EXTENSIONS OF A COMMON FIXED POINT THEOREM FOR FOUR MAPS ON A METRIC SPACE

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1. Introduction. In a recent paper Chatterjee and Singh (1989) established the following extension of a fixed point theorem due to Fisher (1977) and Rao and Rao (1984):

**THEOREM 1.** Let $(X, d)$ be a complete metric space.

Let $T_i : X \rightarrow X, i = 1, 2, 3, 4$ satisfying the following conditions:

$$
\left[ d(T_i T_2 z, T_3 T_4 y) \right]^2 \leq \alpha_1 [d(z, y)]^2 + \alpha_2 d(z, T_i T_2 z) d(y, T_3 T_4 y) \\
+ \alpha_3 d(z, T_3 T_4 y) d(y, T_1 T_2 z) + \alpha_4 d(z, y) d(x, T_i T_2 z) + \alpha_5 d(z, y) d(y, T_3 T_4 y)
$$

for all $z, y$ in $X$ where $\alpha_i \geq 0, i = 1, 2, \cdots, 5$, $\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 < 1$ and $\alpha_1 + \alpha_3 < 1$. Further assume that $T_1 T_2 = T_2 T_1$ and $T_3 T_4 = T_4 T_3$, then $T_i, i = 1, 2, 3, 4$ have a unique common fixed point in $X$.

The object of this paper is to improve the result of Chatterjee and Singh (1989) first by assuming the commutativity of mappings like Chatterjee and Singh and then by assuming a considerably weaker condition of the said commutativity as follows:

Let $(X, d)$ be a metric space and $T_i : X \rightarrow X, i = 1, 2, 3, 4$ are said to be mutually weak-commuting mapping if

$$
d(T_i T_1 T_2 z, T_3 T_4 y) \leq d(T_i T_2 T_1 z, T_3 T_4 y) \text{ and }
$$

$$
d(T_i T_2 z, T_3 T_4 T_1 y) \leq d(T_i T_2 z, T_3 T_4 T_1 y) \text{ for all } z, y \text{ in } X, i = 1, 2, 3, 4. \text{ Our results can be stated in the following form:}
$$
THEOREM 2. Let \((X,d)\) be a complete metric space. Let \(T_i : X \to X, i = 1,2,3,4\) satisfying the conditions:

\[
[d(T_iT_jx, T_3T_4y)]^2 \leq [d(x, y)]^2 + \alpha_2d(x, T_1T_2x)d(y, T_3T_4y)
+ \alpha_3d(x, T_1T_2x)d(x, T_3T_4y) + \alpha_4d(x, T_1T_2x)d(y, T_1T_2x) + \alpha_5d(y, T_3T_4y)d(x, T_3T_4y)
+ \alpha_6d(y, T_3T_4y)d(y, T_1T_2x) + \alpha_7d(x, T_3T_4y)d(x, T_3T_4y)
+ \alpha_8d(x, y)d(y, T_1T_2x) + \alpha_9d(x, y)d(y, T_3T_4y) + d_{11}d(x, y)d(x, T_3T_4y) \cdots
\]

(1)

for all \(x, y\) in \(X\) where \(\alpha_i \geq 0, i = 1,2,\cdots, 11\) with \(\sum_{i=1}^{11} \alpha_i < 1\) and \(\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1\).

Further assume that \(T_1T_2 = T_2T_1\) and \(T_3T_4 = T_4T_3\) then \(T_i, i = 1,2,3,4\) have a unique common fixed point in \(X\).

