Chapter 4

$t$-Pebbling the Product of trees

The $t$-pebbling number, $f_t(G)$, of a connected graph $G$, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, $t$ pebbles can be moved to a specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. We study the $t$-pebbling number of the product of trees.

4.1 Known Results

We find the following definitions, Example 4.1.4, and Theorem 4.1.5 in [32].

Definition 4.1.1 (Path-partition of a rooted tree):
Let $T$ be a tree and $v$ be a vertex of $T$. Let $T_v$ be the rooted tree obtained from $T$ by directing all edges towards $v$, which becomes the root. For a rooted tree $U$, we shall call a vertex $v$ of $U$ a leaf it is of indegree 0. We shall call $v$, a parent of $w$ if there is a directed edge from $w$ to $v$, and an ancestor of $w$ if there is a directed path from $w$ to $v$. We call $v$, a vertex of level $n$ if the directed path
from $v$ to the root has $n$ edges; the height of a tree is the maximum level of its vertices. A path-partition of a rooted tree $U$ is a partition of the edges of $U$ such that each set of edges in the partition forms a directed path.

**Definition 4.1.2** (Maximum Path-partition of a rooted tree):
Path-partitions of a rooted tree $U$ with height $h$ can be formed in the following way. First we consider the subtree $U^1$ of $U$ induced by all leaves of level $h$ and their ancestors and construct a path-partition $P^1$ of $U^1$ such that every path in $P^1$ touches a leaf. Then we let $U^{11}$ be the subtree of $U$ induced by all leaves of level $h$ or $h - 1$ and their ancestors and extend $P^1$ to a path-partition $P^{11}$ of $U^{11}$ by adding paths, which touch the level $h - 1$ leaves of $U$. We continue in this manner until we have a path-partition $P$ of all of $U$. A path-partition constructed in this way is called maximum.

**Definition 4.1.3** (Path-size sequence):
The path-size sequence of a path-partition $\{P_1, P_2, \ldots, P_n\}$ is an $n$-tuple $(a_1, a_2, \ldots, a_n)$, where $a_j$ is the length of $P_j$ (That is, the number of edges in it).

**Example 4.1.4**:
Let us construct a maximum path-partition of the tree $U$ in Figure 4.1. We start with the subtree $U^1$ of $U$ induced by the vertex $i$, the unique vertex of $U$ of level 4, and its ancestors $b, e, f$ and $h$. There is a unique path-partition of $U^1$ such that every path touches a leaf, namely the path-partition with just one path, $\{ib, be, ef, fh\}$. Now we extend this path-partition to a path-partition of the subtree of $U$ induced by the set $\{a, e, i, b, f, h\}$ of all vertices of level 3 or 4 and their ancestors. This produces the path-partition $\{ae\}, \{ib, be,$
Another extension gives us \{\{cg, gh\}, \{ae\}, \{ib, be, ef, fh\}\}, and another extension gives us the maximum path-partition of \(U\), namely \{\{cg, gh\}, \{ae\}, \{ib, be, ef, fh\}, \{dh\}\}. In this case, the maximum path-partition is unique, but this is not always the case. For example, if the vertex \(i\) and the edge \(ib\) were removed from \(U\), \(U\) would have two maximum path-partitions \{\{ae, ef, fh\}, \{be\}, \{cg, gh\}, \{dh\}\} and \{\{be, ef, fh\}, \{ae\}, \{cg, gh\}, \{dh\}\}.

![A rooted tree.](image)

**Theorem 4.1.5** ([32]). Let \(U\) be a rooted tree and \(v\) be the root of \(U\). If the path-size sequence of some maximum path-partition for \(U\) is \((a_1, a_2, \ldots, a_n)\), then \(f(v, U) = \sum_{i=1}^{n} 2^{a_i} - n + 1\).

We find Theorem 4.1.6 in [30].

**Theorem 4.1.6.** Let \(U\) be a rooted tree and \(v\) be the root of \(U\). Let
(a_1, a_2, \ldots, a_n), \text{ be the path-size sequence for some maximum path-partition for } U. \text{ Without loss of generality } a_1 \text{ can be taken to be } h \text{ where } h \text{ is the height of the tree. Then}

\[ f_t(v, U) = t2^h + \sum_{i=2}^{n} 2^{a_i} - n + 1. \]

We find Definition 4.1.7 in [9].

**Definition 4.1.7:**

Given a pebbling of G, a transmitting subgraph of G is a path \( x_0, x_1, \ldots, x_k \) such that there are at least two pebbles on \( x_0 \) and at least one pebble on each of the other vertices in the path, except possibly \( x_k \). In this case, we can transmit a pebble from \( x_0 \) to \( x_k \).

