4.1 Introduction

So long we have been discussing queueing systems with server vacations which are either birth-and-death or non-birth-and-death process. They are in either case Markovian. We shall now re-consider models discussed in Chapter 3 where the distributions of interarrival time and service time do not possess the memoryless property. The process $N(t)$ giving system size at time $t$ will then be no longer Markovian. Two techniques are generally being used to solve such Non-Markovian queueing models with server vacations. Kendall (1951) uses the concept of regeneration point (due to Palm (1943)) and extracts from the process $N(t)$ Markov chains (in discrete time) at suitably chosen regeneration points. This is known as the technique of Imbedded Markov chains. The second important technique (due to Keilson and Kooharian (1960)) known as the supplementary variable technique, involves inclusion of such variable(s) to get complete information on the system. We shall, however, apply only Kendall's method to solve the vacation models.
Non-Markovian queueing models with server vacations have been extensively studied. Almost all the researchers who have studied M/G/1 vacation models have shown an interesting property for system size distributions, known as M/G/1 Stochastic Decomposition Property which states that the stationary number of customers present in the system at a random point in time is distributed as the sum of two independent random variables: (i) the number of Poisson arrivals during a time interval that is itself distributed as the forward recurrence time (i.e., residual life) of a vacation, and (ii) the number of customers present in the corresponding ordinary M/G/1 queueing system. This type of decomposition was first observed by Gaver (1962) and subsequently by Miller (1964), Cooper (1970), Levy and Yechiali (1975), Shantikumar (1980), Scholl and Kleinrock (1983), Ali and Neuts (1984), Federgruen and Green (1984), Neuts and Ramalhoto (1984), Ott (1984) and many others. M/G/1 vacation model was partially used by Cooper to analyze a system of queues served in cyclic order. Levy and Yechiali have studied two different vacation models under FCFS service discipline. In the first model, if a server upon completion of a service finds the queue empty it leaves the system for a vacation (for check up operation in case the server is a machine) and upon termination of that vacation if it finds an empty queue it waits in the system for
an arrival. In the second model, at the epoch of vacation termination if the server finds none waiting for service, another (independent) vacation will start immediately. The server resumes service at the end of a random number of successive vacations, the total period of absence being referred to as 'Vacation Period'. They derived system size distributions using imbedded Markov chain technique observing it at service completion and vacation termination epochs and from that delay time distributions were obtained for both the models. Fuhrmann (1984a) obtained the same result for the same model (second model of Levy and Yechiali) by using an alternative simple and elegant methods in terms of probability generating function (PGF). Ali and Neuts investigated an M/G/1 queueing model with gated vacation and feedback of customers. Their model can be briefly described as follows: customers initially enter a service unit via a waiting room. The customers to be served are stored in a service room which is replenished by the transfer of all those in the waiting room at the points in time when the service room becomes empty. At these epochs of transfer, positive random number of 'overhead customers' are also added to the service room. Algorithmically tractable expressions for the stationary distributions of queue lengths and waiting times at various embedded random epochs like transfer epochs, service completion
epochs have been derived. Cooper (1981) studied the M/G/1 model with gated vacations where he did not allow customer's feedback and his idea of gated vacations is slightly different from that of Ali and Neuts. In his model, an arriving customer who finds the server idle causes a gate to close. When this customer's service is complete, the gate opens and admits into a waiting room all those customers who arrived during this service time, and then closes. When all the customers in the waiting room have been served, the gate opens and admits into the waiting room all those customers who arrived during the collective service times of the preceding group of customers, after which it closes. The process continues in this manner. When the gate opens and finds no waiting customers, it remains open until it closes behind the next arrival. He derived the mean number of customers who enter the waiting room when the gate opens. Scholl and Kleinrock found a relationship between the second moments of the waiting time in an M/G/1 queue with rest (vacation) periods under FCFS, SIRO, LCFS disciplines. Apart from that, they derived Laplace Stieltjes Transform (LST) of the distribution function of the waiting time for non-preemptive LCFS service by using almost exactly the same argument that was employed for ordinary M/G/1 queue with the same service discipline. It has been observed that unlike in the case of FCFS service, M/G/1 Stochastic Decomposition property for waiting time does not hold in the case of LCFS
service discipline. Recently, Fuhrmann and Cooper (1984) considered a class of M/G/1 queueing models with a server who is unavailable over occasional intervals of time. They have obtained by direct and intuitive arguments a fundamental result which states that M/G/1 Decomposition Property holds, in fact, for a very general class of M/G/1 queueing models with any type of vacations. This fundamental result is applicable for finding system size distribution under any service discipline but the distribution of waiting time can be obtained by this result only under FCFS service discipline. This observation has been confirmed by Scholl and Kleinrock also. Shogan (1979) and Shanthikumar (1982) studied non-homogeneous M/G/1 vacation models in which customers arrive at a different rate during vacation periods. Stochastic decomposition property for waiting time has been shown for GI/G/1 vacation models by Doshi (1985) using sample path arguments under an appropriate initial condition and various other assumptions which are fairly general. Lee (1983) studied an M/G/1 finite queue with vacation time under exhaustive and limited service disciplines. His work includes embedded Markov chain of queue length, busy period and waiting time distributions.

In all the vacation models so far discussed, it was assumed that arrivals and services occur one at a time. Of late, Y. Baba (1986) analyzed the M/G/1 vacation model with bulk
arrival where he derived the system size distribution by using supplementary variable technique. The fundamental result of Fuhrmann and Cooper can also be applied for M/G/1 vacation model with batch arrival to derive the distributions of system size and sojourn time, but it could not help us to obtain conditional queueing time distribution of a customer who may arrive at the point of time when the server is on vacation or is busy in the system. This motivated us to reconsider the M/G/1 vacation models studied by Levy and Yechiali (1975) with batch arrivals. We obtained various measures of effectiveness of both the models such as occupation period, cycle time, vacation period, number of customers served in an occupation period and from the distribution of occupation period, LST's of conditional queueing times are derived using almost exactly the same argument employed for ordinary M/G/1 queue by Kleinrock (1975). We next studied the two vacation models of Levy and Yechiali when customers arrive in batches of variable size and are served in batches of maximum capacity. This is a generalization of M/G/1 vacation models with batch arrivals to bulk service case. It is difficult to analyze busy period and delay time distribution for this model, so we derive only system size distribution using imbedded Markov chain technique in a two-dimensional state space by observing it at service completion and vacation termination epochs. It has been shown that many results which have been derived by earlier authors can be obtained from our results as particular cases.
Recently, Daniel and Krishnamoorthy (1986) studied a GI/M/1 vacation model under FGFS service where they defined a Markov chain on a two-dimensional state space representing the queue size and the number of services completed since the most recent resumption of service, prior to arrivals. Using algorithmic approach developed by Neuts and others, they derived the invariant distribution of the chain. We made an attempt to analyze the same model under FCFS exhaustive service discipline (which is a variation from the service discipline called 'limited service discipline considered by Daniel and Krishnamoorthy). Embedded Markov chains are defined on a two-dimensional state space representing the system size and the server's presence in or outside (due to vacation) the system, prior to arrival epochs and random epochs separately. The stationary distributions of system size observed at these two epochs have been derived. They are useful in the theory and applications of queues. For example, the knowledge of the distribution at a pre-arrival epoch is useful to arriving customers and in the analysis of their waiting times, while the distribution at a random epoch is useful to the manager of an organization. For more details on such points, one can see Chaudhry and Templeton (1983). Finally steady state waiting time distribution of an arbitrary customer in the queue (excluding service) has been obtained by using system size distribution at pre-arrival epochs. An interesting feature of
study is that all the results have been expressed in terms of a unique real root of a certain characteristic equation.

It has been observed that stochastic decomposition does not follow in our models.

4.2 $M(X)/G/1$ Vacation Model 1

We consider an $M(X)/G/1$ queueing system with unlimited storage where the stream of arrivals in batches forms a homogeneous Poisson process with rate $\lambda$ and PGF of the batch size $X$ is $C(z) = \sum_{m=1}^{\infty} c_m z^m (\sum_{m=1}^{\infty} c_m = 1)$. Service discipline is FCFS and arriving batches are preordered for service. The service times of the customers i.e. $V_1, V_2, \ldots, V_n, \ldots$ are independent random variables having a common distribution $B(v)$ with first two moments $E(V)$ and $E(V^2)$ respectively. When a service is completed and no customers are present in the system the server becoming idle for lack of work will withdraw from the system for a vacation of duration $U$ having an arbitrary distribution $F(u)$, with first two moments $E(U)$ and $E(U^2)$ respectively. The server returns to the system after completing vacation. If, on return, the server finds customers waiting for service (customers who arrived during the vacation period $U$) it starts servicing immediately and keeps busy until the system becomes empty again and it leaves for another vacation. If no
customers have arrived during the vacation the server shall not leave the system, wait for a batch to arrive when an ordinary $M(X)/G/1$ busy period (BP) starts. At the termination of BP the server takes another vacation.

We now define occupation period and cycle time and determine their distributions.

**Definition 4.1.** We define the occupation period (OP) $T_g$ as the total time elapsing from the movement the server returns from a vacation till it leaves for the next.

**Definition 4.2.** Cycle time $T$ is defined as the interval between the beginnings of two successive vacations.

**Distributions of occupation period and cycle time**

Let $T_1 = \text{interval of time during which the first customer of an arriving batch to the empty system (or the first customer in the batch that first arrives during a vacation, who initiates a BP) is served}$;

$T_2 = \text{interval of time during which the second customer of the batch is served and all the arrivals during his service time are also served}$,

and so on.

$T_1, T_2, \ldots, T_i, \ldots$ represent a sequence of ordinary BP's in an
$M(X)/G/1$ queue and they are independent and identically distributed (i.i.d) random variables (r.v.).

Let $P$ be the time reckoned from the moment the server returns from a vacation (to an empty system) to the moment a batch arrives. Then

$$T_s = P + \sum_{i=1}^{X} T_i, \; N = 0$$

$$= \sum_{i=1}^{N} T_i, \; N > 0$$

(4.1)

where $N$ is the number of customers arrived during a vacation.

Thus OP is not necessarily identical with BP in this model. The cycle time $T$ is then given by $T = T_s + U$.

Theorem 4.1. Arrivals form a compound Poisson process so that if a total of $Q(t)$ customers arrive during $[0,t]$, we have

$$q_j(t) = \text{Prob}(Q(t) = j) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{e^{-\lambda t}}{n!} \{c_j\}^n$$

(4.2)

$j = 0,1,2,\ldots$

where $\{c_0\}^* = \delta_{i0}$ and $\{c_1\}^* = \{c_1\}$, $\{c_1\}^n$ is the n-fold convolution of $\{c_1\}$ with itself.
Proof. $Q(t)$ is the sum of a random number of terms, i.e.

$$Q(t) = \sum_{k=1}^{M(t)} Y_k,$$

where $M(t)$ is the number of batches arrived during $[0,t]$ and is a regular Poisson process with mean $\lambda t$, $\{Y_k\}$ is a sequence of i.i.d discrete r.v.s with probabilities

$$c_m = \text{Prob} \{ Y_k = m \},$$

that is, arrivals occur according to a Poisson process $\{M(t)\}$ but are not necessarily singlets in that their size is $m$ with probability $c_m$. Then by the laws of probability,

$$\text{Prob} \{Q(t) = j\} = \sum_{n=0}^{j} \left[ \text{Prob} \{M(t) = n\} \cdot \text{Prob} \left\{ \sum_{k=1}^{n} Y_k = j \right\} \right]$$

$$= \sum_{n=0}^{j} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left\{ c_j \right\}^{n^*} \quad (j = 0, 1, 2, \ldots)$$

where $\left\{ c_j \right\}^{n^*}$ is the probability that in all $j$ customers arrive in $n$ batches.

Note. Process $Q(t)$ is known as Poisson Cluster Process or Compound Poisson Process or Multiple Poisson Process.

The PGF of $Q(t)$ is
\[ \sum_{j=0}^{\infty} q_j(t) z^j = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} [C(z)]^n \quad (\text{by (4.1)}) \]

\[ = \exp \left[ -\lambda t \left( 1 - C(z) \right) \right] \quad (4.3) \]

Let \( b_j = \text{Prob (j units arrive during a vacation)} \)

\[ = \int_0^\infty q_j(u) \, dF(u) \quad (j = 0, 1, 2, \ldots) \quad (4.4) \]

For a Poisson process with parameter \( \lambda \), the interval up to the occurrence of the next event measured from any point of time (not necessarily from the instant of the previous occurrence) is independent of the elapsed time (since the previous occurrence) and is a r.v. having exponential distribution with mean \( 1/\lambda \).

