Chapter 1

Preliminaries

In this chapter we collect the basic definitions which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Bondy and Murty [2] and Harary [7].

Definition 1.0.1:
A graph $G = (V(G), E(G))$ consists of two finite sets: $V(G)$, the vertex set of the graph, often denoted by just $V$, which is a nonempty set of elements called vertices, and $E(G)$, the edge set of the graph, often denoted by just $E$, which is a possibly empty set of elements called edges, such that each edge $e$ in $E$ is assigned an unordered pair of vertices $(u, v)$, called the end vertices of $e$.

Definition 1.0.2:
If $e$ is an edge with end vertices $u$ and $v$ then $e$ is said to join $u$ and $v$. Note that the definition of a graph allows the possibility of the edge $e$ having identical end vertices, That is, it is possible to have a vertex $u$ joined to itself by an edge—such an edge is called a loop.
Definition 1.0.3:
Let $G$ be a graph. If two (or more) edges of $G$ have the same end vertices these edges are called parallel.

Definition 1.0.4:
A vertex of $G$ which is not the end of any edge is called isolated.

Two vertices which are joined by an edge are said to be adjacent or neighbors.

The set of all neighbors of a fixed vertex $v$ of $G$ is called the neighborhood set of $v$ and is denoted by $\text{Nbhd}(v)$.

Definition 1.0.5:
A graph is called simple if it has no loops and no parallel edges.

Definition 1.0.6:
A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets $V_1$ and $V_2$ and a one-to-one correspondence between the edge sets $E_1$ and $E_2$ in such a way that if $e_1$ is an edge with end vertices $u_1$ and $v_1$ in $G_1$ then corresponding edge $e_2$ in $G_2$ has its end points the vertices $u_2$ and $v_2$ in $G_2$ which correspond to $u_1$ and $v_1$ respectively.

Such a pair of correspondences is called a graph isomorphism.

Definition 1.0.7:
A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. Thus, a graph with $n$ vertices is complete if it has as many edges as possible provided there are no loops and no parallel edges.
If the complete graph has vertices $v_1, v_2, \ldots, v_n$ then the edge set can be given by $E = \{(v_i, v_j) : v_i \neq v_j; i, j = 1, \ldots, n\}$

Definition 1.0.8:
An empty (or trivial) graph is a graph with no edges.

Definition 1.0.9:
Let $H$ be a graph with vertex set $V(H)$ and edge set $E(H)$ and, similarly, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Then we say that $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, we also say that $G$ is a supergraph of $H$.

Definition 1.0.10:
Any graph isomorphic to a subgraph of $G$ is also referred to as a subgraph of $G$.

If $H$ is a subgraph of $G$ then we write $H \subseteq G$. When $H \subseteq G$ but $H \neq G$, that is, $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then $H$ is called a proper subgraph of $G$.

A spanning subgraph (or spanning supergraph) of $G$ is a subgraph (or supergraph) $H$ with $V(H) = V(G)$, That is, $H$ and $G$ have exactly the same vertex set.

The simplest types of subgraph of a graph $G$ are those obtained by the deletion of a vertex or an edge and we now define these.

Definition 1.0.11:
If $G = (V, E)$ and $V$ has at least two elements (that is, $G$ has at least two
vertices), then for any vertex $v$ of $G$, $G - v$ denotes the subgraph of $G$ with vertex set $V - \{v\}$ and whose edges are all those of $G$ which are not incident with $v$, that is, $G - v$ is obtained from $G$ by removing $v$ and all the edges of $G$ which have $v$ as an end. $G - v$ is referred to as a vertex deleted subgraph.

If $G = (V, E)$ and $e$ is an edge of $G$ then $G - e$ denotes the subgraph of $G$ having $V$ as its vertex set and $E - \{e\}$ as its edge set, that is, $G - e$ is obtained from $G$ by removing the edge $e$, (but not its end point(s)). $G - e$ is referred to as an edge deleted subgraph.

The above definition is extended to cater for the deletion of several vertices or edges.

**Definition 1.0.12:**

If $G = (V, E)$ and $U$ is a proper subset of $V$ then $G - V$ denotes the subgraph of $G$ with vertex set $V - U$ and whose edges are all those of $G$ which are not incident with any vertex in $U$.

If $F$ is a subset of the edge set $E$ then $G - F$ denotes the subgraph of $G$ with vertex set $V$ and edge set $E - F$, that is, obtained by deleting all the edges in $F$, but not their end points. $G - U$ and $G - F$ are also referred to as vertex deleted and edge deleted subgraphs (respectively).

