CHAPTER 1.

EQUATIONS OF MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID
IN PARALLEL COORDINATES

I. Coordinate System.

Let \( \Sigma \) be a given analytic regular surface possessing at each point non-zero principal radii of curvature. Let the lines of curvature on \( \Sigma \) be taken as the coordinate curves \( \xi = \text{constant} \) and \( \eta = \text{constant} \). The coordinates \((\xi, \eta, \zeta)\) of any point on \( \Sigma \) are given by

\[
x_0 = \xi, \quad y_0 = \eta, \quad z_0 = \zeta,
\]

where the Jacobian matrix

\[
\begin{pmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{pmatrix}
\]

(1.1)

is of rank two. We further assume that the functions \( \xi, \eta, \zeta \) are such that there is a one-to-one correspondence between the points of the domain on \( \Sigma \), which we consider and the set of values of \( \xi, \eta, \zeta \).

Let the first fundamental quadratic form of the surface be given by

\[
dS^2 = E\,d\xi^2 + 2F\,d\xi\,d\eta + G\,d\eta^2
\]

(1.3)

where

\[
E = \begin{pmatrix}
\frac{\partial x}{\partial \xi} \\
\frac{\partial y}{\partial \xi} \\
\frac{\partial z}{\partial \xi}
\end{pmatrix}^T \begin{pmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{pmatrix} = \begin{pmatrix}
E_{xx} & E_{xy} & E_{xz} \\
E_{yx} & E_{yy} & E_{yz} \\
E_{zx} & E_{zy} & E_{zz}
\end{pmatrix},
\]

(1.4)

and

\[
F = E_{xy} + E_{yx},
\]

\[
G = E_{zz}.
\]

In particular, for coordinate curves \( \xi = \text{constant} \) and \( \eta = \text{constant} \), the elements of length are given by
\[ d\xi^2 = \varepsilon \xi^2 \quad d\eta^2 = \gamma \eta^2 \quad (1.5) \]

\[ \varepsilon \text{ and } \gamma \text{ being positive, if we assume that the coordinate curves are not minimal curves.} \]

If \( X, Y, Z \) be the direction cosines of the normal to the surface \( \Sigma \) at the point \((\xi_0, \eta_0, \zeta_0)\), we have from the conditions that the normal is perpendicular to the directions of \( \xi \) and \( \eta \),

\[ X \frac{\partial \xi}{\partial \xi_0} + Y \frac{\partial \eta}{\partial \xi_0} + Z \frac{\partial \zeta}{\partial \xi_0} = 0 \quad (1.6) \]

so that

\[ X = \frac{1}{H} \begin{vmatrix} \frac{\partial \xi}{\partial \xi_0} & \frac{\partial \xi}{\partial \eta_0} & \frac{\partial \xi}{\partial \zeta_0} \\ \frac{\partial \eta}{\partial \xi_0} & \frac{\partial \eta}{\partial \eta_0} & \frac{\partial \eta}{\partial \zeta_0} \\ \frac{\partial \zeta}{\partial \xi_0} & \frac{\partial \zeta}{\partial \eta_0} & \frac{\partial \zeta}{\partial \zeta_0} \end{vmatrix} \quad Y = \frac{1}{H} \begin{vmatrix} \frac{\partial \xi}{\partial \xi_0} & \frac{\partial \xi}{\partial \eta_0} & \frac{\partial \xi}{\partial \zeta_0} \\ \frac{\partial \eta}{\partial \xi_0} & \frac{\partial \eta}{\partial \eta_0} & \frac{\partial \eta}{\partial \zeta_0} \\ \frac{\partial \zeta}{\partial \xi_0} & \frac{\partial \zeta}{\partial \eta_0} & \frac{\partial \zeta}{\partial \zeta_0} \end{vmatrix} \quad (1.7) \]

where

\[ L = \mathbf{X} \cdot \mathbf{Z} \quad (1.8) \]

We can also write

\[ \mathbf{L} = \begin{bmatrix} \mathbf{X}^T \mathbf{X} \\ \mathbf{Y}^T \mathbf{Y} \end{bmatrix} \]

(1.9)

Let the second fundamental quadratic form of the surface be

\[ \frac{1}{\alpha_0} \begin{vmatrix} \alpha_0 & \beta_0 \\ \beta_0 & \gamma_0 \end{vmatrix} \quad (1.10) \]

Where
Since the coordinate curves are lines of curvature on \( \mathcal{S}_0 \), we have

\begin{equation}
\mathbf{F} = \hat{\mathbf{F}} \quad \text{(1.12)}
\end{equation}

Let \( k_1, k_2 \) be the principal radii of curvature at a point of \( \mathcal{S}_0 \) corresponding to the curves \( z \) constant and \( z \) constant.