Since \( \{M(t)\} \) is a Poisson process with parameter \( \lambda \), using this property, LST of \( P \) is

\[ \Gamma_P(z) = E(e^{-zP}) = \lambda \int_0^\infty e^{-\lambda x} \, e^{-z x} \, dx \]

\[ = \frac{\lambda}{\lambda + z} \]

Conditioning on \( N \) we obtain from (4.1)

\[ \Gamma_{T_S}(z) = E(e^{-zT_S}) \]

\[ = \sum_{m=1}^{\infty} E \left( e^{-z\left( P + \sum_{i=1}^{m} T_i \right)} \right) c_m \text{Prob (N = o)} \]
\[ P(z) = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} E(e^{-zT_1}) b_j \]

\[ = \frac{\lambda}{\lambda + z} \left[ U(\lambda) C \left( T_1(z) \right) - U(\lambda) \right] \]

or finally by using (4.2)

\[ \Gamma_{T_2}(z) = \frac{\lambda}{\lambda + z} \left[ U(\lambda) C \left( T_1(z) \right) \right] \]

\[ + \left[ U \left( (1 - C) \left( T_1(z) \right) \right) - U(\lambda) \right] \]

where \( \Gamma_{T_1}(z) \) is the LST of the ordinary BP \( T_1 \) of \( M^X/G/1 \) queue. Since \( T_2 \) and \( U \) are not independent, LST of \( T \) is not the product of LST's of \( T_1 \) and \( U \). However, LST of \( T \) can be obtained by conditioning on \( U \) and \( N \) and by using (4.1), (4.3), (4.4), which is
\[ \Gamma_T(z) = \mathbb{E} \left[ e^{-(T_y + U)z} \right] \]

\[ = \sum_{i=1}^{\infty} \int_{m=1}^{\infty} \mathbb{E} \left[ e^{-(P + \sum_{j=1}^{m} T_j + u)z} \right] c_i \cdot q_i(u) \, dF(u) \]

\[ + \sum_{i=1}^{\infty} \int_{j=1}^{\infty} \mathbb{E} \left[ e^{-(\sum_{j=1}^{m} T_j + u)z} \right] q_j(u) \, dF(u) \]

\[ = \Gamma_p(z) \sum_{i=1}^{\infty} \int_{m=1}^{\infty} \mathbb{E} \left[ e^{-zT_j} \right]^{m} c_m e^{-\lambda u} \, dF(u) \]

\[ + \sum_{i=1}^{\infty} \int_{j=0}^{\infty} \mathbb{E} \left[ e^{-zT_j} \right] q_j(u) \, dF(u) \]

\[ - \int_{u=0}^{\infty} e^{-uz} e^{-\lambda u} \, dF(u) \]

\[ = \Gamma_p(z) \sum_{i=1}^{\infty} \int_{u=0}^{\infty} e^{-(\lambda + z)u} C \left[ \Gamma_T(z) \right] \, dF(u) \]

\[ + \int_{u=0}^{\infty} \exp \left[ -(z + \lambda(1 - C \left[ \Gamma_T(z) \right]))u \right] \, dF(u) \]

\[ - \int_{u=0}^{\infty} e^{-(\lambda + z)u} \, dF(u) \]

or finally,
\[ \Gamma_T(z) = \Gamma_p(z) \Gamma_u(\lambda + z) C \left[ \Gamma_T_1(z) \right] \]

\[ + \int u \left[ \sum_{j=1}^{\infty} \Gamma_1(z + \lambda(1 - C \left[ \Gamma_T_1(z) \right])) - \Gamma_u(\lambda + z) \right] \]

To determine \( \Gamma_T_1(z) \), we define \( T_1 \) by

\[ T_1 = X_0, \quad j = 0 \]

\[ = X_0 + \sum_{r=1}^{j} X_r, \quad j > 0 \]

where \( X_0 \) is the service time of the customer who initiates \( T_1 \)

and \( X_r \) is the sub-busy period initiated by the \( r \)th customer

among \( j \) arrivals during \( X_0 \). Clearly \( X_1, X_2, \ldots \) are i.i.d. r.v.s
each having the distribution of \( T_1 \).

Then from (4.7) we have

\[ \Gamma_T_1(z) = \mathbb{E} \left[ e^{-z T_1} \right] = \int_{v=q}^{\infty} e^{-z v} q(v) dB(v) \]

\[ + \int_{v=0}^{\infty} \sum_{j=1}^{\infty} e^{-z(v + \sum_{r=1}^{j} X_r)} q_j(v) dB(v) \]

\[ = \int_{v=0}^{\infty} \exp \left[ -(z + \lambda(1 - C \left[ \Gamma_T_1(z) \right]))v \right] dB(v) \]
or finally,

\[ r_T(z) = r_v \left[ z + \lambda (1 - c V(z)) \right] \]  

(4.8)

From (4.8) we have

\[ E(T_1) = E(V) \left[ 1 + \lambda c E(T_1) \right] \]

which gives

\[ E(T_1) = \frac{E(V)}{1 - \lambda c E(V)} \]  

(4.9)

where \( \bar{c} = \sum_{m=1}^{\infty} m p_m \) is the mean batch size.

Using (4.9) we have from (4.5)

\[ E(T_s) = \frac{1}{\lambda} U(\lambda) + U'(\lambda) \bar{c} E(T_1) + E(U) \lambda \bar{c} E(T_1) \]

and finally we have

\[ E(T_s) = \frac{\frac{1}{\lambda} U(\lambda) + \lambda \bar{c} E(U) E(V)}{1 - \lambda \bar{c} E(V)} \]  

(4.10)

Similarly, we have from (4.6)
\[ E(T) = \frac{1}{\lambda} \int_0^\infty (\lambda + B(U) \cdot 1 - Ac \cdot E(V)) \cdot E(U) \]  

(4.11)

Let \( p \) denote the proportion of time the server spends on vacation. Then

\[ P_o = \frac{E(U)}{E(T)} = \frac{\left[1 - \lambda c \cdot E(V)\right] \cdot E(U)}{\frac{1}{\lambda} \int_0^\infty (\lambda + B(U)) \cdot E(U)} \]  

(4.12)

which is also the probability that the server is on vacation.

The proportion of time the server is not busy (is either on vacation or is idle) is

\[ Q_o = \frac{E(U) + \frac{1}{\lambda} \cdot \text{Prob} (N = 0)}{E(T)} = 1 - \lambda c \cdot E(V) \]  

(4.13)

Result (4.13) indicates that the condition for the system to be in a steady state regime is

\[ \rho = \lambda c \cdot E(V) < 1 \]

as the case in the ordinary \( M^{(X)} / G / 1 \) queue.

**Distribution of the number of customers served during busy period**

Let \( N_{BP} \) be the number of customers served during a BP.
We define $N_{BP}$ by

$$N_{BP} = \sum_{i=1}^{X} N_{T_i}, \quad N = 0$$

$$= \sum_{i=1}^{N} N_{T_i}, \quad N > 0$$

(4.14)

where $N_{T_i}$ denotes the number of customers served during an ordinary BP $T_i$. Clearly $N_{T_i}$ represents a sequence of i.i.d. r.v.

Then from (4.14), following the same procedure to derive (4.5) the PGF of $N_{BP}$ is

$$G(z) = E\left[z^{N_{BP}}\right]$$

$$= \Gamma_{U}(\lambda) \cdot C\left[H(z)\right] + \Gamma_{U}\left[\lambda(1-C\left[H(z)\right])\right]$$

(4.16)

where $H(z) = E(z^{N_{T_1}})$

To determine $H(z)$ we consider an ordinary $M^{(X)}/G/1$ queue corresponding to our model. Let $M_i$ denote the number of customers served in $X_i$ where $X_i$ is defined in (4.7). Then

$$N_{T_1} = 1, \quad r = 0$$

$$= 1 + \sum_{i=1}^{r} M_i, \quad r > 0$$

(4.16)

where $r$ customers have arrived during $X_0$.
Since $M_1$ represents a sequence of i.i.d. r.v. each having the distribution of $N_{T1}$, we have from (4.16)

$$H(z) = z \left[ \lambda \left( 1 - C \left[ H(z) \right] \right) \right]$$  \hspace{1cm} (4.17)

The mean number of customers served during a BP is obtained from (4.15) and (4.17) which is

$$E\left[ N_{BP} \right] = \frac{3 \left[ T_U(\lambda) + \lambda E(U) \right]}{1 - \lambda \lambda E(V)}$$

$$= \lambda \lambda E(T)$$ \hspace{1cm} (4.18)

**Distribution of delay (from occupation period)**

**Definition 4.3.** Delay to a customer is defined as the length of time from its arrival to departure (i.e. queuing time plus service time).

To derive delay distribution (from OP) we redefine $T_S$ by

$$T_S = P + \sum_{i=1}^{X} V_i + \sum_{j=1}^{\infty} X_j, \hspace{0.5cm} N = 0$$

$$= \sum_{i=1}^{N} V_i + \sum_{k=1}^{\infty} X_k, \hspace{0.5cm} N > 0$$ \hspace{1cm} (4.19)

where $X_j, (j \geq 1)$ ($X_k, (k \geq 1)$) is the length of time required to service all those customers who have arrived during the
previous interval \( X_{j-1}(X'_{k-1}) \) and we permit the possibility of an infinite sequence of such intervals. Clearly, we define \( X_j(X'_k) = 0 \) for those intervals that fall beyond the termination of this OP; for \( \rho = \lambda c \mathbb{E}(V) < 1 \), we know that with probability 1 there will be a finite \( j_0(k_0) \) for which \( X_j(X'_k) \) and all its successors will be zero. Unlike \( \mathcal{T}_1 \), these intervals \( X_j(X'_k) \) are not i.i.d.

We now obtain the measure of dependence among \( X_j \)'s \( (X'_k \)'s) in terms of LST. Let \( n_j \) be the number of arrivals during \( X_j \). Then \( X_{j+1} \) will be the sum of \( n_j \) service intervals. We have when the size of the arriving batch which initiates the BP is \( X = m \)

\[
E \left[ e^{-2X_j} \Big| X = m, X_{j-1} = y, n_{j-1} = n \right] = \left[ \frac{1}{\mathbb{E}(V)} \right]^n \tag{4.20}
\]

since \( X_j \) is the sum of \( n \) independent service times, all with identical distribution \( \mathbb{E}(V) \). Removing the condition on \( n \) and \( y \) we have from (4.20)

\[
E \left[ e^{-2X_j} \Big| X = m \right] = \sum_{y=0}^{\infty} \sum_{n=0}^{\infty} q_n(y) \left[ \frac{1}{\mathbb{E}(V)} \right]^n \text{Prob}(X_{j-1} \leq y)
\]

and in terms of LST, we have by using (4.3)

\[
\mathbb{E}(X_j|X| = m) = \left[ \mathbb{E}(X_{j-1}) \right] \left[ \lambda(1 - C \frac{1}{\mathbb{E}(V)}) \right] \tag{4.21}
\]
where \( \Gamma_{X_0}(z | m) = \left[ \Gamma_V(z) \right]^m \) \( (4.22) \)

Similarly, we have for \( N > 0 \)

\[ \Gamma_{X_k}(z | N) = \Gamma_{X_{k-1}} \left[ \lambda (1 - C \left[ \Gamma_V(z) \right]^N) | N \right] (k \geq 1) \] \( (4.23) \)

where \( \Gamma_{X_0}(z | N) = \left[ \Gamma_V(z) \right]^N \) \( (4.24) \)

We now condition our calculations on the event that a new ("tagged") arrival (i.e. a batch) occurs during a BP and, in particular, while the BP initiated by an arrival of batch size \( m \) is in its \( j \)th interval (of duration \( X_j \)). Let the queueing time of a batch (i.e. first customer of the batch) be \( W_0 \). This queueing time \( W_0 \) is equal to the remaining time (i.e. residual life) of \( X_j \) plus the sum of the service times of all customers who arrived before he did during \( X_j \).

Let \( Y_j \) be the residual life of \( X_j \) and \( N_j \) be the number of customers arrived during \( X_j - Y_j \). Then

\[ E \left[ e^{-2W_0} | j, m, X_j = y, Y_j = y, N_j = n \right] = e^{-zy} \left[ \Gamma_V(z) \right]^m \] \( (4.25) \)

Removing condition on \( N_j \), we have from \( (4.25) \)
To proceed further, we note that the sequence of intervals $X_{j-1}$, $j > 1$ could generate a renewal process (though the sequence does not itself constitute a renewal process). For, if we define $Z_{j} = X_{j-1}$, $j > 1$, $S_{n} = \sum_{j=1}^{n} Z_{j}$, $n > 1$ and $N(t) = \text{Sup} \{ n: S_{n} \leq t \}$ then $\{N(t), t > 0\}$ is a renewal process which is generated by the sequence $\{Z_{j}, j > 1\}$. Here a renewal means 'completion of service for a batch (more than one batch) arriving in a certain interval $Z_{j}$'. $Z_{j}$ is the interoccurrence time between the $j$th and $(j+1)$th renewals. However, in our problem $X_{j}$'s are not i.i.d. and hence we cannot use

Using the argument given by Ross (see Chapter 3, Ross (1983)) we get the joint PDF of $X_{j}$ and $Y_{j}$ as

$$P(y' < Y_{j} < y' + dy', y < X_{j} < y + dy) = \frac{dy'd\text{Prob}(X_{j} \leq y)}{E(X_{j})} \quad (0 < y < y')$$

 Integrating over $y$ we obtain

$$d\text{Prob}(X_{j} \leq y') = \frac{1 - \text{Prob}(X_{j} \leq y')}{E(X_{j})}$$

so that $\int_{y_{j}}(z) = \left[ 1 - \int_{X_{j}}(z) \right] / z E(X_{j})$
Removing the conditions on $X_j$ and $Y_j$, we obtain from (4.21), (4.26) and (4.27a)

$$E \left[ e^{-ZW_0} \mid j, m \right] =$$

$$= \int_{y=0}^{\infty} \exp \left[ -\lambda(1 - C \left[ \Gamma_V(z) \right] ) \mid m \right] y \frac{d \text{Prob}(X_j \leq y)}{E(X_j)}$$

$$\times \int_{y'=0}^{y} \exp \left[ -(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] ) m y' \right] dy'$$

$$= \int_{y=0}^{\infty} \frac{\exp \left[ -\lambda(1 - C \left[ \Gamma_V(z) \right] ) \mid m \right] y - \exp(-zy \mid m)}{(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] ) E(X_j)} d \text{Prob}(X_j \leq y)$$

and hence by using (4.21)

$$E \left[ e^{-ZW_0} \mid j, m \right] = -\frac{\Gamma_{X_j+1}(z \mid m) - \Gamma_{X_j}(z \mid m)}{(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] ) E(X_j)} \quad (4.28)$$

Similarly, we obtain the corresponding result when the tagged customer arrives during $X_k$, which is

$$E \left[ e^{-ZW_0} \mid k, N \right] = -\frac{\Gamma_{X_k+1}(z \mid N) - \Gamma_{X_k}(z \mid N)}{(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] ) E(X_k)} \quad (4.29)$$
Now we are ready to derive the conditional queueing time of the tagged customer (i.e. first customer of the batch considered). For convenience, we define

\[ B_P^0 = \text{Part of the OP \ when \ server \ is \ not \ busy} \]

\[ B_P^1 = \text{the BP \ which \ starts \ after \ some \ idle \ time \ spent \ by} \]
\[ \text{the server in the system on its return from vacation} \]

and \[ B_P^2 = \text{the BP \ which \ starts \ immediately \ after \ server's} \]
\[ \text{return from vacation.} \]

Thus, virtually \[ B_P^0 = P, B_P^1 = \sum_{i=1}^{X} T_i \] and \[ B_P^2 = \sum_{i=1}^{N} T_i (N \geq 1) \].