THEOREM 3. Let \((X,d)\) be complete metric space. Let \(T_i : X \to X, i = 1,2,3,4\) satisfy the conditions:

\[
[d(T_iT_jx, T_3T_4y)]^2 \leq [d(x, y)]^2 + \alpha_2d(x, T_1T_2x)d(y, T_3T_4y)
+ \alpha_3d(x, T_1T_2x)d(x, T_3T_4y) + \alpha_4d(x, T_1T_2x)d(y, T_1T_2x) + \alpha_5d(y, T_3T_4y)d(x, T_3T_4y)
+ \alpha_6d(y, T_3T_4y)d(y, T_1T_2x) + \alpha_7d(x, T_3T_4y)d(x, T_3T_4y)
+ \alpha_8d(x, y)d(y, T_1T_2x) + \alpha_9d(x, y)d(y, T_3T_4y) + \alpha_{10}d(x, y)d(x, T_3T_4y) + d_{11}d(x, y)d(x, T_3T_4y) \cdots
\]

(1')

for all \(x, y\) in \(X\) where \(\alpha_i \geq 0, i = 1,2,\cdots, 11\). If \(T_i, i = 1,2,3,4\) are mutually weak-commuting mappings and \(\sum_{i=1}^{11} \alpha_i < 1, \alpha_i + \alpha_7 + \alpha_9 + \alpha_{11} < 1\), then \(T_i, i = 1,2,3,4\) have a unique common fixed point in \(X\).

2. Proof of Theorem 2. Let \(x_0\) be an arbitrary point in \(X\) and we define

\[
x_{2n+1} = T_1T_2x_{2n}, \quad n = 0,1,2,\cdots
\]

\[
x_{2n} = T_3T_4x_{2n-1}, \quad n = 1,2,3,\cdots
\]

It follows from (1) that

\[
[d(T_3T_4y, T_1T_2x)]^2 \leq [d(y, x)]^2 + \alpha_3d(y, T_3T_4y)d(x, T_1T_2x)
+ \alpha_4d(y, T_3T_4y)d(x, T_3T_4y) + \alpha_5d(y, T_3T_4y)d(x, T_1T_2x) + \alpha_6d(y, x)d(x, T_3T_4y)
+ \alpha_7d(y, x)d(x, T_3T_4y) + \alpha_{10}d(y, x)d(x, T_1T_2x) + \alpha_{11}d(y, x)d(y, T_1T_2x) \cdots
\]

(2)
By virtue of the symmetric relation of the metric we obtain from (1) and (2) that

\[ [d(T_1T_2x, T_3T_4y)]^2 \leq \alpha_1[d(x, y)]^2 + \alpha_2d(x, T_1T_2x)d(y, T_3T_4y) + \alpha_7d(y, T_1T_2x)d(x, T_3T_4y) + \frac{\alpha_3 + \alpha_4}{2} \{d(x, T_1T_2x)d(x, T_3T_4y) + d(y, T_1T_2x)d(y, T_3T_4y)\} + \frac{\alpha_8 + \alpha_9}{2} \{d(x, T_1T_2x)d(y, T_1T_2x) + d(y, T_1T_2x)d(y, T_3T_4y)\} + \frac{\alpha_3 + \alpha_10}{2} \{d(x, y)d(x, T_1T_2x) + d(y, y)d(y, T_3T_4y)\} \]

Thus by (3) we have

\[ [d(x_{2n+1}, x_{2n})]^2 = [d(T_1T_2x_{2n-1}, T_3T_4x_{2n-1})]^2 \leq \alpha_1[d(x_{2n}, x_{2n-1})]^2 + \alpha_2d(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n}) + \alpha_7d(x_{2n-1}, x_{2n+1})d(x_{2n-1}, x_{2n}) + \frac{\alpha_3 + \alpha_4}{2} \{d(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n})d(x_{2n-1}, x_{2n})\} + \frac{\alpha_8 + \alpha_9}{2} \{d(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n}) + d(x_{2n}, x_{2n})d(x_{2n}, x_{2n})\} + \frac{\alpha_3 + \alpha_10}{2} \{d(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n})d(x_{2n-1}, x_{2n})\} \]

Thus \(d(x_{2n+1}, x_{2n}) \leq Kd(x_{2n}, x_{2n-1})\), where

\[
K^2 = \frac{2\alpha_1 + \alpha_8 + \alpha_{10} + \alpha_4 + \alpha_9 + \alpha_{11} + 2\alpha_2 + \alpha_8 + \alpha_{10} + \alpha_9 + \alpha_4 + \alpha_9 + \alpha_{11} + \alpha_4 + \alpha_8}{4 - 2\alpha_9 + \alpha_{10} + \alpha_4 + \alpha_8 + \alpha_9 + \alpha_{11} + \alpha_4 + \alpha_8} < 1,
\]

by virtue of the relation \(\sum_{i \neq j} \alpha_i < 1\) in the hypothesis of the theorem 2.
Similarly, \( d(x_{2n}, x_{2n-1}) \leq K d(x_{2n-1}, x_{2n-2}) \).