The following theorem of [3] is used here.

**Theorem 4.1.8** ([3]). *A Tree satisfies the 2-pebbling property.*

Definition 3.3.4 discusses the odd 2t-pebbling property of the graph as a whole. We see the odd 2t-pebbling property of a vertex as follows: we say that a vertex \( v \) in a graph \( G \) satisfies the odd 2t-pebbling property if we can put 2t pebbles on \( v \) using pebbling moves from any arrangement of pebbles with at least 2\( f_t(G) - r + 1 \) pebbles, where \( r \) is the number of vertices in the arrangement with an odd number of pebbles.

We will now prove that a tree satisfies the odd 2t-pebbling property.

**Theorem 4.1.9.** *A tree satisfies the odd 2t-pebbling property.*
Proof. Let $T$ be a tree and $v$ be a vertex of $T$. Let $U$ be the rooted tree obtained from $T$ by directing all edges towards $v$, which becomes the root.

Let $(a_1, a_2, \ldots, a_n)$ be the path-size sequence for some maximum path-partition for $U$. Without loss of generality $a_1$ can be taken to be $h$ where $h$ is the height of the tree.

Then by Theorem 4.1.6,

$$f_t(v, U) = t2^h + \sum_{i=2}^{n} 2^{a_i} - n + 1$$

Consider a configuration of $2f_t(v, U) - q + 1$ pebbles where $q$ is the number of vertices with an odd number of pebbles. We use induction on $t$ to prove that $v$ satisfies the odd $2t$-pebbling property. For $t = 1$, the result is true by Theorem 4.1.8. For $t > 1$, the number of pebbles on the tree will be at least

$$2^{h+2} + \sum_{i=2}^{n} 2^{a_i+1} - 2n + 3 - q = 2f_{t-1}(v, U) - q + 1 + 2^{h+1}$$

where $q$ is the number of vertices with an odd number of pebbles. Let $p$ be the number of pebbles on $U$. We claim that there will be at least one $P_i$ with at least $2^{a_i+1}$ pebbles. Otherwise, the total number of pebbles placed on $T$ will be at most

$$2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} - n.$$ 

Then,

$$2f_t(v, U) + 1 \leq p + q \leq 2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} + q - n$$

That is, $t2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} - 2n + 2 + 1 \leq 2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} + q - n$. 


That is, 

\[(t - 1)2^{h+1} - 2n + 3 \leq q - n.\]

That is, 

\[(t - 1)2^{h+1} + 3 \leq q + n.\]

That is, 

\[(t - 1)2^{h+1} + 3 \leq 2|V(U)| \quad \text{since} \quad n \leq |V(U)| \quad \text{and} \quad q \leq |V(U)|.\]

That is, 

\[(t - 1)2^{h} + (3/2) \leq |V(U)| \quad \text{for all} \quad t > 1.\]

This is a contradiction.

So we can put two pebbles on \(v\) using \(2^{n+1}\) pebbles lying on \(P_i\). So at most \(2^{h+1}\) pebbles will be used to put two pebbles on \(v\). Then the remaining number of pebbles on \(U\) will be at least \(2f_{t-1}(v, U) - q + 1\) where \(q\) is the number of vertices with an odd number of pebbles. By induction, these pebbles would suffice to put \(2(t - 1)\) additional pebbles on \(v\).

As \(v\) is arbitrary, every vertex in \(T\) satisfies the odd \(2t\)-pebbling property.

Hence \(T\) satisfies the odd \(2t\)-pebbling property.

\[\square\]

### 4.2 \(t\)-pebbling the product of some graphs

We now discuss some results on the \(t\)-pebbling number of direct product of two graphs.

Conjecture 3.4.3 discusses the \(t\)-pebbling number of the graph as a whole. To discuss the \(t\)-pebbling number of a specific vertex, we state Conjecture 4.2.1 which is a stronger form of Conjecture 3.4.3.

**Conjecture 4.2.1:**

The \(t\)-pebbling number of every vertex \((v, w)\) in \(G \times H\) satisfies \(f_t((v, w), G \times H) \leq f(v, G)f_t(w, H).\)

\[\square\]

We will now show that \(P_2 \times G\) satisfies Conjecture 3.4.3 when \(G\) satisfies
the 2t-pebbling property.

**Theorem 4.2.2.** Let $P_2$ be the path on two vertices $x_1$ and $x_2$ and suppose $G$ satisfies the 2t-pebbling property. Then $f_t(P_2 \times G) \leq 2f_t(G)$.