Since the radius of curvature \( R \) of a normal section of the surface is given by

\begin{equation}
\frac{1}{R} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{ (1.13)}
\end{equation}

the radii of curvature corresponding to the curves \( z \) constant and \( z \) constant respectively are given by

\begin{equation}
R = \frac{1}{k_1} \quad \text{ (1.14)}
\end{equation}

We assume that \( k_1, k_2 \) are continuous possessing continuous derivatives with respect to \( z \).

Let us consider a system of surfaces parallel to the surface \( \mathcal{S}_0 \). If \( z = d \) be the perpendicular distance of the point \((x, y, z)\) on one of the parallel surfaces \( z \) from the given surface \( \mathcal{S}_0 \),
then the equation of $S$ is given by $\xi^2 + \beta^2 = \text{constant}$, and the equation of $S_0$ by $\xi^2 + \beta^2 = \rho$. We consider the developable surfaces formed by the normals to $S_0$ along its lines of curvature. These developables intersect the parallel surfaces along their lines of curvature. Therefore the families of parallel surfaces and the developables of congruence of normals to these surfaces form a triply orthogonal system of surfaces. We can therefore take $\xi, \eta, \zeta$ as the curvilinear orthogonal coordinates of the point $(\xi, \eta, \zeta)$. As there is a one-to-one correspondence between the points of the spaces defined by $(\xi, \eta, \zeta)$ and $(\xi, \eta, \zeta')$, the Jacobian of the transformation

$$\begin{vmatrix}
\xi' & \xi & \eta' \\
\eta & \eta & \zeta' \\
\zeta & \zeta & 1
\end{vmatrix}$$

is neither zero nor infinite.

Let $(x, y, z)$ be the cartesian coordinates of a point on a surface $S$ parallel to $S_0$, whose parallel coordinates are $\xi, \eta, \zeta$, and let $(x_0, y_0, z_0)$ be the cartesian coordinates of the foot of the perpendicular from $(\xi, \eta, \zeta)$ on the surface $S_0$. Then if $\alpha, \beta, \gamma$ be the direction cosines of the normal to $S_0$ at the point $(\xi, \eta, \zeta)$, we have

$$x = x_0 + \alpha \xi, \quad y = y_0 + \beta \eta, \quad z = z_0 + \gamma \zeta. \quad (1.16)$$

If $\rho_1, \rho_2, \rho_3$ be the elements of lengths in the directions $\xi, \eta, \zeta$, we have using (1.15)

$$A = \begin{pmatrix}
\alpha & \beta & \gamma \\
\rho_1 & \rho_2 & \rho_3
\end{pmatrix} \quad (1.17)$$
where

\[ \mathbf{b} = \mathbf{c} \times \mathbf{d} \]  \hspace{1cm} (1.18)

Differentiating the first of the equations (1.6) with respect to \( \xi \) and the second with respect to \( \eta \), we have from (1.11) and (1.18)

\[ \xi' = -\alpha \quad \eta' = -N \]  \hspace{1cm} (1.19)

Differentiating the relation

\[ X' \mathbf{y} + \mathbf{y}' = \mathbf{0} \]

with respect to \( \xi \), we get

\[ X'^{2} = \xi^{2} + \mathbf{y}^{2} \]  \hspace{1cm} (1.20)

which shows that \( \mathbf{y}', \mathbf{y} \) are the direction numbers of a vector perpendicular to the normal to \( \mathbf{e} \). Such a vector is therefore a homogeneous linear function of the two tangential vectors

\[ \mathbf{a}, \mathbf{b} \]

Hence we can write

\[ \mathbf{y} = \alpha \mathbf{a} + \beta \mathbf{b} \]  \hspace{1cm} (1.21)

Multiplying these equations by \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) respectively and adding and using (1.4) with \( \mathbf{F} = \mathbf{c} \) and (1.18) and (1.19) we get

\[ \mathbf{a} = -\mathbf{c} \]  \hspace{1cm} (1.22)

Multiplying the equations (1.21) by \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) respectively, and adding and using (1.4) with \( \mathbf{F} = \mathbf{c} \), we get

\[ \mathbf{a} = -\mathbf{c} \]  \hspace{1cm} (1.23)

Differentiating the second equation of (1.6) with respect to \( \xi \) and using the relation

\[ \mathbf{a} = -\mathbf{c} \]
we get 

\( x^2 + 2px + q = 0 \)  

(1.24)

From (1.23) and (1.24) and we have 

\( x = \frac{-2p \pm \sqrt{4p^2 - 4q}}{2} \)  

(1.25)

Hence we have 

\( x = \frac{-2p \pm \sqrt{4p^2 - 4q}}{2} \)  

(1.26)

Similarly, we can prove that 

\( x = \frac{-2p \pm \sqrt{4p^2 - 4q}}{2} \)  

(1.27)

Substituting from (1.26) and (1.27) in (1.18), we get on 

using (1.4)

\[ C = \frac{a}{k} \]  

and 

\[ C = \frac{a}{k} \]  

(1.28)

Substituting from (1.19) and (1.28) in (1.17), we get 

\[ n = \frac{a - k - b}{k} \]  

(1.29)

Taking the positive square roots, 

\[ n = \frac{a - k - b}{k} \]  

(1.30)

Using the values of \( n \) and \( a \) obtained in (1.14), we can 

write 

\[ n = \frac{a - k - b}{k} \]  

(1.31)
II. Equations of Motion of a Viscous Liquid in Parallel Coordinates.