Removing the condition that our arrival occurs in a particular interval \[ X_i(X_K) \] (still conditioned on our arrival entering during a \[ B_P^1 (B_P^2) \] we have from (4.28), (4.29) and (4.4):

\[
E \left[ e^{-zW_{B_P}} \mid \text{tagged customer enters in } B_P^1 \right] = \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} E \left[ e^{-zW_{B_P}} \mid j, m \right] \frac{E(X_j)}{E(B_P^1)} c_m
\]

\[
= \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \begin{bmatrix} \Gamma X_{j+1}^m(z) - \Gamma X_{j}^m(z) \end{bmatrix} c_m
\]

\[
(z - \lambda + \lambda C \Gamma V(z)) \ E \left[ B_P^1 \right]
\]  

(4.30)

and
\[
E \left[ e^{-zW_0} \mid \text{tagged customer enters in } BP^2 \right] \\
= \sum_{N=1}^{\infty} \sum_{k=0}^{\infty} \left[ \Gamma_{X_k+1}(z \mid N) - \Gamma_{X_k}(z \mid N) \right] b_N \frac{E(X'_k)}{E[BP^2]}
\]

\[
= \sum_{N=1}^{\infty} \sum_{k=0}^{\infty} \int \left[ \Gamma_{X_k+1}(z \mid N) - \Gamma_{X_k}(z \mid N) \right] q_N(u) \, dF(u)
= \frac{\left( z - \lambda + \lambda C \left[ \Gamma(z) \right] \right) E[BP^2]}{(z - \lambda)^2}
\]  

Since \( X_j(X_k^{'}) = 0 \) for \( j \geq j_0 \) (\( k \geq k_0 \)) hence we have

\[
\sum_{j=0}^{\infty} \left[ \Gamma_{X_j+1}(z \mid m) - \Gamma_{X_j}(z \mid m) \right] = 1 - \frac{E \left[ X_0 \right]}{E[BP^2]}
\]

and

\[
\sum_{k=0}^{\infty} \left[ \Gamma_{X_k+1}(z \mid N) - \Gamma_{X_k}(z \mid N) \right] = 1 - \frac{E \left[ X'_0 \right]}{E[BP^2]}
\]

So we arrive at from (4.30) and (4.31) on using (4.22), (4.24', and (4.3)

\[
E \left[ e^{-zW_0} \mid \text{tagged customer enters in } BP^1 \right] \\
= \frac{1 - C \left[ \Gamma(z) \right]}{(z - \lambda + \lambda C \left[ \Gamma(z) \right]) E[BP^1]}
\]  

and
E \left[ e^{-zW_0} \right] \text{ tagged customer enters in } BP^2 \nonumber = \frac{1 - \Gamma_U \left[ \lambda(1 - C \Gamma_V(z)) \right]}{(z - \lambda + \lambda C \Gamma_V(z)) E[BP^2]} \tag{4.33}

where \( E[BP^1] = E(T_1) E(X) = \frac{c E(V)}{1 - \lambda c E(V)} \)

and

\[ E(BP^2) = E(T_1) E(N \mid N > 1) = \frac{\lambda c E(U) E(V)}{(1 - \lambda c E(V))(1 - \Gamma_U(\lambda))} \tag{4.34} \]

Next we consider the event that our arrival occurs during server's vacation. Let \( U' \) be the residual life of vacation \( U \)
and \( N(U - U') \) be the number of customers arrived during \( U - U' \).
To work out the joint PDF of \( U \) and \( U' \) we proceed as follows:

Let us consider a sequence of cycle times \( \{\tau_n\} \). Clearly \( \tau_n \ (n \geq 1) \) are i.i.d. r.v.'s with \( E(\tau_1) = E(T) < \infty \) and form a renewal process \( \{S_n\} \) where \( S_n = \sum_{i=1}^{n} \tau_i \). We note that \( \tau_n \)
consists of two time intervals \( \tau_{n,1} \) and \( \tau_{n,2} \) representing vacation duration and occupation period respectively. Thus \( \tau_n = \tau_{n,1} + \tau_{n,2} \) where \( \{\tau_{n,j} \mid n \geq 1\} \) are i.i.d. r.v.'s \( (j = 1,2) \). No condition is imposed on \( \{\tau_{n,1}, \tau_{n,2}\} \). We call this renewal process an alternating renewal process. The process
is in stage 1 at instant $t$ if $t$ is one of the durations $\{\tau_{n,1}, n \geq 1\}$. Let $N(t)$ be the number of renewal times of $\{\tau_n\}$ and $\gamma_1(t)$ be the residual life of $\tau_{N(t)+1,1}$ if the process is in stage 1 at instant $t$; $\gamma_1(t) = 0$, otherwise.

We are to find the joint PDF of $\tau_{N(t)+1,1}$ and $\gamma_1(t)$. If $F_j(x) = \text{Prob}(\tau_1, j \leq x)$ then from the result of Hsu and He (1939) it follows that joint PDF of $\tau_{N(t)+1,1}$ and $\gamma_1(t)$ as $t \to \infty$ is $(dy' dF_1(y))/E(\tau_{1,1})$.

Treating $\tau_{1,1}$ as $U$ and $\gamma_1(t)$ as $U'$ we get the desired result in the limiting case as

$$P \left[ u' < U < u' + du', u < U < u + du \right] = \frac{du' dF(u)}{E(U)}.$$

Then following steps similar to those used to derive (4.32) we have

$$E \left[ e^{-zU_0} \mid \text{tagged customer arrives during server's vacation} \right]$$

$$= \sum_{n=0}^{\infty} \int_{u'=0}^{\infty} \int_{u=0}^{\infty} E \left[ e^{-zU_0} \mid U = u, U' = u', N(u-u') = n \right]$$

$$= \int_{u'=0}^{\infty} \int_{u=0}^{\infty} x q_n(u-u') \frac{dF(u) du'}{E(U)}$$

$$= \frac{\Gamma_U \left[ \lambda(1 - C \left[ \Gamma_V(z) \right]) \right] - \Gamma_U(z)}{(z - \lambda' + \lambda C \left[ \Gamma_V(z) \right]) E(U)} \quad (4.35)$$
Results (4.32), (4.33) and (4.35) respectively represent the LST of the queueing time of the tagged customer who enters in $BP^1$, $BP^2$ and during server's vacation respectively.

Next we find the LST of the queueing time of a random customer in an arriving batch. If a customer within an arriving batch is selected at random, then the probability of a customer in the $r$th position is

$$\Theta_r = \frac{1 - \sum_{m=1}^{r-1} \frac{c_m}{c}}{c}$$

and the PGF of the distribution is

$$K(z) = \frac{z(1 - C(z))}{c(1 - z)} \quad (4.36)$$

Now the (conditional) queueing time of the $r$th customer in a batch, given that the batch arrives in $BP^1$, is the sum of the corresponding queueing time of the first customer (tagged) of the batch and the service time of the first $(r-1)$ customers. Hence LST of the (conditional) distribution of queueing time ($W$) for a random customer who arrives during $BP^1$ is obtained from (4.32) and (4.36) as

$$E\left[ e^{-zW} \mid \text{random customer enters in } BP^1 \right]$$

$$= E\left[ e^{-zW} \mid \text{tagged customer enters in } BP^1 \right] \left( \sum_{r=1}^{\infty} \Theta_r \left[ \Gamma_W(z) \right]^{r-1} \right)$$

$$= \frac{\Gamma_U(\lambda)(1 - C \left[ \Gamma_V(z) \right])^2}{c(z - \lambda + \lambda C \left[ \Gamma_V(z) \right])(1 - \Gamma_V(z)) E(BF^1)} \quad (4.37)$$
where \( E[BP^1] \) is as given in (4.34).

In an analogous manner, the LST of the queueing time of a random customer who arrives during \( BP^2 \) is obtained from (4.33) as

\[
E\left[ e^{-zW} \mid \text{random customer enters in } BP^2 \right] = \frac{\{1 - \Gamma_U[\lambda(1 - C[\Gamma_V(z)])]\}(1 - C[\Gamma_V(z)])}{c(z - \lambda + \lambda C[\Gamma_V(z)])(1 - \Gamma_V(z))E(BP^2)} \quad (4.38)
\]

where \( E[BP^2] \) is as given in (4.34).

And finally LST of a random customer who arrives during the server's vacation is obtained from (4.34) as

\[
E\left[ e^{-zW} \mid \text{random customer enters during server's vacation} \right] = \frac{\{\Gamma_U[\lambda(1 - C[\Gamma_V(z)])] - \Gamma_U(z)\}(1 - C[\Gamma_V(z)])}{c(z - \lambda + \lambda C[\Gamma_V(z)])(1 - \Gamma_V(z))E(U)} \quad (4.39)
\]

Expressions (4.37), (4.38) and (4.39) are the primary goals we desired to achieve for this model and these could not be obtained by the decomposition principle.
Now LST of \( W \), the queueing time of a random customer, is obtained, on using (4.37) - (4.39) and after some straightforward simplification, as

\[
\Gamma_W(z) = E\left[ e^{-zW} \right]
\]

\[
= E\left[ e^{-zW} \mid \text{random customer arrives during a vacation} \right]
\]

\[
x \cdot \text{Prob}\left[ \text{random customer arrives during vacation} \right]
\]

\[
+ \sum_{i=0}^{2} E\left[ e^{-zW} \mid \text{random customer arrives in the BP}^i \right]
\]

\[
x \cdot \text{Prob}\left[ \text{random customer arrives during the BP}^i \right]
\]

\[
x \cdot \text{Prob}\left[ \text{random customer arrives after server's return from a vacation} \right]
\]

\((W_0 = 0 \text{ when } i = 0)\)

Now

\[
\text{Prob}\left[ \text{arrival enters in BP}^0 \right] = \frac{E\left[ \text{BP}^0 \right]}{E(T_s)} \cdot \text{Prob}(N = 0) = \frac{\sum_{U(\lambda)/\lambda}}{E(T_s)}
\]

\[
\text{Prob}\left[ \text{arrival enters in BP}^1 \right] = \frac{E\left[ \text{BP}^1 \right]}{E(T_s)} \cdot \text{Prob}(N = 0)
\]
\[
\Gamma_W(z) = \frac{z(1-\rho)}{z - \lambda + \lambda \cdot \Gamma_V(z)} \cdot \frac{1 - C \cdot \Gamma_V(z)}{\overline{c}(1 - \Gamma_V(z))} \cdot \\
x \left[ \frac{1 - \Gamma_U(z) + \frac{\overline{U}}{\lambda} \Gamma_U(\lambda)}{z(\frac{1}{\lambda} \Gamma_U(\lambda) + E(U))} \right] \cdot
\]

The first factor on the r.h.s. of (4.41) is the LST of the queuing
time of an arbitrary customer in the ordinary $M^{(X)}/G/1$ queue.

To make an interpretation of the second factor in (4.41) let us define $\hat{W}$ by

$\hat{W} = U'$, if the server is on vacation at customer's arrival point of time.

$= 0$, if the server is idle in the system on its return from vacation at customer's arrival point of time.

Then clearly $W = \hat{W} + W_1$, where $W_1$ is the queueing time of an arbitrary customer in the ordinary $M^{(X)}/G/1$ queue. Thus $\hat{W}$ may be interpreted as "the part of the queueing time $W$ when the server is not busy". Hence when an arbitrary customer joins the system and finds the server busy, $W$ becomes $W_1$.

Therefore,

$$\Gamma_{\hat{W}}(z) = \Gamma_{U'}(z) \cdot \frac{E(U)}{E(U) + \frac{1}{\lambda} \Gamma_U(\lambda)} + \frac{\frac{1}{\lambda} \Gamma_U(\lambda)}{E(U) + \frac{1}{\lambda} \Gamma_U(\lambda)}$$

$$= \frac{1 - \Gamma_U(z) + \frac{1}{\lambda} \Gamma_U(\lambda)}{z(E(U) + \frac{1}{\lambda} \Gamma_U(\lambda))} \quad (4.42a)$$

since from (4.27b) $U'$ being residual life of $U$

$$\Gamma_{U'}(z) = \frac{1 - \Gamma_U(z)}{z E(U)} \quad (4.42b)$$
Thus by (4.42a) the second factor on the r.h.s. of (4.41) is the LST of $\overset{\wedge}{W}$.