Some of the more important subgraphs we shall encounter are the induced subgraphs and we now define these.

**Definition 1.0.13:**

If $U$ is a non empty subset of the vertex set $V$ of the graph $G$ the subgraph $G[U]$
of \( G \) induced by \( U \) is defined to be the graph having vertex set \( U \) and edge set consisting of those edges of \( G \) that have both ends in \( U \).

Similarly if \( F \) is a nonempty subset of the edge set \( E \) of \( G \) the subgraph \( G[F] \) of \( G \) induced by \( F \) is the graph whose vertex set is the set of ends of edges in \( F \) and whose edge set is \( F \).

**Definition 1.0.14:**

Two subgraphs \( G_1 \) and \( G_2 \) of a graph \( G \) are said to be disjoint if they have no vertex in common, and edge disjoint if they have no edge in common.

**Definition 1.0.15:**

Given two subgraphs \( G_1 \) and \( G_2 \) of \( G \), the union \( G_1 \cup G_2 \) is the subgraph of \( G \) with vertex set consisting of all those vertices which are in either \( G_1 \) or \( G_2 \) (or both) and with edge set consisting of all those edges which are in either \( G_1 \) or \( G_2 \) (or both); Symbolically

\[
V(G_1 \cup G_2) = V(G_1) \cup V(G_2)
\]

\[
E(G_1 \cup G_2) = E(G_1) \cup E(G_2).
\]

**Definition 1.0.16:**

A graph \( G = (V, E) \) is said to be \( r \)-partite (where \( r \) is a positive integer) if its vertex set can be partitioned as \( V = V_1 \cup V_2 \cup \cdots \cup V_r \) such that if \((u, v)\) is an edge of \( G \) then \( u \) is in some \( V_i \) and \( v \) is in some other \( V_j \); that is, every one of the induced subgraphs \( \langle V_i \rangle \) is an empty graph. We may denote an \( r \)-partite graph by \( G = (V_1, V_2, \ldots, V_r, E) \), explicitly showing the \( r \)-partition of the vertex set \( V = V_1 \cup V_2 \cup \cdots \cup V_r \).

If an \( r \)-partite graph has all possible edges, that is \((u, v) \in E \) for every \( u \in V_i \)
and every \( v \in V_j \) for all pairs \( V_i, V_j \) then it is called a complete \( r \)-partite graph. If \( |V_i| = n_i \), we denote it by \( K_{n_1,n_2,...,n_r} \).

A 2-partite graph is usually referred to as a bipartite graph. We may denote such a graph by \( G = (V_1, V_2, E) \) where \( V_1 \cup V_2 = V \) is the bipartition of the vertex set \( V \). If a bipartite graph \( G = (V, E) \) with \( V \) partitioned into \( V_1 \cup V_2 \) with \( |V_1| = m \) and \( |V_2| = n \) has all possible edges then it is called a complete bipartite graph and is denoted by \( K_{m,n} \). The complete bipartite graph \( K_{1,n} \) is called an \( n \)-star or an \( n \)-claw.

**Definition 1.0.17:**

An edge \( e \) of a graph \( G \) is said to be incident with the vertex \( v \) if \( v \) is an end vertex of \( e \). In this case we also say that \( v \) is incident with \( e \). Two edges \( e \) and \( f \) which are incident with a common vertex \( v \) are said to be adjacent.

**Definition 1.0.18:**

Let \( v \) be a vertex of the graph \( G \). The degree \( d(v) \) of \( v \) is the number of edges of \( G \) incident with \( v \), counting each loop twice, That is, it is the number of times \( v \) is an end vertex of an edge. When the graph has to be specified we use the notation \( d_G(v) \).

The minimum degree and the maximum degree of a graph \( G \) are usually denoted by the special symbols \( \delta(G) \) and \( \Delta(G) \) respectively.

**Definition 1.0.19:**

If for some positive integer \( k \), \( d(v) = k \) for every vertex \( v \) of the graph \( G \), then \( G \) is called \( k \)-regular.

A regular graph is one that is \( k \)-regular for some \( k \).
A three-regular graph is also called a cubic graph. The complete graph $K_n$ is $(n - 1)$-regular. The complete bipartite graph $K_{n,n}$ on $2n$ vertices is $n$-regular.

