Chapter 3

On \( t \)-pebbling graphs

The \( t \)-pebbling number, \( f_t(G) \), of a connected graph \( G \), is the smallest positive integer such that from every placement of \( f_t(G) \) pebbles, \( t \) pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. In this chapter, we compute the \( t \)-pebbling number of Jahangir graphs; we give alternate proofs for the \( t \)-pebbling numbers of even and odd cycles.

In [17], Lourdusamy has defined the \( 2t \)-pebbling property and the odd \( 2t \)-pebbling property for a graph and has proved that all even cycles satisfy the \( 2t \)-pebbling property. We give an alternate proof for the result that all even cycles satisfy the \( 2t \)-pebbling property. We also prove that all odd cycles satisfy the \( 2t \)-pebbling property. In [17], A. Lourdusamy has generalized Graham’s pebbling conjecture into \( f_t(G \times H) \leq f(G)f_t(H) \) where \( G \) and \( H \) are connected graphs. We call this the \( t \)-pebbling conjecture. In [17], A. Lourdusamy has proved that the \( t \)-pebbling conjecture is true for a graph which is the direct product of a path with an even cycle. We prove that the
The \( t \)-pebbling conjecture is true for a graph which is the direct product of a path with a cycle.

### 3.1 The \( t \)-pebbling number of Jahangir graphs

Graph pebbling is a model for the transmission of consumable resources. Chung [3] defines a pebbling distribution on a connected graph as a placement of pebbles on the vertices of the graph. A pebbling move then consists of removing two pebbles from one vertex, throwing one away, and putting the other pebble on an adjacent vertex. Chung defined the pebbling number of a vertex \( v \) in \( G \) as the smallest number \( f(v, G) \) such that from every placement of \( f(v, G) \) pebbles, it is possible to move a pebble to \( v \) by a sequence of pebbling moves. She also defined the \( t \)-pebbling number of \( v \) in \( G \) as the smallest number \( f_t(v, G) \) such that from every placement of \( f_t(v, G) \) pebbles, it is possible to move \( t \) pebbles to \( v \). Then the \( t \)-pebbling number of \( G \) is the smallest number \( f_t(G) \) such that from any placement of \( f_t(G) \) pebbles, it is possible to move \( t \) pebbles to any specified target by a sequence of pebbling moves. Thus \( f_t(G) \) is the largest value of \( f_t(v, G) \) over all vertices \( v \). The value of \( f_t(G) \) for \( t = 1 \) is the pebbling number of \( G \), denoted by \( f(G) \).

With regard to the \( t \)-pebbling number of graphs, we find the following theorems in [16, 17, 20, 21, 30].

**Theorem 3.1.1 ([30]).** Let \( G \) be a connected graph on \( n \) vertices where \( n \geq 2 \). Let there be a vertex \( v \) such that \( d(v) = n - 1 \). Then \( f_t(v, G) = 2t + n - 2 \).
Theorem 3.1.2 ([30]). Let $K_n$ be the complete graph on $n$ vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.

Theorem 3.1.3 ([16]). Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \ldots, u_{n-1})$ be a cycle of length $n - 1$. Then the $t$-pebbling number of the wheel graph $W_n$ is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.

In order to bring home the notation for complete $r$-partite graph given in [21] we recall Definition 1.0.16 again in Definition 3.1.4.

Definition 3.1.4:
A graph $G = (V, E)$ is called an $r$-partite graph if $V$ can be partitioned into $r$ non-empty subsets $V_1, V_2, \ldots, V_r$ such that no edge of $G$ joins vertices in the same set. The sets $V_1, V_2, \ldots, V_r$ are called partite sets or vertex classes of $G$.

If $G$ is an $r$-partite graph having partite sets $V_1, V_2, \ldots, V_r$ such that every vertex of $V_i$ is joined to every vertex of $V_j$, where $1 \leq i, j \leq r$ and $i \neq j$, then $G$ is called a complete $r$-partite graph. If $|V_i| = s_i$, for $i = 1, 2, \ldots, r$, then we denote $G$ by $K_{s_1, s_2, \ldots, s_r}$.

Notation 3.1.5 ([21]):
For $s_1 \geq s_2 \geq \cdots \geq s_r, s_1 > 1$ and if $r = 2, s_2 > 1$, let $K^*_{s_1, s_2, \ldots, s_r}$ be the complete $r$-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $C_1, C_2, \ldots, C_r$ respectively. Let $n = \sum_{i=1}^{r} s_i$. 


Theorem 3.1.6 ([21]). For $G = K_{s_1, s_2, \ldots, s_r}^*$, 

$$f_t(G) = \begin{cases} 
2t + n - 2, & \text{if } 2t \leq n - s_1 \\
4t + s_1 - 2, & \text{if } 2t \geq n - s_1
\end{cases}$$

Theorem 3.1.7 ([30]). Let $K_{1,n}$ be an $n$-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2.$

Theorem 3.1.8 ([30]). Let $C_n$ denote a simple cycle with $n$ vertices, where $n \geq 3$. Then 

$$f_t(C_n) = \begin{cases} 
t(2^{\frac{n}{2}}), & \text{if } n \text{ is even} \\
1 + (t - 1)(2^{\frac{n}{2}}) + 2 \left\lceil \frac{\sqrt{2} \left\lfloor \frac{n}{2} \right\rfloor}{3} \right\rceil - 1, & \text{if } n \text{ is odd}
\end{cases}$$

Theorem 3.1.9 ([30]). Let $P_n$ be a path on $n$ vertices. Then $f_t(P_n) = t(2^{n-1}).$

Theorem 3.1.10 ([30]). Let $Q_n$ be the $n$-cube. Then $f_t(Q_n) = t(2^n).$

For $t$-pebbling, the readers are invited to go through papers [16, 17, 20, 21, 28, 30].

Now, we determine the $t$-pebbling number of Jahangir graphs.

Theorem 3.1.11. For the Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.