The Stokes-Navier equations of motion of an incompressible viscous fluid under no body forces can be written in the vector form (Milne-Thomson 1972, p. 508)

\( \nabla \cdot \mathbf{u} = - \frac{1}{\rho} \mathbf{f} \) \( \nabla \times \mathbf{u} = \mathbf{v} \times \mathbf{u} \) \( \mathbf{u} = \frac{1}{\mu} \nabla \mathbf{p} - \mathbf{v} \times \mathbf{v} \) \( \mathbf{v} = \frac{1}{\rho} \nabla \times \mathbf{v} \times \mathbf{u} \) (2.1)

where \( \mathbf{v} = \mathbf{r} \) and \( \mathbf{j} \) denotes the vector product of \( \mathbf{r} \) and \( \mathbf{j} \).

Introducing parallel coordinates defined in the preceding section and denoting by \( (\xi, \eta, \zeta) \) the components of velocity in the directions of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) and by \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) the unit vectors in these directions, we have (Milne-Thomson 1972, p. 59)

\[ \mathbf{L} = \mathbf{u} \cdot \mathbf{L} = \begin{pmatrix} \xi \xi & \eta \xi & \zeta \xi \\ \xi \eta & \eta \eta & \zeta \eta \\ \xi \zeta & \eta \zeta & \zeta \zeta \end{pmatrix} \] (2.2)

Using the values of \( \xi, \eta, \zeta \) given in (1.29), we have

\[ \mathbf{L} = \begin{pmatrix} \xi \xi & \eta \xi & \zeta \xi \\ \xi \eta & \eta \eta & \zeta \eta \\ \xi \zeta & \eta \zeta & \zeta \zeta \end{pmatrix} \] (2.3a)

\[ \mathbf{L} = \begin{pmatrix} \xi \xi & \eta \xi & \zeta \xi \\ \xi \eta & \eta \eta & \zeta \eta \\ \xi \zeta & \eta \zeta & \zeta \zeta \end{pmatrix} \] (2.3b)

\[ \mathbf{L} = \begin{pmatrix} \xi \xi & \eta \xi & \zeta \xi \\ \xi \eta & \eta \eta & \zeta \eta \\ \xi \zeta & \eta \zeta & \zeta \zeta \end{pmatrix} \] (2.3c)
The components of \( u^i \) are obtained from the expressions (2.3 a,b.c) for the components of \( u \) by replacing \( u, v, w \) by \( \hat{u}, \hat{v}, \hat{w} \) respectively.

Therefore the Stokes-Navier equations of motion in parallel coordinates of an incompressible viscous fluid under no body forces are

\[(2.4a)\]

\[(2.4b)\]

\[(2.4c)\]
The equation of continuity of an incompressible fluid in orthogonal curvilinear coordinates is

\[
\frac{\partial (\rho u_1)}{\partial x_1} + \frac{\partial (\rho u_2)}{\partial x_2} + \frac{\partial (\rho u_3)}{\partial x_3} = 0
\]

In parallel coordinates this equation reduces to

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0
\]
III. Non-dimensional Forms of the Equations of Motion of an incompressible viscous fluid under no body forces in parallel coordinates.

Let us introduce dimensionless quantities

\[ \kappa, \quad \lambda, \quad \theta = \frac{L}{U}, \quad \rho = \frac{m}{L^2} \]

where \( L \) is a typical length and \( U \) a typical velocity.

Substituting in the Stokes-Navier equations of motion in cartesian coordinates, we get the dimensionless form of these equations in cartesian coordinates. Writing these equations in vector form and transforming to parallel coordinates, these equations are, when we suppress the asterisks:

\[ \begin{align*}
\dot{\kappa} + \rho \nabla^2 \lambda &= \frac{1}{\rho} \nabla \cdot \mathbf{F} \\
\dot{\theta} &= \frac{1}{\rho} \nabla \cdot \mathbf{F} \\
\n\end{align*} \]

where \( \mathbf{F} \) is the body force per unit mass.
where \( Re = \frac{v L}{\nu} \) is the Reynolds number of the flow.

The equation of continuity of an incompressible viscous fluid in dimensionless form in parallel coordinates is

\[
\frac{d}{dt} \int \rho u dA = 0
\]