A vertex with degree zero is called an isolated vertex; a vertex with degree one is a pendant vertex. The unique edge incident with a pendant vertex is a pendant edge. A vertex of odd degree is an odd vertex and a vertex of even degree is an even vertex.

**Definition 1.0.20:**

Let $G$ be a simple graph with $n$ vertices. The complement $\overline{G}$ of $G$ is defined to be the simple graph with the same vertex set as $G$ and where two vertices $u$ and $v$ are adjacent precisely when they are not adjacent in $G$. Roughly speaking then, the complement of $G$ can be obtained from the complete graph $K_n$ by “rubbing out” all the edges of $G$. 
Definition 1.0.21:
Let $G_1$ and $G_2$ be two graphs with no vertex in common. We define the join of $G_1$ and $G_2$, denoted by $G_1 + G_2$, to be the graph with vertex set and edge set given as follows:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2)$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$$

where $J = \{(x_1, x_2) : x_1 \in V(G_1), x_2 \in V(G_2)\}$. Thus $J$ consists of edges which join every vertex of $G_1$ to every vertex of $G_2$.

Definition 1.0.22:
For each $n \geq 4$, the wheel graph, $W_n$, with $n$ vertices, is defined to be the join $K_1 + C_{n-1}$, of an isolated vertex with a cycle of length $n - 1$.

Definition 1.0.23:
A walk in a graph $G$ is a finite sequence $W = v_0e_1v_1e_2v_2 \ldots v_{k-1}e_kv_k$ whose terms are alternately vertices and edges such that, for $1 \leq i \leq k$, the edge $e_i$ has ends $v_{i-1}$ and $v_i$.

Thus each edge $e_i$ is immediately preceded and succeeded by the two vertices with which it is incident.

We say that the above walk $W$ is a $v_0 - v_k$ walk or a walk from $v_0$ to $v_k$. The vertex $v_0$ is called the origin of the walk $W$, while $v_k$ is called the terminus of $W$. Note that $v_0$ and $v_k$ need not be distinct.

The vertices $v_1, v_2, \ldots, v_{k-1}$ in the above walk $W$ are called its internal vertices. The integer $k$, the number of edges in the walk, is called the length of $W$. 
Note that in a walk there may be repetition of vertices and edges. In a simple
graph, a walk \( v_0e_1v_1e_2v_2 \cdots v_{k-1}e_kv_k \) is determined by the sequence \( v_0v_1 \cdots v_k \)
of its vertices since for each pair \( v_{i-1}v_i \) there is only one possible edge with ends
determined by the pair. A trivial walk is one containing no edges.

**Definition 1.0.24:**

Given two vertices \( u \) and \( v \) of a graph \( G \), a \( u-v \) walk is called closed or open
depending on whether \( u = v \) or \( u \neq v \).

**Definition 1.0.25:**

If the edges \( e_1, e_2, \ldots, e_k \) of the walk \( W = v_0e_1v_1e_2v_2 \cdots e_kv_k \) are distinct then
\( W \) is called a trail.

**Definition 1.0.26:**

If the vertices \( v_0, v_1, \ldots, v_k \) of the walk \( W = v_0e_1v_1e_2v_2 \cdots e_kv_k \) are distinct
then \( W \) is called a path.

Clearly any two paths with the same number of vertices are isomorphic.

A path with \( n \) vertices will be denoted by \( P_n \). Sometimes \( P_n \) also stands for
a path of length \( n \).

**Definition 1.0.27:**

A vertex \( u \) is said to be connected to a vertex \( v \) in a graph \( G \) if there is a path
in \( G \) from \( u \) to \( v \).

**Definition 1.0.28:**

A graph \( G \) is called connected if every two of its vertices are connected. A graph
that is not connected is called disconnected.
Given any vertex $u$ of a graph $G$, let $C(u)$ denote the set of all vertices in $G$ that are connected to $u$. Then the subgraph of $G$ induced by $C(u)$ is called the connected component containing $u$, or simply the component containing $u$. The number of components of a graph $G$ is denoted by $\omega(G)$.

**Definition 1.0.29:**
A non-trivial closed trial in a graph $G$ is called a cycle if its origin and internal vertices are distinct. In detail, the closed trial $C = v_1v_2 \cdots v_nv_1$ is a cycle if $C$ has at least one edge and $v_1, v_2, \ldots, v_n$ are distinct vertices.

A cycle of length $k$, that is, with $k$ edges, is called a $k$-cycle. A $k$-cycle is called odd or even depending on whether $k$ is odd or even.