Proof. Consider the Jahangir graph $J_{2,3}$. We prove this theorem by induction on $t$. By Theorem 2.2.3, the result is true for $t = 1$. For $t > 1$, $J_{2,3}$
contains at least 16 pebbles. Using at most 8 pebbles, we can put a pebble
on any desired vertex, say \( v_i \) (1 \( \leq \) \( i \) \( \leq \) 7), by Theorem 2.2.3. Then, the re-
maining number of pebbles on the vertices of the graph \( J_{2,3} \) is at least \( 8t - 8 \). By induction we can put \( (t - 1) \) additional pebbles on the desired vertex \( v_i \) (1 \( \leq \) \( i \) \( \leq \) 7). So, the result is true for all \( t \). Thus, \( f_t(J_{2,3}) \leq 8t \).

Now, we have to show that \( 8t \) pebbles are necessary for moving a pebble
to any desired vertex from any configuration. For that, consider the following
configuration \( C \) such that \( C(v_4) = 8t - 1 \), and \( C(x) = 0 \) where \( x \in V - \{v_4\} \),
then we cannot move \( t \) pebbles to the vertex \( v_1 \). Thus, \( f_t(J_{2,3}) \geq 8t \).

Therefore, \( f_t(J_{2,3}) = 8t \).

**Theorem 3.1.12.** For the Jahangir graph \( J_{2,4} \), \( f_t(J_{2,4}) = 16t \).

**Proof.** Consider the Jahangir graph \( J_{2,4} \). We prove this theorem by induc-
tion on \( t \). By Theorem 2.2.4, the result is true for \( t = 1 \). For \( t > 1 \), \( J_{2,4} \) contains at least 32 pebbles. By Theorem 2.2.4, using at most 16 pebbles,
we can put a pebble on any desired vertex, say \( v_i \) (1 \( \leq \) \( i \) \( \leq \) 9). Then, the
remaining number of pebbles on the vertices of the graph \( J_{2,4} \) is at least
\( 16t - 16 \). By induction, we can put \( (t - 1) \) additional pebbles on the desired
vertex \( v_i \) (1 \( \leq \) \( i \) \( \leq \) 9). So, the result is true for all \( t \). Thus, \( f_t(J_{2,4}) \leq 16t \).

Now, we have to show that \( 16t \) pebbles are necessary for moving a pebble
to any desired vertex from any configuration. For that, consider the following
configuration \( C \) such that \( C(v_6) = 16t - 1 \), and \( C(x) = 0 \) where \( x \in V - \{v_6\} \),
then we cannot move \( t \) pebbles to the vertex \( v_2 \). Thus, \( f_t(J_{2,4}) \geq 16t \).

Therefore, \( f_t(J_{2,4}) = 16t \).
**Theorem 3.1.13.** For the Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.

**Proof.** Consider the Jahangir graph $J_{2,5}$. We prove this theorem by induction on $t$. By Theorem 2.2.5, the result is true for $t = 1$. For $t > 1$, $J_{2,5}$ contains at least 34 pebbles. Using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i$ ($1 \leq i \leq 11$). Then, the remaining number of pebbles on the vertices of the graph $J_{2,5}$ is at least $16t - 14$. By induction, we can put $(t - 1)$ additional pebbles on the desired vertex $v_i$ ($1 \leq i \leq 11$). So, the result is true for all $t$. Thus, $f_t(J_{2,5}) \leq 16t + 2$.

Now, we have to show that $16t + 2$ pebbles are necessary for moving a pebble to any desired vertex from any configuration. For that, consider the following distribution $C$ such that $C(v_6) = 16t - 1$, $C(v_8) = 1$, $C(v_{10}) = 1$ and $C(x) = 0$ where $x \in V - \{v_6, v_8, v_{10}\}$. Then we cannot move $t$ pebbles to the vertex $v_2$. Thus, $f_t(J_{2,5}) \geq 16t + 2$.

Therefore, $f_t(J_{2,5}) = 16t + 2$. $\blacksquare$

**Theorem 3.1.14.** For the Jahangir graph $J_{2,m}$ ($m \geq 6$), $f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m})$.

**Proof.** Consider the Jahangir graph $J_{2,m}$, where $m > 5$. We prove this theorem by induction on $t$. By Theorem 2.2.6, Theorem 2.2.7, and Theorem 2.3.1, the result is true for $t = 1$. For $t > 1$, $J_{2,m}$ contains at least $16 + f(J_{2,m}) = 16 + \begin{cases} 2m + 9, & \text{if } m = 6 \text{ or } 7 \\ 2m + 10, & \text{if } m \geq 8 \end{cases}$ pebbles. Using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i$ ($1 \leq i \leq 2m + 1$). Then, the remaining number of pebbles on the vertices of the graph $J_{2,m}$ is at
least $16t + f(J_{2,m}) - 32$. By induction, we can put $(t - 1)$ additional pebbles on the desired vertex $v_i$ ($1 \leq i \leq 2m + 1$). So, the result is true for all $t$. Thus, $f_t(J_{2,m}) \leq 16(t - 1) + f(J_{2,m})$.

Now, we have to show that $16(t - 1) + f(J_{2,m})$ pebbles are necessary for moving a pebble to any desired vertex from any configuration.

For $m = 6$, consider the following distribution $C$ such that $C(v_6) = 16(t - 1) + 15$, $C(v_{10}) = 3$, $C(v_8) = 1$, $C(v_{12}) = 1$ and $C(x) = 0$ where $x \in V - \{v_6, v_8, v_{10}, v_{12}\}$.

For $m = 7$, consider the following distribution $C$ such that $C(v_6) = 16(t - 1) + 15$, $C(v_{10}) = 3$, $C(v_8) = C(v_{12}) = C(v_{13}) = C(v_{14}) = 1$ and $C(x) = 0$ where $x \in V - \{v_6, v_8, v_{10}, v_{12}, v_{13}, v_{14}\}$.

For $m \geq 8$, if $m$ is even, consider the following distribution $C_1$ such that $C_1(v_{m+2}) = 16(t - 1) + 15$, $C_1(v_{m-2}) = 3$, $C_1(v_{m+6}) = 3$, $C_1(x) = 1$ where $x \notin \{N[v_2], N[v_{m+2}], N[v_{m-2}], N[v_{m+5}]\}$ and $C_1(y) = 0$ where $y \in \{N[v_2], N(v_{m+2}), N(v_{m-2}), N(v_{m+6})\}$.