LST of the distribution of delay ($W_s$) to a random customer is

$$\Gamma_{W_s}(z) = \Gamma_{\overset{\wedge}{W}}(z) \cdot \Gamma_{\overset{\wedge}{W}}(z) \quad (4.43)$$

Mean delay to a random customer is obtained from (4.41) and (4.43), after some simplification as

$$E(W_s) = -\frac{d}{dz} \Gamma_{\overset{\wedge}{W}}(z) \bigg|_{z=0} - \frac{d}{dz} \Gamma_{\overset{\wedge}{W}}(z) \bigg|_{z=0} \cdot$$

$$= E(V) + \frac{\lambda \, c \, E(V^2)}{2(1-\rho)} + \frac{\nu E(V)}{2c(1-\rho)} + \frac{\lambda \, E(U^2)}{E(\Gamma^2(\lambda)) + \lambda \, E(U))} \quad (4.44)$$

where $\nu = \frac{d^2}{dz^2} C(z) \bigg|_{z=1}$

4.3 $\overset{(X)}{M}/G/1$ Vacation Model 2

In this model we, as before, consider an $\overset{(X)}{M}/G/1$ queue with server's vacations where the underlying assumption is that if the server finds the system empty at the end of a vacation, he will take another (independent) vacation which starts immediately and continues in this manner until he finds at least one waiting
customer upon return from vacation. The vacations are i.i.d.
and are independent of the arrival and service process.

**Occupation period and cycle time**

**Definition 4.4.** "Vacation Period" (VP) $T_R$ is defined as the
time elapsed between the moment the server leaves the system
(after a service completion) and the moment he starts servicing
again.

$T_R$ differs from a single vacation of duration $U$.

Unlike in model 1, $T_s$ is the OP in which the server is
always busy, hence in this case we can treat $T_s$ as length of BF.

To find the LST of $T_R$, we note that $T_R$ is the sum of
a random number of successive vacations, the number of vacations
having a geometric distribution,

$$\Gamma_{T_R}(z) = \sum_{k=0}^{\infty} \left( \Gamma_U(z) \right)^{k+1} b_o^k (1 - b_o)$$

$$= \frac{(1 - \Gamma_U(\lambda)) \Gamma_U(z)}{1 - \Gamma_U(\lambda) \Gamma_U(z)} \quad \text{(by using (4.4))} \quad (4.45)$$

(b_o = \Gamma_U(\lambda))

As in model 1, $T_s = \sum_{i=1}^{N} T_i$, $N \geq 0$ \quad (4.46)
Respective first order movements of $T_R$ and $T_S$ are

$$E(T_R) = \frac{E(U)}{1 - \Gamma_U(\lambda)} \quad (4.48)$$

and

$$E(T_S) = \frac{\lambda \bar{\tau} E(V) E(U)}{(1 - \Gamma_U(\lambda))(1 - \rho)} \quad (4.49)$$

Mean Cycle Time is then given by

$$E(T) = E(T_R) + E(T_S) = \frac{E(U)}{1 - \Gamma_U(\lambda)(1 - \rho)} \quad (4.50)$$

Fraction of time the server is in VP is

$$P_0 = \frac{E(T_R)}{E(T)} = 1 - \rho$$

which is the proportion of time the server is idle in an ordinary $M(X)/G/1$ queue and the proportion of time the server is busy is

$$P_1 = \frac{E(T_S)}{E(T)} = \rho = \text{Prob (an arrival enters during a BP)}$$
Distribution of the number of customers served during a busy period

We, as before, define $N_{BP}$ by

$$N_{BP} = \sum_{i=1}^{N} N_{T_i}, N > 0.$$  

Then PGF of $N_{BP}$ is

$$G(z) = \frac{\Gamma_{U}\left[\lambda(1-C[H(z)])\right] - \Gamma_{U}(\lambda)}{1 - \Gamma_{U}(\lambda)} \quad (4.51)$$

where $H(z)$ is given by (4.17).

Mean number of customers served during BP is obtained from (4.51) and (4.17) as

$$\mathbb{E}\left[N_{BP}\right] = \frac{\lambda \bar{c} \mathbb{E}(U)}{(1 - \rho)(1 - \Gamma_{U}(\lambda))} = \lambda \bar{c} \mathbb{E}(T) \quad (4.52)$$

Distribution of delay from occupation period

The analysis is almost identical as in Model 1.

As in Model 1, $T_{S}$ is redefined by

$$T_{S} = \sum_{i=1}^{N} V_i + \sum_{j=1}^{\infty} X_j, N > 0$$

where $X_j$ is defined as before, $X_0 = \sum_{i=1}^{N} V_i$. 

The recurrence relation between $X_j's$ is the same as in (4.23), i.e.

$$\Gamma_{X_j}(z \mid N) = \left[ \Gamma_{X_{j-1}} \left( \lambda(1 - C \left[ \Gamma_V(z) \right]) \mid N \right) \right] \quad (j > 1) \quad (4.53)$$

where

$$\Gamma_{X_0}(z \mid N) = \left[ \Gamma_V(z) \right]^N \quad (4.54)$$

LST of the queueing time of the tagged customer conditioned that it enters during $X_j$ is, as before,

$$E\left[ e^{-zW_0} \mid j \right] = \frac{\Gamma_{X_{j+1}}(z \mid N) - \Gamma_{X_j}(z \mid N)}{(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] )E(X_j)} \quad (4.55)$$

Removing the condition that our arrival occurs in $X_j$ (still conditioned that our arrival enters during a BP) we have from (4.53)-(4.55) and (4.3)

$$E\left[ e^{-zW_0} \mid \text{tagged customer enters in the BP} \right]$$

$$= \sum_{N=1}^{\infty} \sum_{j=0}^{\infty} \sum_{u=0}^{\infty} E\left[ e^{-zW_0} \mid j \right] \frac{E(X_j)}{E(T_S)} \frac{q_N(u)dF(u)}{1 - b_0}
$$

$$= \sum_{N=1}^{\infty} \sum_{j=0}^{\infty} \sum_{u=0}^{\infty} \frac{\Gamma_{X_{j+1}}(z \mid N) - \Gamma_{X_j}(z \mid N)}{(z - \lambda + \lambda C \left[ \Gamma_V(z) \right] )E(T_S)} \frac{q_N(u)dF(u)}{1 - b_0}$$
and finally,

\[
\begin{align*}
E\left[ e^{-zW_o} \mid \text{tagged customer enters during the BP} \right] &= \\
&= \frac{1 - \Gamma_U\left[ \lambda(1-C)\left[ \Gamma_V(z) \right] \right]}{(1 - \Gamma_U(\lambda))(z - \lambda + \lambda C\left[ \Gamma_V(z) \right] )E(T_s)}.
\end{align*}
\]

LST of the queueing time conditioned that our arrival enters during a VP is the same as that when arrival enters during a particular vacation, which is, from (4.34)

\[
\begin{align*}
E\left[ e^{-zW_o} \mid \text{arrival enters during server's VP} \right] &= \\
&= \frac{\Gamma_U\left[ \lambda(1-C)\left[ \Gamma_V(z) \right] \right] - \Gamma_U(z)}{(z - \lambda + \lambda C\left[ \Gamma_V(z) \right] )E(U)}.
\end{align*}
\]

To find LST of (conditional) queueing time \(W\) of a random customer when the customer enters during a BP (or VP)
we can proceed as in Model 1 to obtain,

\[ E \left[ e^{-zW} \mid \text{random customer enters during a BP} \right] \]

\[ = \frac{(1 - \gamma_U[\lambda(1 - C \Gamma_V(z))] \gamma(1 - C \Gamma_V(z)))}{c(1 - \gamma_V(z)(z - \lambda + \lambda C \Gamma_V(z))) B(T_S)(1 - \gamma_U(\lambda))} \] \hspace{1cm} (4.58)

and

\[ E \left[ e^{-zW} \mid \text{random customer enters during a VP} \right] \]

\[ = \frac{(\gamma_U[\lambda(1 - C \Gamma_V(z))] - \gamma_U(z))(1 - c \Gamma_V(z))}{c(1 - \gamma_V(z)(z - \lambda + \lambda C \Gamma_V(z))) B(U)} \] \hspace{1cm} (4.59)

where \( B(T_S) \) is given by (4.49).

And finally LST of the (unconditional) queueing time of a random customer is obtained by using (4.58), (4.59) and (4.49) as

\[ \Gamma_V(z) = E \left[ e^{-zW} \right] \]

\[ = E \left[ e^{-zW} \mid \text{random customer arrives during a VP} \right] \cdot \text{Prob (random customer arrives during a VP)} \]

\[ + E \left[ e^{-zW} \mid \text{random customer arrives during a BP} \right] \cdot \text{Prob (random customer arrives during a BP)} \]
and LST of the distribution of delay, $W_s$ to a random customer is

$$
\Gamma_{W_s}(z) = \Gamma_{W}(z) \Gamma_{V}(z)
$$

$$
= \frac{z(1 - \rho)}{z - \lambda + \lambda c \left[ \Gamma_{V}(z) \right]} \frac{1-c \left[ \Gamma_{V}(z) \right]}{c(1 - \Gamma_{V}(z))} \left[ 1 - \Gamma_{U}(z) \right] \frac{1}{zE(U)}
$$

From (4.61), we have, mean delay to a customer is

$$
E(W_s) = E(V) + \frac{\lambda c E(V^2)}{2(1 - \rho)} + \frac{\mu E(V)}{2c(1 - \rho)} + \frac{E(U^2)}{2E(U)}
$$

$$
= E\left[ \text{delay to a customer for an ordinary } M^{(X)}/G/1 \text{ queue} \right] + E\left[ \text{Residual life of a vacation} \right]
$$
4.4 $M^{(X)}/G^S/I$ Vacation Model 1

We consider a $M^{(X)}/G^S/I$ vacation model where underlying assumption (excepting the service) is same as in $M^{(X)}/G/I$ vacation model discussed in section 4.2. Unlike in $M^{(X)}/G/I$ vacation model, customers are served in batches of maximum capacity $S$ under 'usual bulk service rule', that is, if at the service epoch the queue consists of more than $S$ customers, the facility takes $S$ units for service, but if exactly $S$ units or less are waiting, the whole queue is taken for service. Service discipline is FCFS and arriving batches are preordered for service. For simplicity, the service time distribution is assumed to be independent of the batch size.

Embedded Markov chain of queue length.

Let us suppose we examine the system at epochs $t_0, t_1, t_2, \ldots$, of service completion or vacation termination. To distinguish between these two types of transition instants, we define the state space of the system as

$$\{(i,j) : i = 0,1 ; j = 0,1,2, \ldots \} ,$$

$i = 0$ corresponds to a vacation termination instant and $i = 1$ corresponds to a service completion instant and $j$ is the number of customers in the system at one of these instants.

If $\tau(t)$ denotes the system state or system size at time $t$, then the points $t_n$ are the regeneration points of the
process \( \{ \tau(t) \} \) and the sequence of r.v. \((i_n, j_n) = \tau(t_n + \sigma)\) defines a semi-Markov chain with a law of transition given by

\[
(i_{n+1}, j_{n+1}) = (1, \xi) \quad \text{if and only if} \quad j_n \in \delta_1
\]

\[
= (1, j_n + \xi - 3) \quad \text{if} \quad j_n \in \delta_r (r \geq 2)
\]

\[
= (1, \xi) \quad \text{if} \quad (i_n, j_n) = (0, 0), m \in \delta_1
\]

\[
= (1, m + \xi - 3) \quad \text{if} \quad (i_n, j_n) = (0, 0), m \in \delta_r (r > 2)
\]

\[
= (0, N) \quad \text{if} \quad (i_n, j_n) = (1, 0)
\]

where \( \xi \) denotes the number of customers arrived during a service time, \( N \), the number of customers present at the end of a vacation, \( m \) the number of customers in an arriving batch and we group the number of customers waiting for service or arriving in a batch into mutually disjoint classes

\[
\delta_r (r \geq 1) \text{ where } \delta_r = [(r-1)S + 1, rS].
\]

Let \( a_r = \text{Prob} (\xi = r) \int_0^{\infty} q_r(v) dB(v) \quad (r = 0, 1, 2, \ldots) \) (4.63)

where \( q_r(t) \) is defined by (4.2).

In the steady state the limiting distributions

\[
p_{i,j} = \lim_{n \to \infty} \text{Prob} \left[ i_n = i, j_n = j \right], (i = 0, 1; j = 0, 1, 2, \ldots)
\]
satisfies the following equations

\[ p_{0,j} = p_{1,0} b_j, \quad j = 0, 1, 2, \ldots \]  
\[(1.64a)\]

\[ p_{1,j} = p_{0,0} \left[ \sum_{m=1}^{S-1} c_m a_j + \sum_{m=S}^{S+j} c_m a_{j-m+S} \right] \]
\[ + \sum_{r=1}^{S-1} p_r a_j + \sum_{r=S}^{S+j} p_r a_{j-r+S}, \quad j = 0, 1, 2, \ldots \]  
\[(1.64b)\]

and \[ \sum_{i,j} p_{i,j} = 1 \]  
\[(4.64c)\]

where we define

\[ p_{i,j} = p_{0,j} + p_{1,j} \]  
\[(4.65)\]

We define the PGF's

\[ A(z) = \sum_{j=0}^{\infty} a_j z^j, \quad B(z) = \sum_{j=0}^{\infty} b_j z^j \]  
\[(4.66a)\]

\[ P_i(z) = \sum_{j=0}^{\infty} p_{i,j} z^j (i = 0, 1), \quad P(z) = P_0(z) + P_1(z) \]  
\[(4.66b)\]

where \( b_j \) is defined by (4.4).