**Proof.** Without loss of generality assume that the target vertex is $(x_1, y)$ for some $y$. If $p_1 + \frac{p_2 - q_2}{2} \geq f_t(G)$ then we can use Lemma 3.4.4 to put $f_t(G)$ pebbles on $\{x_1\} \times G$. Since this subgraph is isomorphic to $G$, we can then put $t$ pebbles on $(x_1, y)$. Also since $G$ satisfies the 2t-pebbling property, if $\frac{p_2 + q_2}{2} > f_t(G)$ we can put 2t pebbles on $(x_2, y)$ and then we can use a pebbling move to t-pebble $(x_1, y)$. Hence the only distributions from which we cannot t-pebble the target satisfy the inequalities

\[
p_1 + \frac{p_2 - q_2}{2} < f_t(G)
\]
\[
\frac{p_2 + q_2}{2} \leq f_t(G)
\]

But adding these together shows that $p_1 + p_2 < 2f_t(G)$. Thus some configuration of pebbles from which we cannot t-pebble some target must begin with fewer than $2f_t(G)$ pebbles. So we get $f_t(P_2 \times G) \leq 2f_t(G)$.

**Theorem 4.2.3.** Suppose $G$ satisfies the 2t-pebbling property. Let $P_3 = \{x_1, x_2, x_3\}$ be the path on three vertices. Consider the graph $P_3 \times G$. Then $f_t(\{x_2\} \times G) \leq 3f_t(G)$.

**Proof.** Since $G$ has the 2t-pebbling property, we can put 2t pebbles on $(x_1, y)$ unless $\frac{p_1 + q_1}{2} \leq f_t(G)$. By Lemma 3.4.4 and Theorem 4.2.2 we can put
Let us now generalize the definition of \( f(v, G) \) found in [32].

**Definition 4.3.1:**

If we have a digraph \( G \) with some pebbles placed on it, we let \( p \) be the total number of pebbles on \( G \) and \( r \) be the number of vertices of \( G \) with an odd number of pebbles. If \( G \) is a digraph and \( v \) is a vertex of \( G \), we say that \( f_t(v, G) \leq \alpha \) if

1. For all \( g \geq 1 \), if \( p \geq g\alpha \), then \( gt \) pebbles can be moved to \( v \).
2. For all \( g \geq 2 \), if \( p + r > g\alpha \), then \( gt \) pebbles can be moved to \( v \).

This definition is similar to Definition 3.3.4 and allows us to prove Theorem 4.3.2, below, which discusses the \( t \)-pebbling number of products.

In terms of the terminology of Definition 3.3.4, a graph which follows Definition 4.3.1 can also be called a graph with the odd 2\( t \)-pebbling property.
We now generalize the work of David Moew’s [32] in the setting of \( t \)-pebbling.

**Theorem 4.3.2.** Let \( U \) be a rooted tree with root \( v \) and let \( G \) be a digraph with \( w \) a vertex of \( G \). If \( f_t(w, G) \leq \alpha \), then

\[
f_t((v, w), U \times G) \leq f(v, U)f_t(w, G).
\]

**Proof.** The proof is by induction on \( h \), the height of \( U \). If \( h \) is zero, the result is trivial. Otherwise let \( P \) be a maximum path-partition of \( U \), and let \( U^1 \) be the subtree of \( U \) induced by the set of all vertices of level less than \( h \). Then, \( \{p_0 \cap E(U^1) \neq \varnothing / P_0 \in P\} = P^1 \) say, is a path-partition of \( U^1 \). If \( h = 1 \), we let \( P^1 \) contain one length zero path at \( v \). Let \( v_1, v_2, \ldots, v_n \) be the vertices in \( U \) which are parents of leaves of level \( h \). Then in \( P^1 \) there is a path to each \( v_j \), \( P_j \) say; let \( P_j \) have \( a_j \) edges. Let \( Q_1, Q_2, \ldots, Q_m \) be the remaining paths in \( P^1 \); and let \( Q_j \) have \( b_j \) edges. Now \( U \) can be obtained from \( U^1 \) by adding leaves to \( v_1, v_2, \ldots, v_n \). Suppose we add \( L \) leaves in all. Then since \( P \) is maximum, every path in \( P \) must touch a leaf, and the path-partition \( P \) must consist of the paths \( Q_1, Q_2, \ldots, Q_m \), combined with the paths \( P_1, P_2, \ldots, P_n \), each prefixed by an edge to \( v_j \) from one of its leaves, and \( L - n \) one edge paths to the \( v_j \)'s from their other leaves. Also, since \( P \) is maximum, the prefixed \( P_j \)'s together with these \( L - n \) one-edge paths must form a path-partition for the subtree \( U_0 \) of \( U \) induced by the vertices of level \( h \) and their ancestors. Hence if we let \( U_0^1 \) be the subtree of \( U^1 \) induced by the \( v_j \)'s and their ancestors, the \( P_j \)'s must form a path-partition of \( U_0^1 \).
Furthermore, the leaves \( \{v_1, v_2, \ldots, v_n\} \) of \( U_0^1 \) are all on the same level and each \( P_j \) touches the corresponding \( v_j \). Hence \( \{P_1, P_2, \ldots, P_n\} \) is in fact a maximum path-partition of \( U_0^1 \). This implies that \( P^1 \), which consists of the \( P_j \)'s and the \( Q_j \)'s, is a maximum path-partition of \( U^1 \).