If $m$ is odd, then consider the following configuration $C_2$ such that $C_2(v_{m+1}) = 16(t - 1) + 15$, $C_2(v_{m-3}) = 3$, $C_2(v_{m+5}) = 3$, $C_2(x) = 1$ where $x \notin \{N[v_2], N[v_{m+1}], N[v_{m-3}], N[v_{m+5}]\}$ and $C_2(y) = 0$ where $y \in \{N[v_2], N(v_{m+1}), N(v_{m-3}), N(v_{m+5})\}$.

Then, we cannot move $t$ pebbles to the vertex $v_2$ of $J_{2,m}$ for all $m \geq 6$.

Thus, $f_t(J_{2,m}) \geq 16(t - 1) + \begin{cases} 2m + 9, & \text{if } m = 6 \text{ or } 7 \\ 2m + 10, & \text{if } m \geq 8 \end{cases}$.

That is, $f_t(J_{2,m}) \geq 16(t - 1) + f(J_{2,m})$.

Therefore, $f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m})$. \hfill \blacksquare
Clarke, Hochberg and Hurlbert [4] defined the concept of pebbling number of a graph through the concept of bad pebbling distribution as follows:

Suppose $D$ is a distribution of pebbles on the vertices of $G$ and there is some choice of a root vertex $r$ such that it is impossible to move a pebble to $r$, then we say that $D$ is a bad pebbling distribution [4]. We denote by $D(v)$ the number of pebbles on a vertex $v$ in $D$ and let $|D|$ be the total number of pebbles in $D$, that is, $|D| = \sum_{v \in V(G)} D(v)$. This yields another way to define $f(G)$ as one more than the maximum $k$ such that there exists a bad pebbling distribution $D$ of size $k$. We generalize this in the setting of $t$-pebbling and we define the $t$-pebbling number of a graph in a similar way.

We give alternate proof for the $t$-pebbling number of cycles.

If $D_t$ is a distribution of pebbles on the vertices of $G$ and there is some choice of a vertex $r$ ($r$ is any specified root vertex or target vertex) such that it is impossible to move $t$ pebbles to $r$ then we say that $D_t$ is a bad $t$-pebbling distribution. We denote $D_t(v)$ the number of pebbles on vertex $v$ in $D_t$ and let $|D_t|$ be the total number of pebbles in $D_t$, that is $|D_t| = \sum_{v \in V(G)} D_t(v)$.

**Definition 3.2.1:**

We define the $t$-pebbling number of a graph $G$, $f_t(G)$, to be one more than the maximum $k$ such that there exists a bad $t$-pebbling distribution $D_t$ of size $k$.

We now proceed to give alternate proofs for the $t$-pebbling numbers of even and odd cycles.
Theorem 3.2.2. The pebbling number of $C_{2k}$ satisfies $f_t(C_{2k}) = t2^k$.

Proof. Let $V(C_{2k}) = \{x_0, x_1, \ldots, x_{2k-2}, x_{2k-1}\}$ and $E(C_{2k}) = \{x_0x_1, x_1x_2, \ldots, x_{2k-1}x_0\}$.

Without loss of generality we assume that $x_0$ is the target vertex. Clearly $\text{diam}(C_{2k}) = d(x_0, x_k) = k$ so $x = t2^k - 1$ is the maximum number of pebbles that can be placed at $x_k$ such that $t$ pebbles could not be moved to $x_0$. Suppose $x$ pebbles are placed at $x_k$. Since $\text{diam}(C_{2k}) = k$, if a pebble is placed on any vertex of $C_{2k}$, it would be possible to put $t$ pebbles on $x_0$ with $x$ pebbles at $x_k$.

Since the $t$-pebbling number is one more than the maximum numbers of pebbles that can be placed on the graph such that $t$-pebbles could not be moved to every vertex in the graph, the $t$-pebbling number is $x + 1$.

Theorem 3.2.3. The $t$-pebbling number of $C_{2k+1}$ satisfies

$$f_t(C_{2k+1}) = \frac{2^{k+1} - (-1)^{k+2}}{3} + (t - 1)2^k.$$

Proof. Let $V(C_{2k+1}) = \{x_0, x_1, \ldots, x_{2k-1}, x_{2k}\}$.

Let $E(C_{2k+1}) = \{x_0x_1, x_1x_2, \ldots, x_{2k-1}x_{2k}, x_{2k}x_0\}$.

Let $x_0$ be the target vertex.

Clearly $\text{diam}(C_{2k+1}) = d(x_0, x_k) = d(x_0, x_{k+1}) = k$.

Let us consider a configuration of $y$ pebbles at $x_k$ and $z$ pebbles at $x_{k+1}$ such that

$$y + \frac{z}{2} < t(2^k) \quad (3.1)$$
and \( \frac{y}{2} + z < t(2^k) \) \hspace{1cm} (3.2)

Clearly we cannot put \( t \)-pebbles on \( x_0 \). We also observe that if we place an additional pebble on any vertex of \( C_{2k+1} \), we can put \( t \) pebbles on \( x_0 \). Then for the above configuration the inequalities would become

\[
2y + z \leq t(2^{k+1}) - 1, \\
y + 2z \leq t(2^{k+1}) - 1.
\]

So we get \( y + z \leq \left\lfloor \frac{t(2^{k+2} - 2)}{3} \right\rfloor \).

Thus the maximum number of pebbles placed altogether on \( x_k \) and \( x_{k+1} \) in order that we may not put \( t \) pebbles on \( x_0 \) is \( \left\lfloor \frac{t(2^{k+2}) - 2}{3} \right\rfloor \).