We multiply equations (1.64a) and (1.64b) by appropriate powers of \( z \), add them, use equations (4.65) and (4.66) and get
\[ P_1(z) = \left\{ b_o(C(z) - 1) + E(z) - 1 + \sum_{r=1}^{S-1} \left( z^S - z^r \right) \left( b_o c_r + b_r \right) \right\} p_1,0 \]
\[ + \sum_{r=1}^{S-1} \left( z^S - z^r \right) p_1,0 \left\{ A(z) \left/ (z^S - A(z)) \right. \right\} \]

(4.67a)

(In deriving the (4.67a), we use the fact that

\[ \sum_{j=0}^{S+j} \sum_{m=S}^{\infty} c_m a_{j-m+S} z^j = \sum_{m=S}^{\infty} c_m z^{-S} \sum_{j=0}^{\infty} a_{j-m+S} z^{j-m+S} \]
\[ = \frac{1}{z^S} A(z) (C(z) - \sum_{m=1}^{S-1} c_m z^m) \]

and

\[ P(z) = \left\{ b_o A(z) (C(z) - 1) + E(z) - 1 + \sum_{r=1}^{S-1} \left( z^S - z^r \right) \left( b_o c_r + b_r \right) A(z) \right\} p_1,0 \]
\[ + \sum_{r=1}^{S-1} \left( z^S - z^r \right) p_1,0 \left\{ A(z) \left/ (z^S - A(z)) \right. \right\} \]

(4.67b)

From (4.3), (4.4), (4.63) and (4.66a) we have

\[ A(z) = \gamma V \left[ \lambda (1 - C(z)) \right], \quad B(z) = \gamma U \left[ \lambda (1 - C(z)) \right] \]

(4.68)

Now to evaluate the \( S \) unknown probabilities \( p_1,0, p_1,1, \ldots, p_1,S-1 \) in \( P_1(z) \) we apply Rouche's theorem (which is stated in theorem 4.2 without proof) to the denominator in (4.67a).

Theorem 4.2 (Rouche's theorem): If \( f(z) \) and \( g(z) \) are analytic functions of \( z \) inside and on a closed contour \( C \), and also if
\(|f(z)| < |g(z)| \) on \( C \), then \( g(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( C \).

We take

\[ g(z) = z^S, \quad f(z) = -A(z) = -\int_v \left[ \lambda(1-C(z)) \right] \cdot \]

That both \( f(z) \) and \( g(z) \) are analytic inside and on the unit circle \(|z| = 1 + \varepsilon (\varepsilon > 0)\) is clearly understood. Moreover, on \(|z| = 1 + \varepsilon\), we have

\[
|f(z)| = \left| \int_v \left[ \lambda(1-C(z)) \right] \cdot \right|
\]

\[
= \left| \int_0^\infty \frac{B(v)\,dv}{\lambda(1-C(z))v} \right|
\]

\[
\leq \int_0^\infty \frac{|B(v)|\,dv}{\lambda(1-C(z))v} \]

\[
= \int_0^\infty \frac{|B(v)|\,dv}{\lambda(1-C(z))v} \leq 1
\]

Thus we find that all the conditions of Rouche's theorem are satisfied. Hence the denominator of \( P_1(z) \) has \( S \) zeros \( z_i \) (\( i = 1, 2, \ldots, S-1 \)) and \( z_S = 1 \) inside \(|z| = 1 + \varepsilon (\varepsilon > 0)\) is
an arbitrary number). Since $P_1(z)$ is analytic in $|z| \leq 1 + \epsilon$, the numerator also vanishes at these zeros. In fact, numerator of $P_1(z)$ is evidently zero at $z = z_0$. Thus we have the following $(S-1)$ equations in $S$ unknowns $p_{1,r} (r = 0,1,\ldots,S-1)$

$$S-1 \sum_{r=1}^{S-1} (z_i^S - z_i^r)p_{1,r} = p_{1,0} \left[ b_0(1-C(z_i)) + 1-B(z_i) \right] - \sum_{r=1}^{S-1} (z_i^S - z_i^r)(b_0c_r + b_r) \quad (i = 1,2,\ldots,S-1)$$

(4.69)

We can treat the system (4.69) as a set of $(S-1)$ equations in $(S-1)$ unknowns $p_{1,r} (r = 1,\ldots,S-1)$ assuming $p_{1,0}$ known. These equations are linearly independent if their determinant $\Delta \neq 0$, where

$$\Delta = \begin{vmatrix}
z_1^{S-1} - z_1 & z_2^S - z_2 & \cdots & z_1^S - z_1 \\
z_2^{S-1} - z_2 & z_2^S - z_2 & \cdots & z_2^S - z_2 \\
\vdots & \vdots & \ddots & \vdots \\
z_{S-1}^{S-1} - z_{S-1} & z_{S-1}^S - z_{S-1} & \cdots & z_{S-1}^S - z_{S-1}
\end{vmatrix}$$

(we take common factor $z_i (i = 1,2,\ldots,S-1)$ out from the $i$th row)
We subtract each column of the second determinant starting with the second from the column to its left and use simple properties of determinant:

\[
\frac{S-1}{\prod_{i=1}^{S-1} (z_i(z_i - 1))} = \left| \begin{array}{cccc}
1 & z_1 & z_1^2 & \cdots & z_1^{S-2} \\
n & z_2 & z_2^2 & \cdots & z_2^{S-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n & z_{S-1} & z_{S-1}^2 & \cdots & z_{S-1}^{S-2}
\end{array} \right|
\]

(We subtract each column of the second determinant starting with the second from the column to its left and use simple properties of determinant)

\[
= \frac{S-1}{\prod_{i=1}^{S-1} (z_i(z_i - 1))} \left( \prod_{j=i+1}^{S-1} (z_j - z_i) \right)
\]

(the final result is obtained through successive reduction of the size of the determinant by subtracting each row of the third determinant, starting with the second, from the previous row and taking common factors out in each case).

Since \(|z_1| < 1 + \varepsilon\), \(\Delta\) can be zero only when \(z_1 = 0\) for at least one \(i\) or (ii) \(z_i = z_j\) for some \(i\) and \(j\) \((1 \leq i < j \leq S-1)\). Now \(z_1 = 0\) can be a zero of \(z^S - A(z)\) only when \(a_0 = 0\), and this can not be true since there is a non zero probability that no customer arrives during a service time. Hence \(z_1 \neq 0\). Thus \(\Delta \neq 0\) if \(z_i \neq z_j\) \((1 \leq i < j \leq S-1)\) or if the \(S-1\) zeros of the denominator of \(P_{1}(z)\) are all simple. In this case only the probabilities \(P_{1,r}(r = 1, 2, \ldots, S-1)\) can be uniquely determined in terms of \(P_{1,0}\). And this condition is satisfied
for the system $M^X/X^S/1$ in which service time distribution is $k$-Erlang (convolution of $k$ independent exponential distributions with common mean $1/\mu$).

The system $M^X/X^S/1$

For the system $M^X/X^S/1$, service time has $k$-Erlang distribution given by

$$dE(v) = \frac{\mu^k v^{k-1} e^{-\mu v}}{(k-1)!} dv, \quad v > 0$$

so that $\gamma_v(z)$ becomes

$$\gamma_v(z) = \left(\frac{\mu}{z + \mu}\right)^k$$ (1.70)

For this system, the denominator of $P_1(z)$ becomes

(since $\rho = \text{traffic intensity} = \frac{\lambda k c}{S\mu}$)

$$z^S - \left(\frac{\mu}{\lambda + \mu - \lambda c(z)}\right)^k = z^S - \left[ 1 + \frac{S\rho}{kc} (1-c(z)) \right]^{-k}$$

which has $S-1$ zeros $z_i (i = 1, 2, \ldots, S-1)$, inside $|z| = 1$ other than one zero $z_S = 1$. That the zeros are simple is easy to see. For, if $z = \bar{z}$ is a multiple zero with $|\bar{z}| \leq 1$

$$\bar{z}^S = \left[ 1 + \frac{S\rho}{kc} (1-c(\bar{z})) \right]^{-k}$$
Differentiating both sides,

\[
\frac{-S-1}{z} = \frac{1}{c} \rho C'(z) \left[ 1 + \frac{Sp}{kc}(1-C(z)) \right]^{-k-1}
\]

Dividing the former by the latter gives

\[
\frac{z}{C'(z)\rho} \left[ 1 + \frac{Sp}{kc}(1-C(z)) \right] = 1
\]

and hence

\[
\rho \left[ \frac{z}{C'(z)} - \frac{S}{kc} \frac{1-C(z)}{KC} \right] = 1
\]

Since \( \rho > 0 \), \( \frac{z}{C'(z)} - \frac{S}{k cousin}(1-C(z))/KC \) is positive and hence

\[
\frac{z}{C'(z)} + \frac{S}{k cousin}(1-C(z))/KC > 0
\]

so that we have

\[
\frac{z}{C'(z)} + \frac{S}{k cousin}(1-C(z))/KC < 1 + \frac{S}{k cousin}
\]

for \( |z| \leq 1 \)

which requires that \( \rho \geq 1 \), a condition which is excluded in the hypothesis, since we know that a steady state solution exists for an infinite queue only when \( \rho < 1 \).

Now to obtain explicit expression of \( P_1(z) \) for \( M^{(X)}/E_k^{S}/1 \) queue we determine the values of \( p_{1,r} \) \( (r = 1, 2, \ldots, S-1) \) exclusively in terms of \( p_{1,0} \). Writing
d_i = d(z_i) = b_0(1-\theta(z_i)) + 1-\beta(z_i) - \sum_{r=1}^{S-1} (z_i^3 - z_i^r)(b_0c_r + b_r)

we have from (4.69)

\[ \sum_{r=1}^{S-1} (z_i^3 - z_i^r) p_{1,r} = d_1 p_{1,0} \quad (i = 1, 2, \ldots, S-1) \]

Since \( \Delta \neq 0 \), Cramer's rule can be applied whence we have

\[ p_{1,r} = w_r p_{1,0} \quad (r = 1, 2, \ldots, S-1) \quad (4.71) \]

where \( w_r = \Delta_r / \Delta \), \( \Delta_r \) is obtained from \( \Delta \) by replacing its

rth column by the column \( (d_1, d_2, \ldots, d_{S-1})^T \). Now the only remaining

unknown \( p_{1,0} \) can be found from the normalizing condition (4.64c).

Using (4.64c), (4.62a) and (4.71) we have by L'Hospital's rule

\[ l = \sum_{i,j} p_{1,i} j = P(1) \]

\[ = \frac{(S + \lambda c E(U) - \lambda k c / \mu + b_0 c + \sigma)}{S - \lambda k c / \mu} p_{1,0} \quad (4.72) \]

and

\[ p_{1,i} = \sum_{j=0}^{\infty} \cdot p_{1,j} = p_{1,i} (1) = \frac{(b_0 c + \lambda c E(U) + \sigma)}{S - \lambda k c / \mu} p_{1,0} \quad (4.73) \]

where \( \sigma = \sum_{r=1}^{S-1} (S-r) \beta_r \), \( \beta_r = b_0c_r + b_r + w_r \) \( (r = 1, 2, \ldots, S-1) \)

\( p_{1,i} \) is the probability that a service completion takes place at a
transition instant while $p_{1,j}/p_1$ is the probability that a departing batch leaves behind $j$ customers in the system. Thus the distribution of queue size observed at the departure epochs is given by the PGF $P_1(z)/p_1$. Mean number of customers in the system is $L = p_1(1)/p_1$. Required calculations for $P_1'(1)$ result in

$$L = \frac{\lambda^2 c^2 k(\lambda E(U) + \mu + \frac{1}{2} \lambda^2 c E(U^2))}{S - \lambda c k/\mu}$$

$$+ \frac{(\lambda E(U) + \mu/\lambda)\nu + \alpha + 2\alpha \lambda c k/\mu}{2(S - \lambda c k/\mu)}$$

$$+ \frac{(\lambda^2 c k(k+1)/\mu^2)(\lambda E(U) + \mu)}{2(S - \lambda c k/\mu)^2}$$

$$\frac{\{\sigma + \frac{\lambda E(U) + \mu}{\mu} - S(S-1)\} + \lambda^2 c^2 k(k+1)/\mu}{2(S - \lambda c k/\mu)^2} \eta$$

where $\alpha = \sum_{r=1}^{S-1} \left[ S(S-1)-r(r-1) \right] \beta_r$

$$\nu = \frac{d^2}{dz^2} C(z) \bigg|_{z=1} \quad \text{and} \quad \eta = p_{1,0}/p_1.$$  \hspace{1cm} (d.74)

The system $M(X)/M^S/1$

This is just the previous system when $k = 1$. Explicit expression for $P_1(z)$, $P(z)$, $p_{1,0}$, and $p_1$ are

$$\text{...}$$
Substitution of $k = 1$ in (4.74) gives the mean queue size observed at departure epochs.

4.5 $M^X/G^S/I$ Vacation Model 2

In this model we, as in Model 1, consider a $M^X/G^S/I$ queue with server's vacations where the underlying assumption is that if the server finds the system empty at the end of a vacation, he will take another vacation which starts immediately and continue in this manner until he finds at least one waiting customer upon return from a vacation. The vacations are i.i.d. r.v. and are independent of arrival and service processes.
Embedded Markov chain of queue length

In order to derive the distribution of queue size, we define an imbedded Markov chain with a state space similar to the one introduced in Model 1 (Section 4.4) except for the (0,0) state which does not exist in this model. We have

\[(i_{n+1}, j_{n+1}) = (1, \xi) \text{ if and only if } j_n \in \xi_1\]

\[= (1, j_n + \xi - 3) \quad j_n \in \delta_r (r \geq 2)\]

\[= (0, N^*) \quad (i_n, j_n) = (1, 0)\]

where \(N^*\) is the number of customers present at the end of a VI defined in Section 4.3.