Let \( f_t((v, w), U \times G) \leq X \) and \( f_t((v, w), U^1 \times G) \leq X^1 \). By induction,

\[
f_t((v, w), U^1 \times G) \leq f((v, U^1) f_t(w, G).
\]

That is, \( X^1 = \left( \sum_{i=1}^{n} 2^{a_i} + \sum_{j=1}^{m} 2^{b_j} - (m + n) + 1 \right) \alpha. \)

Note that \( P \) consists of paths of lengths \( a_1 + 1, a_2 + 1, \ldots, a_n + 1, b_1, b_2, \ldots, b_m \) and \( L - n \) paths of length 1. Hence

\[
X = \left( 2(L - n) + \sum_{i=1}^{n} 2^{a_i+1} + \sum_{j=1}^{m} 2^{b_j} - (m + L) + 1 \right) \alpha.
\]

\[
= X^1 + \left( L + \sum_{i=1}^{n} 2^{a_i} - n \right) \alpha.
\]

Now setting \( G \) equal to the trivial graph and using the induction hypothesis, we see that \( f_t(v, U^1_0) \leq Q t \) where \( Q = \sum_{i=1}^{n} 2^{a_i} - n + 1. \)

Then \( X - X^1 = (L + Q - 1) \alpha. \)

Let \( g \geq 1. \) If \( p \geq gX \), we will prove that \( gt \) pebbles can be moved to \((v, w)\). It is enough to prove this for \( g = 1, \) since for \( g > 1 \) we can perform \( g \) steps, each one looking at \( X \) pebbles on \( U \times G \) and rearranging these pebbles to put \( t \) pebbles on \((v, w)\).

Let \( l_1, l_2, \ldots, l_L \) be the level-\( h \) vertices of \( U \), and let \( p^1 \) be the number of pebbles in \( U^1 \times G \) and \( p_k \) be the number of pebbles in \( \{l_k\} \times G \), where \( k = 1, 2, \ldots, L. \) Let \( r^1 \) be the number of vertices with an odd number of pebbles
in $U^1 \times G$ and $r_k$ be the number of vertices with an odd number of pebbles in $\{l_k\} \times G$, where $k = 1, 2, \ldots, L$. If $p^1 + \sum_{k=1}^{L} \frac{p_k-r_k}{2} \geq X^1$, then we are done, since for each $k$ and vertex $y \in G$, we can take two pebbles off $(l_k, y)$ and put one pebble on some $(v_i, y)$. Therefore for each $k$, we take $p_k - r_k$ pebbles from $\{l_k\} \times G$ and move $\frac{p_k-r_k}{2}$ pebbles into $U^1 \times G$. Now the total number of pebbles on $U^1 \times G$ is at least $X^1$ and so we can put the pebbles on $(v, w)$, by the induction hypothesis.

Otherwise, since $p \geq X$, we have

$$\sum_{k=1}^{L} \frac{p_k + r_k}{2} = p - p^1 - \sum_{k=1}^{L} \frac{p_k - r_k}{2} > X - X^1,$$

$$\sum_{k=1}^{L} (p_k + r_k) > 2(X - X^1) = 2(L + Q - 1)\alpha \tag{4.1}$$

Now for each $k$, if $p_k + r_k > s\alpha$ for some $s \geq 2$, we can take pebbles from $\{l_k\} \times G$ and put $st$ pebbles on $(l_k, w)$, by hypothesis. Hence if $\frac{p_k+r_k-2\alpha}{2\alpha} > s \geq 0$, we can move $(2s+2)t$ pebbles to $(l_k, w)$ and then $(s+1)t$ pebbles from $(l_k, w)$ into $\{v_1, v_2, \ldots, v_n\} \times \{w\}$, and so by making $s$ as large an integer as possible, we can put at least $\frac{(p_k+r_k-2\alpha)t}{2\alpha}$ pebbles on $U^1_0 \times \{w\}$. By doing this for all $k$, we can put at least $\sum_{k=1}^{L} \frac{(p_k+r_k-2\alpha)t}{2\alpha}$ pebbles in $U^1_0 \times \{w\}$.