Since the \( t \)-pebbling number is one more than the maximum number of pebbles placed on \( C_{2k+1} \) such that \( t \) pebbles could not be moved to every vertex in the graph, the \( t \)-pebbling number of \( C_{2k+1} \) is

\[
f_t(C_{2k+1}) = \left\lfloor \frac{t(2^{k+2}) - 2}{3} \right\rfloor + 1
\]

\[
= \left\lfloor \frac{2^{k+2} - 2}{3} + (t - 1)\frac{2^{k+2}}{3} \right\rfloor + 1
\]

\[
= \begin{cases} 
\left\lfloor \frac{2^{k+2} - 2}{3} + (t - 1)\frac{2^{k+2}}{3} \right\rfloor + 1 & \text{if } k \text{ is odd} \\
\left\lfloor \frac{2^{k+2} - 4}{3} + (t - 1)\left( \frac{2^{k+2}}{3} + \frac{2}{3} \right) \right\rfloor + 1 & \text{if } k \text{ is even}
\end{cases}
\]

\[
= \begin{cases} 
\left\lfloor \frac{2^{k+2} - 2}{3} \right\rfloor + 1, & \text{if } k \text{ is odd} \\
\left\lfloor \frac{2^{k+2} - 4}{3} \right\rfloor + 1, & \text{if } k \text{ is even}
\end{cases}
\]
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\[ \begin{align*}
&= \begin{cases} 
\frac{2^{k+2}-2}{3} + (t - 1)2^k + 1, & \text{if } k \text{ is odd} \\
\frac{2^{k+2}-4}{3} + (t - 1)2^k + 1, & \text{if } k \text{ is even}
\end{cases} \\
&= \begin{cases} 
\frac{2^{k+2}-2+3}{3} + (t - 1)2^k, & \text{if } k \text{ is odd} \\
\frac{2^{k+2}-4+3}{3} + (t - 1)2^k, & \text{if } k \text{ is even}
\end{cases} \\
&= \begin{cases} 
\frac{2^{k+2}+1}{3} + (t - 1)2^k, & \text{if } k \text{ is odd} \\
\frac{2^{k+2}-1}{3} + (t - 1)2^k, & \text{if } k \text{ is even}
\end{cases} \\
&= \frac{2^{k+2} - (-1)^{k+2}}{3} + (t - 1)2^k
\end{align*} \]

\section{3.3 The 2t-pebbling property of Cycles}

Chung defined the two pebbling property of a graph, and Wang \cite{37} extended Chung’s definition to the odd two-pebbling property. In [3, 37] we find the following definitions.

**Definition 3.3.1** ([3]):

Suppose \( p \) pebbles are placed on a graph \( G \) in such a way that \( q \) vertices of \( G \) are occupied, that is, there are exactly \( q \) vertices which have one pebble or more. We say the graph \( G \) satisfies the 2-pebbling property if we can put two pebbles on any specified vertex of \( G \) starting from every configuration in which \( p \geq 2f(G) - q + 1 \) or equivalently \( (p + q) > 2f(G) \).

**Definition 3.3.2** ([37]):

Suppose \( p \) pebbles are placed on a graph \( G \) in such a way that there are exactly \( r \) vertices which have odd number of pebbles. We say the graph \( G \) satisfies the
odd 2-pebbling property if we can put two pebbles on any specified vertex of $G$ starting from every configuration in which $p \geq 2f(G) - r + 1$ or equivalently $(p + r) > 2f(G)$.

Let us now generalize these concepts.

**Definition 3.3.3 ([17]):**
Given the $t$-pebbling of $G$, let $p$ be the number of pebbles on $G$, let $q$ be the number of vertices with at least one pebble. We say that $G$ satisfies the $2t$-pebbling property if it is possible to move $2t$ pebbles to any specified target vertex of $G$ starting from every configuration in which $p \geq 2f_t(G) - q + 1$ or equivalently $p + q > 2f_t(G)$ for all $t$.

If $q$ stands for the number of vertices with an odd number of pebbles, we call the property, the odd $2t$-pebbling property.

**Definition 3.3.4 ([17]):**
We say a graph satisfies the odd $2t$-pebbling property for all $t$. If, for any arrangement of pebbles with at least $2f_t(G) - r + 1$ pebbles, where $r$ is the number of vertices in the arrangement with an odd number of pebbles, it is possible to put $2t$ pebbles on any target vertex using pebbling moves.

It is easy to see that a graph which satisfies the $2t$-pebbling property also satisfies the odd $2t$-pebbling property for all $t$.

For $t = 1$, Definition 3.3.3 gives the two pebbling property and Definition 3.3.4 gives the odd two-pebbling property.

In this section let us discuss the $2t$-pebbling property of cycles.
Notation 3.3.5:
Let the vertices of $C_m$ be $\{x_0, x_1, \ldots, x_{m-1}\}$ in order.

Without loss of generality, assume $x_0$ is the target vertex in $C_m$. Given a configuration of pebbles on $C_m$, we let $p_i$ represent the number of pebbles on $x_i$. If $m$ is even, we suppose $m = 2k$, and if $m$ is odd, we let $m = 2k + 1$. In either case, we define the vertex sets $A$ and $B$ by $A = \{x_1, x_2, \ldots, x_{k-1}\}$; $B = \{x_{m-1}, x_{m-2}, \ldots, x_{m-k+1}\}$. Let $p_A$ and $p_B$ denote the number of pebbles on $A$ and $B$ respectively.

![Figure 3.1: Even and odd cycles.](image)

Lemma 3.3.6. Let $S$ be a subset of vertices in a graph $G$. Suppose we have a configuration on $G$ with $q$ occupied vertices in which each $x_i$ has $p_i$ pebbles. Then

$$\sum_{x_i \in S} p_i + \sum_{x_i \notin S} p_i \geq p + 2 - |S|.$$
Proof. We can rewrite the left side of the inequality as

\[ \sum_{x_i \in S} p_i + \sum_{x_i \notin S} p_i + \sum_{x_i \notin S} p_i \geq p + \sum_{x_i \notin S} p_i \]

So it is sufficient to show \( p + \sum_{x_i \notin S} p_i \geq p + q - |s| \).

That is, it is enough to show that \( \sum_{x_i \notin S} p_i \geq q - |s| \).

That is, it is enough to show that \( q \leq \sum_{x_i \notin S} p_i + |s| \).

We observe that on the right hand side, all the vertices in \( S \) are counted (occupied or not), and each occupied \( x_i \) not in \( S \) is counted \( p_i \) times, \( p_i \geq 1 \).