Clearly, \(\text{Prob}(N = j) = b_j/(1 - b_0), j = 1, 2, \ldots\)

The limiting probabilities \(p_{i,j}\) satisfy the following equations

\[p_{0,j} = (b_j/(1 - b_0))p_{1,0}, j = 1, 2, \ldots \quad (4.80a)\]

\[p_{1,j} = \sum_{r=1}^{S-1} p_{r} a_j^r + \sum_{r=3}^{S+j} p_{r} a_j^{r+S}, j = 0, 1, 2, \ldots \quad (4.80b)\]

Following the same technique as in Model 1, we arrive at, using (4.80a), (4.80b) and (4.66),
\[ P_0(z) = \frac{B(z) - b_0}{1 - b_0} p_{1,0} \] (4.21a)

\[ P_1(z) = \frac{\{ [B(z) - 1 + \sum_{r=1}^{S-1} (z^S - z^r)b_r] p_{1,0} + (1 - b_0) \sum_{r=1}^{S-1} (z^S - z^r)p_{1,r}] A(z_o) \}}{(z^S - A(z))(1 - b_0)} \] (4.21b)

and

\[ P(z) = P_0(z) + P_1(z) \]

\[ \frac{\left[ z^S (B(z) - b_0) - A(z)(1 - b_0) + \sum_{r=1}^{S-1} (z^S - z^r)b_r A(z) \right] p_{1,0} + (1 - b_0) \sum_{r=1}^{S-1} (z^S - z^r)p_{1,r} A(z) }{(z^S - A(z))(1 - b_0)} \] (4.21c)

The unknown constants \( p_{i,r} (r = 0,1,\ldots,S-1) \) can be determined from the following equations.

\[ \sum_{r=1}^{S-1} (z^S - z^r)p_{i,r} = \left[ 1 - B(z_i) - \sum_{r=1}^{S-1} (z^S - z^r)b_r \right] p_{i,0} (i = 1,2,\ldots,S-1) \] (4.22)

along with the normalizing condition

\[ \sum_{i,j} p_{i,j} = 1 \]

As we see in Model 1, unique determination of \( p_{i,r} (r = 0,1,\ldots,S-1) \) in this model is possible only when \( z_i (i = 1,2,\ldots,S-1) \) are simple zeros which holds good for \( M(X)/E^S_k/1 \) system.
Writing

\[ d_1^l = d^l(z_1) = 1 - E(z_1) - \sum_{p=1}^{S-1} (z_1 - z_{1-p}) b_r \quad (i = 1, 2, \ldots, S-1) \]

we have

\[ p_{1,r} = \frac{1}{r} p_{1,0} \quad (r = 1, 2, \ldots, S-1) \]

where \( w_r = \Delta_r^1/\Delta \) (\( \Delta \) is defined as in Section (4.4)) and \( \Delta_r^1 \) is the corresponding quantity for \( \Delta_r \) when \( d_1 \) is replaced by \( d_1^l \).

For \( M(X)/E_k/1 \) system, using L'Hospital's rule, we have from (4.81c)

\[ 1 = \sum_{i,j} p_{i,j}^l = P(1) \]

\[ = \frac{\left[ (1-b_0)(S-\lambda ck/\mu) + \lambda \bar{c} E(U) + \sigma^l \right] p_{1,0}}{(1-b_0)(S-\lambda ck/\mu)} \quad (4.84) \]

\[ p_{1,0} = P(1) - p_{1,0} \]

\[ = \frac{\left[ \lambda \bar{c} E(U) + \sigma^l \right] p_{1,0}}{(1-b_0)(S-\lambda ck/\mu)} \quad (4.85) \]

where \( \sigma^l = \sum_{r=1}^{S-1} \left[ S(S-1) - r(r-1) \right] b_r^l \)

\[ b_r^l = b_r + (1-b_0) w_r^l \quad (r = 1, 2, \ldots, S-1) \quad (4.86) \]
From (4.25) and (4.36) we have

\[ P_{1,0} = \frac{(1-b_c)(S-\lambda c k/\mu)}{(1-b_0)(S-\lambda c k/\mu) + \lambda E(U) + \sigma l} \]  
(4.87)

\[ P_1 = \frac{\lambda E(U) + \sigma l}{(1-b_0)(S-\lambda c k/\mu) + \lambda E(U) + \sigma l} \]  
(4.88)

Mean number of customers in the system observed at departure epochs works out as

\[
L = \left\{ \frac{\lambda^2 c^2(kE(U)/\mu + E(U')^2/2)}{S - \lambda c k/\mu} + \frac{\lambda E(U) + \alpha l + 2\sigma l \lambda c k/\mu}{2(S - \lambda c k/\mu)} \right\} \frac{\eta}{1-b_0} 
\]  
(4.89)

where \( \alpha l = \sum_{r=1}^{S-1} \left[ S(S-1) - r(r-1) \right] \beta_r \) , \( \eta = \frac{P_{1,0}}{P_1} \).

For \( M^X/M^S/1 \) system explicit expressions for \( P_1(z) \) and \( P(z) \) are

\[
P_1(z) = \frac{\mu \left[ B(z) - 1 + \sum_{r=1}^{S-1} (z^{S-r} - z^r) \beta_r \right] P_{1,0}}{\left\{ \lambda (1-C(z))z^S - \mu (1-z^S) \right\} (1-b_0)} 
\]  
(4.90)

and
\[ P(z) = \frac{\left\{ \mu + \lambda(1-C(z)) \{B(z) - b_0 \} z^S + \sum_{r=1}^{S-1} (z^S - z^r) B_r - (1-b_0) \mu \right\} P_{1,0}}{\left[ \lambda(1-C(z)) - \mu(1-z^S) \right] (1-b_0)} \]  

where

\[ P_{1,0} = \frac{(1-b_0)(S-\lambda c/\mu)}{(1-b_0)(S-\lambda c/\mu) + \lambda c B(U) + \sigma^2} \]  

Substitution of \( k = 1 \) in (4.89) gives the mean queue size observed at departure epochs.

**Particular cases**

Several particular cases discussed here by relaxing assumptions about various processes characterizing the models.

(i) When service occurs one at a time and the arrival takes place in batches of variable size in \( M(X)/G/1 \) system.

**Model 1**

From (4.67),

\[ P_1(z) = \frac{A(z)b_0(C(z)-1) + B(z)-1}{z - A(z)} P_{1,0} \]  

\[ P(z) = \frac{zB(z)-A(z)+b_0 A(z)(C(z)-1)}{z - A(z)} P_{1,0} \]  

where \( A(z) \) and \( B(z) \) are given by (4.68).
The only unknown $p_{1,0}$ can be evaluated from the normalizing equation $P(1) = 1$. Using L'Hôpital's rule, we have

$$ p_{1,0} = \frac{1 - \lambda \bar{c}E(V)}{1 - \lambda \bar{c}E(V) + \bar{c}(b_0 + \lambda E(U))} \quad (4.94) $$

$$ p_1 = P(1) $$

$$ = \frac{\bar{c}(b_0 + \lambda E(U))}{1 - \lambda \bar{c}E(V)} \quad (4.95) $$

Expected system size observed at departure epochs, $L = \frac{p_1(1)}{p_1}$, is given by

$$ L = \frac{\lambda \bar{c}E(V) + \frac{\lambda^2 \bar{c}E(V^2)}{2(1-\lambda \bar{c}E(V))} + \frac{\lambda^2 \bar{c}E(U^2)}{2(b_0 + \lambda E(U))}}{\frac{\bar{c}(b_0 + \lambda E(U))}{1 - \lambda \bar{c}E(V)}} \quad (4.96) $$

**Model 2**

From (4.81), we have

$$ P_1(z) = \frac{A(z)(B(z)-1)}{z - A(z)} \cdot \frac{P_{1,0}}{1 - b_0} \quad (4.97) $$

$$ P(z) = \frac{z(B(z)-b_0)-A(z)(1-b_0)}{z - A(z)} \cdot \frac{P_{1,0}}{1 - b_0} \quad (4.98) $$

As in Model 1, the only unknown $p_{1,0}$ is evaluated from $P(1) = 1$ as

$$ p_{1,0} = \frac{(1-b_0)(1-\lambda \bar{c}E(V))}{(1-b_0)(1-\lambda \bar{c}E(V)) + \lambda \bar{c}E(U)} \quad (4.99) $$
and \( p_{1*} = P_{1}(l) = P(l) - p_{1,0} \)

\[
= \frac{\lambda E(U)}{(1-b_0)(1-\lambda E(V))} p_{1,0} \tag{4.100}
\]

Expected number of customers in the system at departure epochs works out as

\[
L = \lambda E(V) + \frac{\lambda^2 E^2(V)}{2(1-\lambda E(V))} + \frac{\nu}{2c(1-\lambda E(V))} + \frac{\lambda E(U^2)}{2E(U)} \tag{4.101}
\]

It may be mentioned that \( L \neq \lambda E(W_s) \) since for bulk arrival process, the number of customers left by a departing customer is not exactly the number of customers arrived during his waiting time. If, however we could derive mean queue length (\( L \)) observed at any random point of time by other method (like supplementary variable technique) then the equality \( L = \lambda E(W_s) \) does hold.

(ii) When service occurs one by one and the arrival takes place in batches of variable size in \( M^X/M/1 \) system.

**Model 1**

From (4.93)-(4.95) we have

\[
P_1(z) = \frac{\mu [b_0(C(z)-1) + b(z)-1]}{\lambda(1-C(z)) - \mu(1-z)} p_{1,0} \tag{4.102}
\]

and
P(z) = \frac{\frac{Z}{\mu} B(z)(\mu + \lambda(1-C(z))) + b_o(C(z)-1)-1}{\lambda(1-C(z))z - \mu(1-z)} p_{1,0} \\

where \( p_{1,0} = \frac{1 - \frac{\lambda c}{\mu}}{1 - \frac{\lambda c}{\mu} + \lambda \bar{E}(U) + \frac{b_o}{\mu}} \) \hspace{1cm} (4.104)

and \( p_l = \frac{\lambda \bar{E}(U)}{1 - \frac{\lambda c}{\mu}} p_{1,0} \) \hspace{1cm} (4.105)

Mean queue size observed at departure epochs is obtained from (4.96) as

\[ L = \frac{\lambda c}{\mu} + \frac{\lambda^2 c^2}{\mu^2(1 - \frac{\lambda c}{\mu})} + \frac{\lambda \bar{E}(U)}{2(1 - \frac{\lambda c}{\mu})} \]

Model 2

From (4.97)-(4.100) we have for the \( M^X/M/1 \) system,

\[ P_l(z) = \frac{\frac{\mu}{\lambda} B(z)-1 p_{1,0}}{\lambda(1-C(z))z - \mu(1-z)} (1-b_o) \] \hspace{1cm} (4.107)

\[ P(z) = \frac{[\{\mu + \lambda(1-C(z))\} (b_o - \frac{\lambda}{\mu}) - (1-b_o)\mu]}{\lambda(1-C(z))z - \mu(1-z)} (1-b_o) \] \hspace{1cm} (4.108)

where \( p_{1,0} = \frac{(1-b_o)(1-\frac{\lambda c}{\mu})}{(1-b_o)(1-\frac{\lambda c}{\mu}) + \frac{\lambda \bar{E}(U)}{\mu}} \) \hspace{1cm} (4.109)

and \( p_l = \frac{\lambda \bar{E}(U)}{(1-b_o)(1-\frac{\lambda c}{\mu}) + \lambda \bar{E}(U)} \) \hspace{1cm} (4.110)
Mean queue size is obtained from \((4.101)\) as

\[
L = \frac{\lambda c}{\mu} + \frac{\lambda c^2}{\mu(1 - \frac{\lambda c}{\mu})} + \frac{\nu}{2c(1 - \frac{\lambda c}{\mu})} + \frac{\lambda c^2 E(U)}{2E(U)} \tag{4.111}
\]

(iii) When service occurs one by one and arrivals take place in batches of fixed size \(N_1\).

Here \(C(z) = \frac{N_1}{z}, \nu = N_1(N_1 - 1), \bar{c} = N_1\).

Equations \((4.93)-(4.111)\) give the corresponding results for these values of \(C(z), \nu\) and \(\bar{c}\).

(iv) When service occurs in batches of maximum capacity \(S\) and arrival occurs one at a time. Here \(C(z) = z, \nu = 0, \bar{c} = 1, c_r = 0, r > 1, c_1 = 1\). Substituting these values of \(C(z), \nu, \bar{c}\) in equations \((4.76)-(4.79)\) and \((4.90)-(4.92)\) we have the corresponding results for \(M/M/S/1\) models which follows trivially.

Remarks

We have seen for both the models that determination of \(p_r (r = 1, 2, \ldots, S-1)\) is possible uniquely only when the determinant \(\Delta\) is non zero i.e. zeros of \(z^S - A(z)\) are all simple which is the case when service time distribution is Erlangian. Whether the same is possible for a general service time distribution is yet to be investigated. In the case of service time distributions when zeros of \(z^S - A(z)\) are not all simple, we can not employ
Cramer's rule to solve equations (4.69) and (4.82). However, in this case there are many numerical methods, like Gauss elimination, Crout's method, Gauss-Seidel iteration, Relaxation method etc. which can be used to solve for the unknowns \( p_{1r} \). When the batch size distribution, service time distribution are known we can find the zeros of \( z^S - A(z) \) if \( S \) is assumed to have a given value by Muller's method \((\text{Conte and Boor (1972)})\). When zeros of \( z^S - A(z) \) are not simple, modified Newton-Raphson's formula viz.