A 3-cycle is often called a triangle.

Clearly any two cycles of the same length are isomorphic. An $n$-cycle, that is, a cycle with $n$ vertices, will sometimes be denoted by $C_n$.

Note that a loop is just a 1-cycle. Also given a pair of parallel edges $e_1, e_2$ with distinct end vertices $v_1$ and $v_2$ we can form the cycle of length 2. Conversely, the two edges of any cycle of length 2 are a pair of parallel edges.

**Definition 1.0.30:**
For any two vertices $u$ and $v$ connected by a path in a graph $G$, we define the distance between $u$ and $v$, denoted by $d(u, v)$, to be the length of a shortest $u-v$ path. If there is no path connecting $u$ and $v$ we define $d(u, v)$ to be infinite.

**Definition 1.0.31:**
Let $G$ be a connected graph with vertex set $V$. For each $v \in V$, the eccentricity of $v$, denoted by $e(v)$, is defined by $e(v) = \max\{d(u, v) : u \in V, u \neq v\}$. 
The radius of $G$, denoted by $\text{rad}(G)$, is defined by

$$\text{rad}(G) = \min\{e(v) : v \in V\},$$

while the diameter of $G$, denoted by $\text{diam}(G)$, is defined by

$$\text{diam}(G) = \max\{e(v) : v \in V\}.$$ 

Thus the diameter of $G$ is given by $\max\{d(u, v) : u, v \in V\}$.

**Definition 1.0.32:**

Let $u$ and $v$ be two vertices of a graph $G$. A collection $\{P_1, P_2, \ldots, P_n\}$ of $u-v$ paths is said to be internally disjoint if, given any distinct pair $P_i$ and $P_j$ in the collection, $u$ and $v$ are the only vertices $P_i$ and $P_j$ have in common.

**Definition 1.0.33:**

A graph is called acyclic if it contains no cycles.

**Definition 1.0.34:**

A graph $G$ is called a tree if it is a connected acyclic graph.

**Definition 1.0.35:**

An edge $e$ of a graph $G$ is called a bridge (or a cut-edge) if the subgraph $G - e$ has more connected components than $G$ has.
Definition 1.0.36:
A vertex $v$ of a graph $G$ is called a cut vertex (or articulation point) of $G$ if $\omega(G - v) > \omega(G)$.

In other words, a vertex $v$ is a cut vertex of $G$ if its deletion disconnects some connected component of $G$, thereby producing a subgraph having more connected components than $G$ has.

Definition 1.0.37:
Let $G$ be a simple graph. The (vertex) connectivity of $G$, denoted by $\kappa(G)$, is the smallest number of vertices in $G$ whose deletion from $G$ leaves either a disconnected graph or $K_1$.

Definition 1.0.38:
A simple graph $G$ is called $n$-connected (where $n \geq 1$) if $\kappa(G) \geq n$.

Definition 1.0.39:
A directed graph $D = (V, A)$ consists of two finite sets $V$, the vertex set, a nonempty set of elements called the vertices of $D$ and $A$, the arc set, a (possibly empty) set of elements called the arcs of $D$, such that each arc $a$ in $A$ is assigned an ordered pair of vertices $(u, v)$.

If $a$ is an arc, in the directed graph $D$, with associated ordered pair of vertices $(u, v)$, then $a$ is said to join $u$ to $v$, $u$ is called the origin or the initial vertex or the tail of $a$, and $v$ is called the terminus or the terminal vertex or the head of $a$.

Definition 1.0.40:
Let $D$ be a digraph. Then a directed walk in $D$ is a finite sequence $W =$
v_0a_1v_1 \cdots a_kv_k, whose terms are alternately vertices and arcs such that for i = 1, 2, \ldots, k, the arc a_i has origin v_{i-1} and terminus v_i.

As in graphs, this directed walk W is often written simply as its sequence of vertices W = v_0v_1 \cdots v_k, the number k of arcs in W is called the length of W. There are similar definitions for directed trails, directed paths and directed cycles.

**Definition 1.0.41:**
Let v be a vertex in the digraph D. The indegree id(v) of v is the number of arcs of D that have v as its head, that is, the number of arcs that "go to" v. Similarly, the outdegree od(v) of v is the number of arcs of D that have v as its tail, that is, the number of arcs that "go out" of v.

**Notation 1.0.42:**
Let \lfloor x \rfloor denote the largest integer \leq x and let \lceil x \rceil denote the smallest integer \geq x.