Hence each occupied vertex is counted at least once, whether it is in \( S \) or not, and so the total is at least \( q \).

\[ \sum_{x_i \notin S} p_i \geq p + q - |s| \]

\[ q \leq \sum_{x_i \notin S} p_i + |s| \]

\[ q \leq \sum_{x_i \notin S} p_i + |s| \]

\[ q \leq |V(C)| \leq f(C) \]

**Theorem 3.3.7.** All cycles satisfy the 2t-pebbling property if the target vertex has a pebble.

**Proof.** Consider a configuration of pebbles on \( C \) in which \( p \) pebbles occupy \( q \) vertices where \( p + q \geq 2f_t(C) + 1 \).

Then \( p + q \geq 2f(C) + 2^{k+1}(t - 1) + 1 \).

That is, \( p \geq f(C) + 2^{k+1}(t - 1) + 1 \) where the inequality follows because \( q \leq |V(C)| \leq f(C) \).

**Case 1:** \( t = 1 \).

Then \( p \geq f(C) + 1 \).

If \( x_0 \) has a pebble the remaining \( f(C) \) pebbles would suffice to put an additional pebble on \( x_0 \).

**Case 2:** \( t > 1 \).

Then regardless of whether cycle is even or odd, there are at least \( (t - 1) \) pebbles available on \( x_0 \).
1) \(2^{k+1}\) pebbles on the graph.

We claim that there are at least \((t-1)2^{k+1}\) pebbles on \(A \cup \{x_0, x_k\}\), which is isomorphic to \(p_{k+1}\). Suppose not.

Then there are at most \((t-1)2^k\) pebbles on \(A \cup \{x_0, x_k\}\) and there are at most \((t-1)2^k\) pebbles on the other path. So there are at most \((t-1)2^{k+1}\) pebbles on both paths, and that there are at most \((t-1)2^{k+1}\) pebbles on \(C\). This is a contradiction to \(p \geq f(C) + (t-1)2^{k+1} + 1\).

Hence there are at least \((t-1)2^k\) pebbles on \(A \cup \{x_0, x_k\}\). So we can move \(2(t-1)\) pebbles to \(x_0\). Using the remaining \(f(C)\) pebbles we can move an additional pebble to \(x_0\). Therefore, the number of pebbles on \(x_0\) is \(2t\).

Lourdusamy [17] has proved that even cycles satisfy the \(2t\)-pebbling property. We now give an alternate proof for the result.

**Theorem 3.3.8.** The even cycle \(C_{2k}\) satisfies the \(2t\)-pebbling property for all \(k \geq 2\).

**Proof.** Consider a configuration of pebbles on \(C_{2k}\) in which \(p\) pebbles occupy \(q\) vertices where \((p + q) \geq 2f_t(C_{2k}) + 1\).

Since \(q \leq |V(C_{2k})| \leq f(C_{2k}) \leq f_t(C_{2k})\), we get

\[ p + q \geq q + f_t(C_{2k}) + 1 \]

That is, \(p \geq f_t(C_{2k}) + 1\)

That is, \(p \geq t(2^k)\).

By Theorem 3.3.7, we may assume that the target vertex \(x_0\) is not occupied. So \(p_0 = 0\).

We claim that we can move at least \(t\) pebbles to \(x_0\) using only the pebbles
of either $A \cup \{x_k\}$ or $B \cup \{x_k\}$.

Suppose not, then

\[
\frac{p_k - 1}{2} + p_A \leq t(2^{k-1}) - 1
\]
\[
\frac{p_k - 1}{2} + p_B \leq t(2^{k-1}) - 1
\]

For, if one of these inequalities fails, we could put $t(2^{k-1})$ pebbles on either $A$ or $B$ and these would be sufficient to put $t$ pebbles on $x_0$. Now adding together these inequalities we get

\[
p_k - 1 + p_A + p_B \leq t(2^k) - 2
\]

That is, $p_k + p_A + p_B \leq t(2^k) - 1$.

That is, $p < t(2^k)$.

This is a contradiction to $p > t(2^k)$.

Hence without loss of generality we assume that the pebbles on $A \cup \{x_k\}$ are sufficient to put $t$ pebbles on $x_0$. We break the rest of the proof into two cases. Either the pebbles on $A$ are sufficient by themselves to put $t$ pebbles on $x_0$ or the pebbles on $x_k$ are required as well to put $t$ pebbles on $x_0$.

**Case 1:** Suppose the pebbles on $A$ are sufficient to put $t$ pebbles on $x_0$.

We then try to use the pebbles on $B \cup \{x_k\}$ to put $t$ more pebbles on $x_0$. If the pebbles on $B \cup \{x_k\}$ are not sufficient to put $t$ more pebbles on $x_0$ then,

\[
p_k + 2p_{k+1} + 4p_{k+2} + \cdots + 2^{k-1}p_{2k-1} \leq t(2^k) - 1
\] (3.3)

Since we have assumed the pebbles on $A$ are sufficient to put $t$ pebbles
on \( x_0 \), we can use the pebbles on \( A \cup \{x_k\} \) to put \( 2t \) pebbles on \( x_0 \) unless

\[
p_k + 2p_{k-1} + 4p_{k-2} + \cdots + 2^{k-1}p_1 \leq t(2^{k+1}) - 1 \quad (3.4)
\]

Adding (3.3) and (3.4) we get,

\[
2^{k-1}p_1 + 2^{k-2}p_2 + \cdots + 2^2p_{k-2} + 2p_{k-1} + p_k + p_k + 2p_{k+2} + \cdots + 2^{k-1}p_{2k-1} \leq 2(2^{k-1} + 2^k - 1).
\]

Now dividing by two we get an inequality in which each pebble is counted once and every pebble except for those on \( x_{k-1}, x_k \) and \( x_{k+1} \) is counted at least twice. Thus we have,

\[
2p_1 + \cdots + 2p_{k-2} + p_k + p_{k+1} + 2p_{k+2} + \cdots + 2p_{2k-1} \leq 3t(2^{k-1}) - 1
\]

Applying Lemma 3.3.6 with \( S = \{x_{k-1}, x_k, x_{k+1}\} \) we find the left side is at least as large as \( p + q - 3 \).