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots
\]

may be applied, where \( p \) is the multiplicity of a zero of \( f(z) = z^S - A(z) \), \( x_0 \) being the starting value of the zero of \( f(z) \).

Recently Chaudhry et al. (1987) by using Muller's method presented a unified approach to numerically evaluate the steady-state probabilities for number in the system. Their approach may be used to determine the roots of the equation \( f(z) = 0 \).

4.6 GI/M/1 Vacation Model

We consider a GI/M/1 queueing model with server's vacation under exhaustive service rule. We assume that service time distribution is exponential with mean \( 1/\mu \), interarrival time is a r.v. \( U \), having a general distribution with mean \( 1/\lambda \), successive vacations are i.i.d. r.v. \( V_1, V_2, \ldots \), each having a general distribution with mean \( 1/\tau \). Vacations, service times and
interarrival times are mutually independent r.v. Server after returning from a vacation shall join the system only when the system is found to be non-empty. Both the arrivals and services occur one a time.

For notational convenience we adopt the following notation:

\[ \hat{V} \] = Residual life of a vacation of duration \( V \) measured from an arrival epoch.

\[ T \] = Length of the interval between a random epoch and the preceding arrival epoch.

\[ Z^+_1, Z^+_2 \] = Excess of \( U(T) \) over \( \hat{V} \).

\[ S_j (j \geq 1) \] = Sum of \( j \) consecutive service times

\[ F (\cdot) = \] The distribution function of the r.v. \( \mathcal{X} \) \( (F (\cdot) = 0 \) for \( \mathcal{X} \leq 0) \)

\[ f (\cdot) = \] Density function of the r.v. \( \mathcal{X} \) whenever exists

\[ \tilde{F} (\cdot) = \] Laplace Transform of \( f (\cdot) \)

From (4.27) it is easy to show that

\[ F_X (x) = \gamma \int_0^x (1 - F_X (y)) \, dy \]

\[ F_T (x) = \lambda \int_0^x (1 - F_U (y)) \, dy \]
Distributions of $Z_1^+$ and $Z_2^+$ are given by following lemma.

**Lemma 4.1.** Distribution functions of $Z_1^+$ and $Z_2^+$ are respectively given by

\[
F_{Z_1^+}(x) = \frac{\int_{s=0}^{x} \int_{y=0}^{\infty} f_U(y)f_{U_0}(s+y)dy \, ds}{\Pr(\hat{V} < U)}
\]

\[
F_{Z_2^+}(x) = \frac{\int_{s=0}^{x} \int_{y=0}^{\infty} f_{\hat{V}}(y)f_{\hat{V}_0}(s+y)dy \, ds}{\Pr(\hat{V} < T)}
\]

**Proof.** We shall prove this for the r.v. $Z_1^+$ and the same for the r.v. $Z_2^+$ can be similarly proved.

We have $Z_1^+ = U - \hat{V}, 0 < Z_1^+ < \infty$.

If we take $X = U - \hat{V}, Y = \hat{V}$ then the joint density function of $X,Y$ is

\[
f_{X,Y}(x,y) = f_{U,\hat{V}}(u,\hat{v}) \left| \frac{\partial(u,\hat{v})}{\partial(x,y)} \right|
\]

where

\[
\begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial \hat{v}}{\partial x} \\
\frac{\partial \hat{u}}{\partial y} & \frac{\partial \hat{v}}{\partial y}
\end{vmatrix} = 1
\]
so that

\[ f_{X,Y}(x,y) = f_{U,V}(u,v) \]

\[ = f_u(u)f_v(v) \quad (\text{since } U, V \text{ are independent r.v.}) \]

\[ (0 < u < \infty, 0 < v < \infty) \]

\[ = f_U(x+y)f_V(y) \quad (-\infty < x < \infty, 0 < y < \infty) \]

Integrating over \( y \) from 0 to \( \infty \) we have.

\[
\begin{align*}
f_X(x) &= \int_0^\infty f_{X,Y}(x,y) \, dy \\
&= \int_{y=0}^\infty f_U(x+y)f_V(y) \, dy \quad (-\infty < x < \infty) \\
\end{align*}
\]

\[
\begin{align*}
\Pr(V < U) &= \int_0^\infty (1 - F_U(x)) f_V(x) \, dx \\
&= 1 - e^{-\theta} \\
\end{align*}
\]

where \( \theta = \Pr(V > U) = \int_0^\infty F_U(x) f_V(x) \, dx \)

Now

\[
\begin{align*}
f_{Z_1^+}(x) \, dx &= \Pr(X < Z_1^+ < x+dx) \\
&= \Pr(X < X < x+dx | X > 0) \\
&= \frac{f_X(x)dx}{\Pr(V < U)} \quad (0 < x < \infty)
\end{align*}
\]
Clearly, \( \int_{x=0}^{\infty} f_{Z+}(x) \, dx = 1 \)

Hence

\[
F_{Z+}(x) = \int_{s=0}^{x} f_{Z+}(s) \, ds
\]

\[
= \frac{\int_{s=0}^{\infty} \int_{y=0}^{\infty} f_Z(y) f_U(s+y) \, dy \, ds}{\text{Prob}(V < U)}
\]

which completes the proof.

Since the service time distribution is exponential with parameter \( \mu \), \( S_j \), the sum of \( j \) successive service times is a two parameter \((j, \mu)\) Gamma variate with distribution function

\[
F_{S,j}(x) = I_x(\mu, j),
\]

\( I_x(\mu, j) \) is the incomplete gamma function defined by

\[
I_x(\mu, j) = \int_{0}^{x} f_{S,j}(y) \, dy
\]

where

\[
f_{S,j}(y) = \frac{j^y j! \mu^j e^{-\mu y}}{(j-1)!} \quad (j \geq 1)
\]
Embedded Markov chain of queue length

(i) Pre-arrival epoch

Let us assume that \( t_n, n = 1, 2, \ldots, (t_0 = 0) \) be the epoch at which \( n \)th customer arrives and define \( T_{n+1} = t_{n+1} - t_n \). We examine the system at time \( t_n - 0 \), that is at pre-arrival epoch. We define the two-dimensional state space of the system as

\[
\{(i,j): i = 0, 1; j = 0, 1, 2, \ldots\}
\]

where \( i = 0 \) corresponds to the case when server is on vacation and \( i = 1 \) to the case when server is in the system, \( j \) is the number of customers in the system at any of these cases.

If \( \tau(t) \) denotes the system state observed at time \( t \), then the sequence of r.v. \( (i_n, j_n) = \tau(t_n - 0) \) defines a semi-Markov chain with a law of transition defined by

\[
(i_n, j_n) = (1, r) \quad \text{and} \quad E_{n+1} = r > 1
\]

\[
(i_{n+1}, j_{n+1}) = (1, 1) \quad \text{or} \quad (i_n, j_n) = (0, r) \quad \text{and} \quad V < T_{n+1}, C_{n+1} = r > 0
\]

For \( j \geq 2 \),

\[
(i_n, j_n) = (1, j+r-1) \quad \text{and} \quad E_n = r > 0
\]

\[
(i_{n+1}, j_{n+1}) = (1, j) \quad \text{or} \quad (i_n, j_n) = (0, j+r-1) \quad \text{and} \quad V < T_{n+1}, C_{n+1} = r > 0
\]
\[(i_{n+1}, j_{n+1}) = (0, 0) \quad \text{or} \quad (i_{n+1}, j_{n+1}) = (0, j) \quad \text{and} \quad V < T_{n+1}, \]

For \( j \geq 1, \)

\[(i_{n+1}, j_{n+1}) = (0, j) \quad \text{or} \quad (i_{n+1}, j_{n+1}) = (0, j-1) \quad \text{and} \quad V > T_{n+1}, \]

where \( B_{n+1} \) is the number of customers served during \( T_{n+1} \) and \( C_{n+1} \) is the corresponding quantity during the time \( T_{n+1} - V \), given that \( V \) terminates before \( T_{n+1} \).

As \( T_n, B_n \) and \( C_n \) are independent for all \( n \) we can take \( T_n = \theta, B_n = \beta, C_n = \gamma \) for all \( n \) and their distributions are given by

\[ g_r = \text{Prob} \left[ B = r \right] = \int_{x=0}^{\infty} \frac{e^{-\mu x} (\mu x)^r}{r!} dF_\theta(x), \quad r = 0, 1, 2, \ldots \]

\[ h_r = \text{Prob} \left[ C = r \right] = \int_{x=0}^{\infty} \frac{e^{-\mu x} (\mu x)^r}{r!} dF_\beta(x), \quad r = 0, 1, 2, \ldots \]

Define the PGF
\[ G(z) = \sum_{r=0}^{\infty} g_r z^r = \tilde{P}_U(\mu(1-z)) \quad (\text{4.112}) \]

\[ H(z) = \sum_{r=0}^{\infty} h_r z^r = \tilde{P}_Z(\mu(1-z)) \]

In the steady state, limiting distributions

\[ \bar{p}_{ij} = \lim_{n \to \infty} \text{Prob} [i_n = i, j_n = j] \quad (i = 0, 1, j = 0, 1, 2, \ldots) \]

satisfies the following equations

\[ \bar{p}_{ll} = \sum_{r=1}^{\infty} \bar{p}_{lr} g_r + (1-\Theta) \sum_{r=0}^{\infty} \bar{p}_{or} h_r \quad (\text{4.113a}) \]

\[ \bar{p}_{lj} = \sum_{r=0}^{\infty} \bar{p}_{l,j+r-1} g_r + (1-\Theta) \sum_{r=0}^{\infty} \bar{p}_{o,j+r-1} h_r \quad (j \geq 2) \quad (\text{4.113b}) \]

\[ \bar{p}_{\infty} = \sum_{j=1}^{\infty} \bar{p}_{1,j} \hat{g}_j + (1-\Theta) \sum_{j=0}^{\infty} \bar{p}_{o,j} \hat{h}_j \quad (\text{4.113c}) \]

\[ \bar{p}_{o,j} = \Theta \bar{p}_{o,j-1} \quad (j \geq 1) \quad (\text{4.113d}) \]

where

\[ \hat{g}_j = \sum_{k=j+1}^{\infty} g_k = 1 - \sum_{k=0}^{j} g_k \quad (\text{4.114a}) \]

\[ \hat{h}_j = \sum_{k=j+1}^{\infty} h_k = 1 - \sum_{k=0}^{j} h_k \]
PFG of $g$ and $h$ are respectively

$$G(z) = \frac{1-G(z)}{1-z}, \quad H(z) = \frac{1-H(z)}{1+z}$$

(Eq. 4.114b)

Equations (4.113d) yields

$$\hat{P}_{0j} = e^j \hat{P}_{\infty}, \quad j = 0,1,2, \ldots$$

(Eq. 4.115)

Denoting the displacement operator $E$ (so that $E^r(p_{ij}) = p_{i,j+r}$ etc.) and using (4.112) and (4.115) we have from (4.113b)

$$\left\{ E - G(E) \right\} \hat{P}_{1j} = (1-\theta)^j \theta^H(\theta), \quad j \geq 1$$

(Eq. 4.116)

The characteristic equation of the differential equation (4.116) is

$$R(z) = z - G(z) = 0$$

It can be easily shown that when $G'(1) = -\mu P_y(0) = \mu/\lambda > 1$ i.e. $\lambda/\mu < 1$, $R(z)$ has only one zero inside $|z| = 1$. We assume that $\rho = \lambda/\mu < 1$, then if the root of $R(z) = C$ inside $|z| = 1$ is denoted by $\alpha$ and the roots on and outside $|z| = 1$ are denoted by $\alpha_i (i = 1,2, \ldots)$ then the solution of (4.116) is

$$\hat{P}_{1j} = A_1 \alpha_1^j + \sum_{i} A_i \alpha_i^j + \frac{\theta(1-\theta)H(\theta)}{\theta - G(\theta)} \hat{P}_{\infty}$$

where $A, A_i$ are constants.
Since $\sum_{j=1}^{\infty} P_{ij} < 1$, $A_i = 0$ for all $i$. Thus when $\rho < 1$, we get

$$
- \frac{3 \cdot 6 \cdot (1 - \theta) H(\theta)}{\rho} \pi_i = - A \rho^i + \ldots
$$

$$(4.117)$$

Substituting for $P_{1j}$ in (4.113c) and (4.114a), (4.114b) and (4.115) we have

$$
A = \frac{(1 - \theta) H(\theta)}{G(\theta) - \theta} \pi_{oo}
$$

$$(4.118)$$

which could be obtained by using the fact that $P_{1j}$ vanishes for $j = 0$.