Now

$$\sum_{k=1}^{L} \frac{(p_k + r_k - 2\alpha)t}{2\alpha} = \sum_{k=1}^{L} \frac{(p_k + r_k)t - 2\alpha L t}{2\alpha} > \frac{2\alpha(L + Q - 1) - 2\alpha L t}{2\alpha} = (Q - 1)t.$$
So we can put \( t \) pebbles on \((v, w)\), since \( f_t(v, U^1_0) \leq Q_t \).

Now we will prove that for all \( g \geq 2 \), if \( p + r > gX \), then we can put \( gt \) pebbles on \((v, w)\). It is enough to prove this for \( g = 2 \), since for \( g > 2 \), we can first take one pebble on each vertex with an odd number of pebbles, and augment this set with pairs of pebbles until we have \( 2X + 1 - r \) pebbles.

Then we can put \( 2t \) pebbles on \((v, w)\) by rearranging these pebbles. Then there will be at least \((g - 2)X\) pebbles which can be used to move \((g - 2)t\) additional pebbles to \((v, w)\).

If \( p^1 + r^1 > 2X^1 \), then by induction hypothesis, we can put \( 2t \) pebbles on \((v, w)\).

Otherwise, \( p^1 + r^1 \leq 2X^1 \), \( 2X^1 - r^1 \geq p^1 \). If \( p^1 \geq X^1 \), then we can put \( t \) pebbles on \((v, w)\) in \( U^1 \times G \). Also \( p + r - p^1 - r^1 > 2(X - X^1) \). So (4.1) holds and as above, we can move \( Q_t \) pebbles into \( U^1_0 \times \{w\} \). Therefore we can put \( t \) additional pebbles on \((v, w)\). Now let \( p^1 < X^1 \). Then we claim that we can move \( X^1 - p^1 \) pebbles into \( U^1 \times G \) which can be used to put \( t \) pebbles on \((v, w)\) in \( U^1 \times G \) and still sufficient number of pebbles can be left over in \((U \setminus U^1) \times G\) to put \( t \) additional pebbles on \((v, w)\). If

\[
\sum_{k=1}^{L} \frac{p_k - r_k}{2} \geq X^1 - p^1,
\]

then we can move \( X^1 - p^1 \) pebbles into \( U^1 \times G \). (4.2) will be true if

\[
\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} r_k + 2p^1 \geq 2X^1.
\]

(4.3)
Now

\[
\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} r_k + 2p^1 \geq p^1 + r^1 + \sum_{k=1}^{L} p_k - \sum_{k=1}^{L} r_k = p + r - 2\sum_{k=1}^{L} r_k \geq p + r - 2L|V(G)|
\]

Since \(f_t(w, G) \leq \alpha\), \(\alpha \geq |V(G)|\),

\[
\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} r_k + 2p^1 \geq p + r - 2L\alpha.
\]

But \(X - X^1 \geq \alpha L\). So \(\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} r_k + 2p^1 \geq p + r - 2X + 2X^1 > 2X - 2X^1 + 2X^1\).

So (4.3) holds.

After moving \(c\) pebbles out of the \(\{l_k\} \times G\)'s, the \(r_k\)'s will still be the same as before, because we remove pebbles by twos, but the sum of \(p_k\)'s will decrease by \(2c\). Hence to have sufficient number of pebbles left over in \((U \setminus U^1) \times G\) to put \(t\) additional pebbles on \((v, w)\), we must have

\[
\sum_{k=1}^{L} (p_k + r_k) - 2(X^1 - p^1) > 2(L + Q - 1)\alpha.
\]

That is, we must have

\[
\sum_{k=1}^{L} (p_k + r_k) + 2p^1 > 2X^1 + 2(L + Q - 1)\alpha
\]

(4.4)
We know $2(X - X^1) = 2(L + Q - 1)$ and so $2X = 2X^1 + 2(L + Q - 1)$. Now

$$\sum_{k=1}^{L} (p_k + r_k) + 2p^1 \geq \sum_{k=1}^{L} (p_k + r_k) + p^1 + r^1$$

$$= p + r > 2X.$$ 

So (4.4) follows and so we are done. \hfill \blacksquare

**Theorem 4.3.3.** Let $U$ and $W$ be rooted trees with roots $v$ and $w$ respectively. Then $f_t((v, w), U \times W) \leq f(v, w)f_t(w, W)$.

**Proof.** Follows from Theorem 4.3.2 and Theorem 4.1.9. \hfill \blacksquare