Therefore, we can move \( 2t \) pebbles to \( x_0 \) unless

\[
p + q \leq 3t(2^{k-1}) + 2.
\]

That is, \( 2f_t(C_{2k}) + 1 \leq 3t(2^{k-1}) + 2 \).

That is, \( t(2^{k+1}) + 1 \leq 3t(2^{k-1}) + 2 \).

This is a contradiction to \( k \geq 2 \).

Hence \( 2t \) pebbles can be moved to \( x_0 \).
Case 2: The pebbles on $x_k$ are required as well to put $t$ pebbles on $x_0$.

Apart from the pebbles of $A$, if the pebbles on $x_k$ are required for the first $t$ pebbles, we transfer enough pebbles from $x_k$ to $A$ to put $t$ pebbles on $x_0$. This costs at most $t(2^k) - 2p_A$ pebbles from $x_k$. We then use the remaining pebbles on $x_k$ along with the pebbles on $B$ to put $t$ more pebbles on $x_0$. We can do this as long as

$$p_k - (t(2^k) - 2p_A) + 2p_B \geq t(2^k) \quad \text{or}$$

$$p_k + 2p_A + 2p_B \geq t(2^{k+1})$$

Applying Lemma 3.3.6, $S = \{x_k\}$, the left side of this inequality is at least $p + q - 1 \geq 2f_t(C_{2k}) + 1 - 1 = t(2^{k+1})$ which is always true. Hence we can move $2t$ pebbles to $x_0$.

We now proceed to prove that the odd cycles satisfy the $2t$-pebbling property. First we will show that $C_3$ possesses this property.

**Theorem 3.3.9.** The cycle $C_3$ satisfies the $2t$-pebbling property.

**Proof.** Consider a configuration of pebbles on $C_3$ in which $p$ pebbles occupy $q$ vertices where $(p + q) \geq 2f_t(C_3) + 1$.

By Theorem 3.3.7, we may assume that $p_0 = 0$.

First we prove the result for $t = 1$. We claim that either we can put 2 pebbles on $x_0$ using the pebbles on $\{x_1, x_2\}$ or we can put a pebble on $x_0$ using the pebbles on $x_1$. Suppose not.

Then $p_1 + 2p_2 \leq 7$. 

Also \( p_1 \leq 1 \).

Adding and dividing by two, we get \( p_1 + p_2 \leq 4 \).

Applying Lemma 3.3.6, with \( S = \{ x_1, x_2 \} \).

We see that \( p + q - 2 \leq 4 \).

That is, \( p + q \leq 6 \).

But \( p + q \geq 2f(C_3) + 1 = 7 \).

Thus we get \( 7 \leq p + q \leq 6 \).

This is a contradiction.

Thus we can put two pebbles on \( x_0 \) in all cases by using either pebbles on \( \{ x_1, x_2 \} \) or by using the pebbles on \( x_1 \) for the first pebble and those on \( x_2 \) for the second pebble.

Let us now prove theorem, for \( t > 1 \). If \( q = 1 \), then \( p \geq 2(2t + 1) \). These pebbles are placed either at \( x_1 \) or \( x_2 \). Then it is easy to see that we can move \( 2t \) pebbles to \( x_0 \). Let us now consider the case \( q = 2 \).

In this case we distribute at least \( 4t + 1 \) pebbles on \( x_1 \) and \( x_2 \). Let us now assume that we distribute exactly \( 4t + 1 \) pebbles on \( x_1 \) and \( x_2 \). In distributing these \( 4t + 1 \) pebbles, we may assume without loss of generality, that the different choices of pebbles for \( x_1 \) are \( 1, 2, \ldots, t, t + 1, \ldots, 2t \) and the corresponding choices of pebbles for \( x_2 \) are \( 4t, 4t - 1, \ldots, 3t + 1, 3t, 3t - 1, \ldots, 2t + 1 \). We observe that the choices of pebbles for \( x_1 \) are of two clumps each of length \( t \). Similarly the choices of pebbles for \( x_2 \) are also of two clumps each of length \( t \). The pairs of pebbles on \( (x_1, x_2) \) are \( (p_1, p_2) = (1, 4t), (2, 4t - 1), \ldots, (t - 3, 3t), (t, 3t + 1), (t + 1, 3t), (t + 2, 3t - 1), \ldots, (2t, 2t + 1) \).

Now it is easy to see that from the choices of pebbles on \( (x_1, x_2) \), we can move \( 2t \) pebbles to \( x_0 \).
Hence $C_3$ satisfies the $2t$-pebbling property for all $t$. ■

We now prove that all odd cycles other than $C_3$ also satisfy the $2t$-pebbling property.

**Theorem 3.3.10.** The cycle $C_{2k+1}$ satisfies the $2t$-pebbling property for all $t$ where $k \geq 2$.

**Proof.** Consider a configuration of pebbles on $C_{2k+1}$ in which $p$ pebbles occupy $q$ vertices where $(p + q) \geq 2f_t(C_{2k+1}) + 1$.

By Theorem 3.3.7, we may assume that $x_0$ has zero pebbles.

If either,

\[
p_k + 2p_{k+1} + \cdots + 2^k p_{2k} \geq t(2^{k+2}) \quad \text{or} \quad p_{k+1} + 2p_k + \cdots + 2^k p_1 \geq t(2^{k+2})
\]

we can use either the pebbles of $B \cup \{x_{k+1}, x_k\}$ or $A \cup \{x_k, x_{k+1}\}$ to put $2t$ pebbles on $x_0$. For other configurations of pebbles we note that we can move at least $t$ pebbles to $x_0$ using only the pebbles of either $A \cup \{x_k\}$ or $B \cup \{x_{k+1}\}$. Otherwise we have

\[
p_k + 2p_{k-1} + \cdots + 2^{k-1} p_1 \leq t(2^k) - 1,
\]

\[
p_{k+1} + 2p_{k+2} + \cdots + 2^{k-1} p_{2k} \leq t(2^k) - 1.
\]

Adding these inequalities together we get,

\[2^{k-1} p_1 + \cdots + 2p_{k-1} + p_k + p_{k+1} + 2p_{k+2} + \cdots + 2^{k-1} p_{2k} \leq t(2^{k+1}) - 2.\]
Applying Lemma 3.3.6, with $S = \{x_k, x_{k+1}\}$, we see that the left hand side is at least $p + q - 2$. So we get $p + q - 2 \leq t(2^{k+1}) - 2$.