Thus $P_{1j} = A(\alpha^j - \theta^j)$, $j > 1$

$$(4.119)$$

where $\pi_{oo}$ is determined from normalizing equation

$$
\sum_{j=0}^{\infty} \pi_{0j} + \sum_{j=1}^{\infty} \pi_{1j} = 1
$$

as

$$
\pi_{oo} = \left[ \frac{(1 - \theta) H(\theta)}{G(\theta) - \theta} \left( \frac{\alpha}{1 - \alpha} - \frac{\theta}{1 - \theta} + \frac{1}{1 - \theta} \right) \right]^{-1}
$$

$$(1.120)$$

Note that in ordinary GI/M/1 system steady state arrival point system size has a geometric distribution while the same is not true when server vacations are considered.
Mean number of customers in the system obtained from arrival point system size distribution is, by using (4.115), (4.118) and (4.119)

\[
E(N) = \sum_{j=0}^{\infty} p_j + \sum_{j=1}^{\infty} j p_j
\]

\[
= p_0 \sum_{j=0}^{\infty} e^j + A \sum_{j=1}^{\infty} j (e^j - e^0)
\]

or finally, \( E(N) = \frac{(1-e)H(e)}{G(e) - e} \left[ \frac{\alpha}{(1-\alpha)^2} - \frac{e}{(1-e)^2} \right] + \frac{e}{(1-e)^2} \sum_{j=0}^{\infty} e^j
\]

(ii) Random epoch

To obtain the limiting probabilities of the system size at a random epoch, we observe the system at time \( t \) which precedes an arrival epoch. Then, the sequence of r.v. \((i_t, j_t) = \tau(t)\) \((t_n \leq t < t_{n+1}, n = 0, 1, 2, \ldots)\) and \((i_n, j_n) = \tau(t_n - \alpha)\) are related by

\[
(i_t, j_t) = (1, 1) \iff (i_n, j_n) = (1, r) \enspace \text{and} \enspace E_t^* = r \geq 1
\]

or

\[
(i_t, j_t) = (1, 1) \iff (i_n, j_n) = (0, r) \enspace \text{and} \enspace V < T_t, C_t^* = r \geq 0
\]

For \( j \geq 2 \)

\[
(i_t, j_t) = (1, j) \iff (i_n, j_n) = (1, j+r-1) \enspace \text{and} \enspace E_t^* = r \geq 0
\]

or

\[
(i_t, j_t) = (1, j) \iff (i_n, j_n) = (0, j+r-1) \enspace \text{and} \enspace V < T_t, C_t^* = r \geq 0
\]
(i_n, j_n) = (1, j) and T_t > S_j+1, j > 1
(i_t, j_t) = (0, 0) or
(i_n, j_n) = (0, j) and V < T_t, T_t - V > S_j+1, j > 0

(i_t, j_t) = (0, j) \iff (i_n, j_n) = (0, j-1) and V > T_t, j > 1

where B_{t^*} is the number of customers served during T_t = t-t_n
and C_{t^*} is the corresponding quantity during T_t - V, given that
V ends before T_t. B_{t^*}, C_{t^*} and T_t are independent for all n
and hence independent of t.

The limiting probabilities \( p_{ij} \), where

\[
 p_{ij} = \lim_{n \to \infty} \text{Prob} \left[ i_t = i, j_t = j \right]
\]

satisfies the following equations

\[
 p_{11} = \sum_{r=1}^{\infty} \sum_{r=0}^{\infty} \frac{p_r}{r!} g_r^* + (1 - \Theta^*) \sum_{r=0}^{\infty} p_{r,0} h_r^* \tag{4.122a}
\]

\[
 p_{1j} = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{p_{l,j}}{r!} g_r^* \cdot (1 - \Theta^*) \sum_{r=0}^{\infty} p_{r,j+1} h_r^* \quad (j > 2) \tag{4.122b}
\]

\[
 p_{00} = \sum_{j=1}^{\infty} \sum_{j=0}^{\infty} \frac{p_{l,j}}{r!} g_r^* + (1 - \Theta^*) \sum_{r=0}^{\infty} p_{r,0} h_r^* \tag{4.122c}
\]

\[
 p_{0j} = \frac{p_{l,0}}{p_{r,j-1}} \Theta^*, \quad j > 1 \tag{4.122d}
\]
where

\[ g_r^* = \text{Prob} \left[ B^* = r \right] = \int_0^\infty e^{-\mu y} \frac{(\mu y)^r}{r!} d F_T(y) \]

\[ h_r^* = \text{Prob} \left[ C^* = r \left| V < T \right. \right] = \int_0^\infty e^{-\mu y} \frac{(\mu y)^r}{r!} d F_{Z^2}(y) \]  \hspace{1cm} (4.123)

\[ e_r^* = 1 - \sum_{k=0}^\infty e_k^* , \quad h_r^* = 1 - \sum_{k=0}^\infty h_k^* \]

and

\[ e^* = \text{Prob} \left( V > T \right) = \int_0^\infty F_T(x) f_T(x) \, dx \]  \hspace{1cm} (4.124)

PGF of \( g_r^* , h_r^* , e_r^* \) and \( h_r^* \) are respectively

\[ G^*(z) = \tilde{F}_T(\mu \left( 1 - z \right)) = \frac{\lambda}{\mu} \frac{1 - G(z)}{1 - z} \]

\[ H^*(z) = \tilde{F}_{Z^2}(\mu \left( 1 - z \right)) = \frac{\lambda}{\mu} \frac{(1 - e)(1 - H(z))}{(1 - e^*)(1 - z)} \]

\[ G(z) = \frac{1 - G^*(z)}{1 - z} , \quad H(z) = \frac{1 - H^*(z)}{1 - z} \]  \hspace{1cm} (4.125)

From (4.122d) and (4.115) we have

\[ p_{0j} = \theta^* e^{j-1} \frac{1}{p_{oo}} , \quad j \geq 1 \]  \hspace{1cm} (4.126)

and from (4.122b) we have on using (4.115), (4.119) and (4.124)
\[ p_{lj} = A(\alpha^{j-l} \sum_{r=0}^{\infty} \gamma_r a_r^* - \alpha^{j-l} \sum_{r=0}^{\infty} \gamma_r^* b_r^*) \]

\[ + \left[ (1 - \theta^*) \alpha^{j-l} \sum_{r=0}^{\infty} h_r^* e_r \right] p_{oo} \]

\[ = A \left[ \alpha^{j-l'} G^*(\alpha) - \alpha^{j-l'} G^*(\theta) \right] + \alpha^{j-l'} (1 - \theta^*) H^*(\theta) p_{oo} \]

or finally using (4.125)

\[ p_{lj} = A \left[ \alpha^{j-l'} - \alpha^{j-l'} \right] \frac{\lambda}{\mu} + \frac{\lambda}{\mu} \alpha^{j-l'} \bar{p}_{oo}, \quad j \geq 1 \quad (4.127) \]

where we have used the fact that \( G(\alpha) = \alpha \) since \( \alpha \) is the root of the equation \( Z - G(z) = 0 \).

From (4.122c) we have on using (4.115), (4.119), (4.124) and (4.125)

\[ p_{oo} = A(\sum_{j=1}^{\infty} \gamma_j^* a_j^1 - \sum_{j=1}^{\infty} \gamma_j^* e_j^1) + (1 - \theta^*) \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \theta^j h_j^* \]

\[ = A(G^*(\alpha) - G^*(\theta)) + (1 - \theta^*) \bar{p}_{oo} H^*(\theta) \]

\[ = A\left( \frac{1 - G^*(\alpha)}{1 - \alpha} - \frac{1 - G^*(\theta)}{1 - \theta} \right) + (1 - \theta^*) \frac{1 - H^*(\theta)}{1 - \theta} \bar{p}_{oo} \]
\[ P_{oo} = A(1 - \frac{\lambda}{\mu}) \left( \frac{1}{1 - \alpha} - \frac{1}{1 - \theta} \right) + \frac{1}{1 - \theta} (1 - \theta^* - \frac{\lambda}{\mu}) P_{oo} \]  

or finally,

\[ \frac{1}{1 - \alpha} - \frac{1}{1 - \theta} \]

\[ E(N) = \sum_{j=0}^{\infty} j P_{oj} + \sum_{j=1}^{\infty} j P_{lj} \]

\[ = e^{\frac{\lambda}{\mu}} \left( \frac{1}{1 - \alpha} - \frac{1}{1 - \theta} \right) + \frac{1}{1 - \theta} \sum_{j=1}^{\infty} e^j \sum_{j=0}^{\infty} \frac{\lambda}{\mu} \sum_{j=0}^{\infty} \alpha^j \]

\[ = \frac{\lambda}{\mu} A \left[ \frac{1}{(1 - \alpha)^2} - \frac{1}{(1 - \theta)^2} \right] + \frac{1}{1 - \theta} (\theta^* + \frac{\lambda}{\mu}) P_{oo} \]  

(4.129)
Waiting-time (in queue) distribution

Let $W$ be the r.v. denoting waiting time of a customer in the queue (excluding service). An arriving customer on its arrival may find the system in one of the following two states

(i) $(0,j)$ for $j > 0$ and (ii) $(1,j)$ for $j > 1$

When the customer on its arrival finds the system in state $(0,0)$, he has to wait for a time $\hat{V}$.

When the arriving customer finds the system in state $(0,j)$ for $j > 0$ he has to wait for a time $\hat{V} + S_j$.

When the arriving customer finds the system in state $(1,j)$ for $j > 1$ he has to wait for a time $S' + S_{j-1}$ ($S_0 = 0$) where $S'$ is the remaining service time of the customer already in service.

Thus distribution function of $W$ is

$$F_W(t) = \text{Prob}(\hat{V} \leq t)p_{00} + \sum_{j=1}^{\infty} \text{Prob}(\hat{V} + S_j \leq t)p_{0j}$$

$$+ \sum_{j=1}^{\infty} \text{Prob}(S' + S_j \leq t)p_{1j} \quad (4.130)$$

Now

$$\text{Prob}(\hat{V} + S_j \leq t) = \int_{x=0}^{t} \text{Prob}(S_j \leq t-x) f_{\hat{V}}(x) \, dx$$

$$= \int_{x=0}^{t} F_{\hat{V}}(t-x) f_{S_j}(x) \, dx$$
Hence \[ \sum_{j=1}^{\infty} \text{Prob} \left( V + S_j \leq t \right) p_{0j} \]

\[ = \frac{1}{p_{00}} \sum_{j=1}^{\infty} e^{j} t \int_{0}^{t-x} \frac{\mu^j x^{j-1} e^{-\mu x}}{(j-1)!} \, dx \]

\[ = \mu e \frac{1}{p_{00}} \int_{0}^{t} \left( F_{\theta}(t-x) - e^{-\mu(1-\theta)x} \right) \, dx \]

\[ = \frac{\Theta}{1-\theta} \left( F_{\theta}(t) - e^{-\mu(1-\theta)t} \xi(t) \right) - \mu(1-\theta) \xi(t) - e^{-\mu(1-\theta)x} \xi(t) \right) \]

where \[ \xi(t) = \int_{0}^{t} e^{\frac{\mu(1-\theta)x}{t}} f_{\theta}(x) \, dx \]

Since service time distribution is exponential, \( S' \) is the r.v. having same distribution as that of service time with parameter \( \mu \), so that \( S' + S_j \) is a gamma variate with parameters \( \mu, j+1 \).

Hence \[ \sum_{j=1}^{\infty} \text{Prob} \left( S' + S_{j-1} \leq t \right) p_{1j} \]

\[ = A \sum_{j=1}^{\infty} \Gamma_{\theta}(\mu, j) (\alpha^j - \epsilon^j) \] (4.132)

Using (4.131) and (4.132) we have from (4.130)
and the corresponding density function is

\[ f_W(t) = \begin{cases} f_V(t) + \mu \theta e^{-\mu(1-\theta)t} \xi(t) \\ + \frac{\mu(1-\theta)H(\theta)}{G(\theta) - \theta} \left[ \alpha e^{-\mu(1-\alpha)t} - \theta e^{-\mu(1-\theta)t} \right] \end{cases} \]  

It may be noted that stochastic decomposition principle does not apply in this model. Expected waiting time in the queue is, from (4.134).
\[ E(W) = \int_{t=0}^{\infty} t f_W(t) \, dt \]

\[ = \left\{ E(\hat{V}) + \frac{\theta}{1-\theta} \int_{t=0}^{\infty} \left( t + \frac{1}{\mu(1-\theta)} \right) f_{\hat{V}}(t) \, dt \right\} - P_\infty \]

\[ + \frac{(1-\theta)H(\theta)}{G(\theta)-\theta} \left[ \frac{\alpha}{1-\alpha} \int_{0}^{\infty} e^{-\mu(1-\theta)t} \, dt \right] \]

\[ = \left\{ \frac{1}{1-\theta} \left[ E(\hat{V}) + \frac{\theta}{\mu(1-\theta)} \right] \right\} - P_\infty \]

\[ + \frac{(1-\theta)H(\theta)}{\mu(G(\theta)-\theta)} \left[ \frac{\alpha}{(1-\alpha)^2} - \frac{\theta}{(1-\theta)^2} \right] \]

\[ = \left\{ \frac{1}{1-\theta} \left[ E(\hat{V}) + \frac{\theta}{\mu(1-\theta)} \right] \right\} - P_\infty \]

(4.135)

where \( E(\hat{V}) \), the expected residual life of a vacation, is obtained, by using (4.42b), as

\[ E(\hat{V}) = \frac{E(\hat{V}^2)}{2E(\hat{V})} \quad \text{[See p.173, Kleinrock (1975)]} \]

Relations (4.121) and (4.135) yield

\[ E(W) = \frac{1}{\mu} E(N) + \frac{E(\hat{V})}{1-\theta} \]  

(4.36)
Remark

It is apparent that the numerical evaluation of the results obtained depends on the computation of the unique real root \( \alpha \) (0 < \( \alpha < 1 \)) of the equation \( z - \int_{0}^{1} F_u(\mu(1-z)) = 0 \). Numerical evaluation of the root \( \alpha \) is a simple problem. When \( GI = B_k \), the root \( \alpha \) is listed in Kambo and Chaudhry (1984). It may be noted here that it is a simple matter to check that \( \alpha = \rho \) is the root when \( GI = M \).