So \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is a complete metric space there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \).

Now by considering \( [d(T_1T_2z, x_{2n})]^2 \) we get from (1) on letting \( n \to \infty \) that \( (1 - \alpha_4) [d(T_1T_2z, z)]^2 \leq 0 \) which implies that \( T_1T_2z = z \), since \( \alpha_4 < 1 \). Similarly by considering \( [d(x_{2n+1}, T_3T_4z)]^2 \) we conclude from (1) by letting \( n \to \infty \) that \( T_3T_4z = z \), since \( \alpha_9 < 1 \).

Thus

\[
T_1T_2z = z = T_3T_4z. 
\] (4)

Lastly,

\[
[d(T_1z, z)]^2 = [d(T_1T_1z, T_2T_2z)]^2 = [d(T_1T_2T_1z, T_2T_2z)]^2 \\
\leq \alpha_1[d(T_1z, z)]^2 + \alpha_2d(T_1z, T_1T_1z)d(z, T_3T_4z) + \alpha_3d(T_1z, T_1T_2T_1z)d(T_1z, T_3T_4z) \\
+ \alpha_4d(T_1z, T_2T_2z)d(z, T_1T_2T_1z) + \alpha_5d(T_3T_4z, T_3T_4z)d(z, T_1T_2T_1z) \\
+ \alpha_6d(T_1z, z)d(T_1z, T_3T_4z) + \alpha_7d(T_1z, z)d(z, T_1T_2T_1z) \\
+ \alpha_8d(T_3T_4z, z)d(T_1z, T_3T_4z) + \alpha_9d(T_1z, z)d(T_1z, T_3T_4z) \\
+ \alpha_{10}d(T_1z, z)d(T_1z, T_3T_4z)
\]

i.e., \((1 - \alpha_1 - \alpha_7 - \alpha_9 - \alpha_{11})[d(T_1z, z)]^2 \leq 0 \) which implies that \( T_1z = z \), since \( \alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1 \). Thus from (4) \( T_1T_2z = T_2T_1z = T_3T_4z = z \).

Similarly by considering \( [d(z, T_3z)]^2 \) we conclude from (1) that \( T_3z = z \), since \( \alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1 \) and using (4) we get \( T_3T_4z = T_4T_3z = T_4z = z \). Hence \( T_1z = T_2z = T_3z = T_4z = z \), i.e., \( z \) is a common fixed point of \( T_i, i = 1, 2, 3, 4 \).

Since the proof of the fact that \( z \) is the unique common fixed point of \( T_i, i = 1, 2, 3, 4 \) is a routine one, we omit it here.

3. Proof of Theorem 3. Let \( x_0 \) be an arbitrary point of \( X \) and we define

\[
x_{2n+1} = T_1T_2x_{2n}, \quad n = 0, 1, 2, \ldots
\]
\[
x_{2n} = T_3T_4x_{2n-1}, \quad n = 1, 2, \ldots
\]

Then proceeding exactly as in the proof of Theorem 2, we get \( T_1T_2z = T_3T_4z = z \).