Substituting $(p + q) \geq 2f_t(C_{2k+1}) + 1$, we get

$$2 \left( \frac{2^{k+2} - (-1)^{k+2}}{3} + (t - 1)2^k \right) + 1 \leq t(2^{k+1}).$$

This gives $2^{k+3} + 3 \leq 2(-1)^{k+2} + 3(2^{k+1})$, which is a contradiction to $k \geq 2$. Hence we can move at least $t$ pebbles to $x_0$ using only the pebbles of either $A \cup \{x_k\}$ or $B \cup \{x_{k+1}\}$.

Hence without loss of generality, we assume the pebbles on $A \cup \{x_k\}$ are sufficient to put at least $t$ pebbles on $x_0$. We now have two cases: either the pebbles on $A$ are sufficient by themselves or the pebbles on $x_k$ are required as well.

**Case 1**: Suppose the pebbles of $A$ are sufficient by themselves for the first $t$ pebbles.

Then we can use the pebbles of $B \cup \{x_{k+1}, x_k\}$ to put $t$ additional pebbles on $x_0$. This succeeds unless

$$p_k + 2p_{k+1} + 2^2p_{k+2} + \cdots + 2^kp_{2k} \leq t(2^{k+1}) - 1 \quad (3.5)$$

Similarly we can use the pebbles of $A \cup \{x_k\}$ to put $2t$ pebbles on $x_0$ unless,

$$p_k + 2p_{k-1} + \cdots + 2^{k-1}p_1 \leq t(2^{k+1}) - 1 \quad (3.6)$$
Adding together these inequalities (3.5), (3.6) and dividing by two gives,

\[ 2^{k-2}p_1 + 2^{k-3}p_2 + \cdots + 2p_{k-2} + p_{k-1} + p_k + p_{k+1} + 2p_{k+2} + \cdots + 2^{k-1}p_{2k} \leq t(2^{k+1}) - 1. \]

Applying Lemma 3.3.6 with \( S = \{x_{k-1}, x_k, x_{k+1}\} \), we see that the left side is at least \( p + q - 3 \). So we get

\[ p + q - 3 \leq t(2^{k+1}) - 1. \]

Substituting \( p + q \geq 2f_t(C_{2k+1}) + 1 \), we get

\[ 2 \left( \frac{2^{k+2} - (-1)^{k+2}}{3} + (t - 1)2^k \right) + 1 \leq t(2^{k+1}) + 2 \]

This gives, \( 2^{k+3} + 3 \leq 2(-1)^{k+2} + 3(2^{k+1}) + 2 \), which is a contradiction to \( k \geq 2 \).

**Case 2:** The pebbles on \( x_k \) are required as well for the first \( t \)-pebbles.

If the pebbles on \( x_k \) are required for the first \( t \) pebbles, we transfer enough pebbles from \( x_k \) to \( A \) to put \( t \) pebbles on \( x_0 \). This costs at most \( t(2^k) - (2p_{k-1} + 2^2p_{k-2} + \cdots + 2^{k-1}p_1) \) pebbles from \( x_k \). If the pebbles of \( B \) are sufficient to put \( t \) additional pebbles then we are done. Otherwise, we can use the remaining pebbles on \( x_k \) along with the pebbles on \( B \cup \{x_{k+1}\} \) to put \( t \) additional pebbles on \( x_0 \). We can do this as long as

\[ p_k - (t2^k - (2p_{k-1} + 2^2p_{k-2} + \cdots + 2^{k-1}p_1)) + 2p_{k+1} + 2^2p_{k+2} + \cdots + 2^{k-1}p_{2k} \geq t(2^{k+1}). \]
That is,

\[ p_k + 2p_{k-1} + 2^2p_{k-2} + \cdots + 2^{k-1}p_1 \]
\[ + 2p_{k+1} + 2^2p_{k+2} + \cdots + 2^k p_{2k} \geq t(2^{k+1} + 2^k) \]  \hspace{1cm} (3.7)

Also,

\[ p_{k+2} + 2p_{k+3} + \cdots + 2^{k-2}p_{2k} \leq t(2^{k-1}) - 1. \]

That is,

\[ -2p_{k+2} - 2^2p_{k+3} - \cdots - 2^{k-1}p_{2k} \geq -t(2^k) + 2 \]  \hspace{1cm} (3.8)

Adding (3.7) and (3.8) we get,

\[ 2^{k-1}p_1 + 2^{k-2}p_2 + \cdots + 2^{2p_{k-2}} + 2p_{k-1} + p_k \]
\[ + 2p_{k+1} + 2p_{k+2} + 2^2p_{k+3} + \cdots + 2^{k-1}p_{2k} \geq t(2^{k+1}) + 2 \]

Applying Lemma 3.3.6, with \( S = \{x_k\} \), we see that the left side is at least \( p + q - 1 \).

But \( p + q \geq 2f_t(C_{2k+1}) + 1. \)

Hence it is enough to prove that

\[ 2f_t(C_{2k+1}) \geq t(2^{k+1}) + 2. \]

That is,

\[ 2 \left( \frac{2^{k+2} - (-1)^{k+2}}{3} + (t - 1)2^k \right) \geq t(2^{k+1}) + 2. \]

That is, \( 2^{k+1} \geq 2(-1)^{k+2} + 6 \), which is true for all \( k \geq 2. \)
Hence if the pebbles on $x_k$ are required for the first $t$ pebbles then the remaining pebbles on $x_k$ along with the pebbles of $B \cup \{x_{k+1}\}$ can be used to put $t$ additional pebbles on $x_0$.

