methods of non-linear theory may be applied, as well as phase-space technique, finite difference, and mathematical modelling.