Now we shall use the notion of mutually weak – commuting mappings as mentioned in Art. 1. Here we have
[d(T_1 z, z)]^2 = [d(T_1 T_2 T_3 T_4 z)]^2 \leq [d(T_1 T_2 T_3 z, T_4 T_4 z)]^2 \\
\leq \alpha_1 [d(T_1 z, z)]^2 + \alpha_2 d(T_1 T_2 T_3 z d(z, T_3 T_4 z)] + \alpha_3 d(T_1 z, T_1 T_2 T_3 z) d(T_1 z, T_3 T_4 z) \\
+ \alpha_4 d(T_1 z, T_1 T_2 T_3 z) d(z, T_1 T_2 T_3 z) + \alpha_5 d(z, T_3 T_4 z) d(T_1 z, T_3 T_4 z) \\
+ \alpha_6 d(z, T_3 T_4 z) d(z, T_1 T_2 T_3 z) + \alpha_7 d(T_1 z, T_3 T_4 z) d(z, T_1 T_2 T_3 z) \\
+ \alpha_8 d(T_1 z, z) d(T_1 T_2 T_3 z) + \alpha_9 d(T_1 z, z) d(z, T_1 T_2 T_3 z) \\
+ \alpha_{10} d(T_1 z, z) d(z, T_2 T_4 z) + \alpha_{11} d(T_1 z, z) d(T_1 z, T_2 T_4 z).

i.e. \((1-\alpha_1-\alpha_7-\alpha_9-\alpha_{11}) [d(T_1 z, z)]^2 \leq 0\) which implies that \(T_1 z = z\), since \(\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1\). Similarly, it can be shown that \(T_2 z = z\).

Again by considering \([d(z, T_3 z)]^2\) and \([d(z, T_4 z)]^2\) separately we can prove in the same manner that \(T_3 z = T_4 z = z\).

Hence \(T_1 z = T_2 z = T_3 z = T_4 z = z\), i.e. \(z\) is a common fixed point of \(T_i, i = 1, 2, 3, 4\).

We omit the proof of the fact that \(z\) is the unique common fixed point of \(T_i, i = 1, 2, 3, 4\) as before.

4. Since \(\max\{\alpha_1, \alpha_2, \ldots, \alpha_{11}\} \leq \alpha_1 + \alpha_2 + \cdots + \alpha_{11}\) for non-negative reals \(\alpha_i, i = 1, 2, \ldots, 11\) we obtain the following corollary from Theorem 2.

**COROLLARY**  In a complete metric space \((X, d)\) if there exist four self mappings \(T_i, i = 1, 2, 3, 4\) and satisfy the relations:

\[
[d(T_1 T_2 x, T_3 T_4 y)]^2 \leq \max \{\alpha_1 [d(x, y)]^2, \alpha_2 d(x, T_1 T_2 x) d(y, T_3 T_4 y), \\
\alpha_3 d(x, T_1 T_2 x) d(z, T_3 T_4 y) \alpha_4 d(x, T_1 T_2 z) d(y, T_1 T_2 x) \alpha_5 d(y, T_3 T_4 y) d(z, T_1 T_2 x), \\
\alpha_6 d(x, T_3 T_4 y) d(y, T_1 T_2 y), \alpha_7 d(x, T_3 T_4 y) d(y, T_1 T_2 z), \alpha_8 d(x, y) d(x, T_1 T_2 z), \\
\alpha_9 d(x, y) d(y, T_1 T_2 y), \alpha_{10} d(x, y) d(y, T_3 T_4 y), \alpha_{11} d(x, y) d(z, T_3 T_4 y)]
\]

for all \(x, y\) in \(X\), \(\alpha_i \geq 0, i = 1, 2, \ldots, 11\) with \(\sum_{i=1}^{11} \alpha_i < 1\) and \(\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1\). Further assume that \(T_1 T_2 = T_2 T_1\) and \(T_3 T_4 = T_4 T_3\), then \(T_i, i = 1, 2, 3, 4\) have a unique common fixed point in \(X\).

**REMARK**

(i) If we take \(T_2 = T_4 = I\) and \(\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_9 = \alpha_{10} = \alpha_{11} = 0\), then Theorem 2 reduces to Fisher (1977).
(ii) If we take $T_2 = T_4 = P$ and $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_9 = \alpha_{10} = \alpha_{11} = 0$, then Theorem 2 reduces to Rao and Rao (1984).

(iii) If we put $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_9 = \alpha_{10} = \alpha_{11} = 0$, then theorem 2 reduces to Chatterjee and Singh (1989).

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References


Some Extensions of the Fixed Point Theorems of Brouwer and Darbo

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1. Introduction: The following fixed point theorems may be regarded as extensions of the well-known Brouwer fixed point theorem.