Otherwise, the pebbles of $x_k$ along with the pebbles of $A$ are sufficient to put $2(t-1)$ pebbles on $x_0$, and the remaining pebbles on $x_k$ along with the pebbles of $B \cup \{x_{k+1}\}$ can be used to put two pebbles on $x_0$. That is, we keep moving pebbles from $x_k$ to $A$ till we get enough pebbles to put $2(t-1)$ pebbles on $x_0$ and it can be easily seen that the remaining pebbles on $x_k$ along with the pebbles of $B \cup \{x_{k+1}\}$ can be used to put two pebbles on $x_0$. Hence the cycle $C_{2k+1}$ satisfies the $2t$-pebbling property for all $t$ where $k \geq 2$.

Hence we have proved that all cycles satisfy the $2t$-pebbling property. Thus it follows that all cycles satisfy the odd $2t$-pebbling property.

### 3.4 The $t$-pebbling conjecture of product of a path with a cycle

We now discuss some results on the $t$-pebbling number of direct product of two graphs. In direct product of two graphs we take the following notation from [9].

**Notation 3.4.1 ([9]):**

We write $\{x\} \times H$ (respectively $G \times \{y\}$) for the subgraph of vertices whose projection onto $V_G$ is the vertex $x$ (respectively whose projection onto $V_H$ is $y$).

If the vertices of $G$ are labeled $x_i$ then for any distribution of pebbles on $G \times H$,
we write $p_i$ for the number of pebbles on $\{x_i\} \times H$ and $q_i$ for the number of occupied vertices of $\{x_i\} \times H$.

Chung [3] credited Conjecture 3.4.2 to Graham.

**Conjecture 3.4.2 (Graham):**
For any connected graphs $G$ and $H$, we have $f(G \times H) \leq f(G)f(H)$ where $G \times H$ represents the direct product of graphs.

Lourdusamy [17] generalized Graham’s conjecture as follows:

**Conjecture 3.4.3 (The $t$-pebbling Conjecture):**
For any connected graphs $G$ and $H$, we have $f_t(G \times H) \leq f(G)f_t(H)$ where $G \times H$ represents the direct product of graphs $G$ and $H$.

We take Lemma 3.4.4 from [9]. It describes how many pebbles we can transfer from one copy of $H$ to an adjacent copy of $H$ in $G \times H$. It is also called transfer Lemma.

**Lemma 3.4.4 (Transfer Lemma).** Let $(x_i, x_j)$ be an edge in $G$. Suppose that in $G \times H$, we have $p_i$ pebbles occupying $q_i$ vertices of $\{x_i\} \times H$. If $(q_i - 1) \leq k \leq p_i$ and if $k$ and $p_i$ have the same parity then $k$ pebbles can be retained on $\{x_i\} \times H$ while moving $\frac{p_i - k}{2}$ pebbles onto $\{x_j\} \times H$. If $k$ and $p_i$ have opposite parity we must leave $k + 1$ pebbles on $\{x_i\} \times H$, so we can only move $\frac{p_i - (k + 1)}{2}$ pebbles onto $\{x_j\} \times H$. In particular we can always move at least $\frac{p_i - q_i}{2}$ pebbles onto $\{x_j\} \times H$. 


In [17], Lourdusamy has proved that the $t$-pebbling conjecture is true for product of a path with a graph satisfying the $2t$-pebbling property.

We will now give his theorem with proof here, for easy reading of thesis.

**Theorem 3.4.5** ([17]). Let $P_m$ be a path on $m$ vertices. When $G$ satisfies the $2t$-pebbling property, $f_t(P_m \times G) \leq 2^{m-1} f_t(G)$.

**Proof.** Let $P_m = \{x_1, x_2, \ldots, x_m\}$ be a path on $m$ vertices. The proof is by induction on $m$. For $m = 1$, theorem is trivial. Let $y \in G$.

Let $(x_1, y)$ be the target vertex. By Lemma 3.4.4, we can transfer $\frac{p_m - q_m}{2}$ pebbles from $\{x_m\} \times G$ to $\{x_{m-1}\} \times G$. If $p_1 + p_2 + p_3 + \cdots + p_{m-1} + \frac{p_m - q_m}{2} \geq 2^{m-2} f_t(G)$, then we can use induction to put $f_t(G)$ pebbles on $\{x_1\} \times G$. We can then put $t$ pebbles on $(x_1, y)$ since $\{x_1\} \times G$ is isomorphic to $G$. Also, since $G$ satisfies the $2t$-pebbling property, if $\frac{p_m + q_m}{2} > 2^{m-2} f_t(G)$, then we can put $t(2^{m-1})$ pebbles on $(x_m, y)$ and then we can put $t$ pebbles on $(x_1, y)$ using the path $P_{m-1}$. Hence the only distributions from which we cannot $t$-pebble the target satisfy the inequalities

$$p_1 + p_2 + \cdots + p_{m-1} + \frac{p_m - q_m}{2} < 2^{m-2} f_t(G)$$

$$\frac{p_m + q_m}{2} < 2^{m-2} f_t(G)$$

But adding these together we get that $p_1 + p_2 + \cdots + p_m < 2^{m-1} f_t(G)$. Thus some configuration of pebbles from which we cannot $t$-pebble $(x_1, y)$ must begin with fewer than $2^{m-1} f_t(G)$ pebbles. By symmetry, we can $t$-pebble $(x_m, y)$ using any configuration of $2^{m-1} f_t(G)$ pebbles on $P_m \times G$.

Now let $(x_i, y)$ be the target vertex for $i = 2$ to $m - 1$. Then without loss
of generality, there are at least $2^{m-2}f_t(G)$ pebbles on \{x_1, x_2, \ldots, x_{m-1}\} \times G$

(Otherwise there are at least $2^{m-2}f_t(G)$ pebbles on \{x_2, \ldots, x_m\} \times G). So by

induction we can $t$-pebble the target vertex.

\[\]

**Corollary 3.4.6.** Let $P_m$ be a path on $m$ vertices. Let $C_n$ be a cycle on $n$

vertices. Then $f_t(P_m \times C_n) \leq 2^{m-1}f_t(C_n)$ for all $t$.

**Proof.** Follows from Theorem 3.3.8, Theorem 3.3.9, Theorem 3.3.10 and

Theorem 3.4.5

\[\]