**Theorem-1:** (S. Reich [1])

Let \( f : B^n \rightarrow \mathbb{R}^n \) be a continuous function such that for every \( y \in S^{n-1} \) there is no \( m > 1 \) with \( f(y) = my \), where \( B^n = \{x \in \mathbb{R}^n, ||x|| \leq 1\} \) and \( S^{n-1} = \{x \in \mathbb{R}^n, ||x|| = 1\} \). Then \( f \) has a fixed point.

**Theorem-2:** (D.K.Bayen and S.K.Chatterjea [2])

Let \( f : B^n \rightarrow \mathbb{R}^n \) be a continuous function such that for every \( y \in S^{n-1} \) there is no \( m < 1 \) with \( f(y) = my \). Then \( f \) has a fixed point.

Similarly the following theorem of A.Vignoli [3] may be regarded as an extension of the well-known Darbo's fixed point theorem.

**Theorem-3:** Let \( f : B(0,R) \rightarrow X \) be \( \alpha \)-contractive with constant \( k \) \((0 \leq k < 1)\) and let \( f \) satisfy the following condition on \( \partial B(0,R) \):

(a) if \( f(x) = mx \) for some \( x \in \partial B(0,R) \) then \( m \leq 1 \), where \( X \) is real Banach space, \( B(0,R) = \{x \in X : ||x|| \leq R\} \) and \( \partial B(0,R) = \{x \in X : ||x|| = R\} \). Then there exists \( x \in B(0,R) \) such that \( f(x) = x \).

Noticing the above extensions of Brouwer fixed point theorem and Darbo's fixed point theorem we are inclined to discuss the following mappings:

(I) Let \( f : B(O,R) \rightarrow X \) be \( \alpha \)-contractive with constant \( k \) \((0 \leq k < 1)\) and let \( f \) satisfy the following condition on \( \text{Int} B(O,R) \):

a) if \( f(x) = mx \) for some \( x \in \text{Int} B(O,R) \) then \( m \leq 1 \).
(II) Let \( f : B(0,R) \rightarrow X \) be \( \alpha \)-contractive with constant \( k (0 < k < 1) \)
and let \( f \) satisfy the following condition on \( \partial B(0,R) \):
(a) if \( f(x) = mx \) for some \( x \in \partial B(0,R) \) then \( m > 1 \).

(III) Let \( f : B(0,R) \rightarrow X \) be \( \alpha \)-contractive with constant \( k (0 < k < 1) \)
and let \( f \) satisfy the following condition on \( \text{Int} B(0,R) \):
(a) if \( f(x) = mx \) for some \( x \in \text{Int} B(0,R) \) then \( m > 1 \).

2. Discussion of the Problems:

First we shall consider the mapping mentioned in (I). We define
a radial retraction \( r : X \rightarrow B(0,R) \) by

\[
r(x) = \begin{cases} 
  x, & \text{if } ||x|| \leq R \\
  \frac{R - x}{||x||}, & \text{if } ||x|| > R.
\end{cases}
\]

Since \( r \) is \( \alpha \)-nonexpansive, the composite mapping \( rof : B(0,R) \rightarrow B(0,R) \)
is \( \alpha \)-contractive. Thus by Darbo's fixed point theorem there exists
\( x \in B(0,R) \) such that \( rof(x) = x \).

Thus \( x = rof(x) = \begin{cases} 
  f(x), & \text{if } ||f(x)|| \leq R \\
  \frac{f(x)}{||f(x)||}, & \text{if } ||f(x)|| > R.
\end{cases} \)

Suppose now \( x \in \text{Int} B(O,R) \) and \( x \) is not a fixed point of \( f \). Then we have
\( ||f(x)|| > R \), i.e. \( \frac{||f(x)||}{R} > 1 \).

Hence \( x = R \cdot \frac{f(x)}{||f(x)||} \)
or, \( f(x) = \frac{||f(x)||}{R} x \)

\( = mx \), where \( m = \frac{||f(x)||}{R} > 1 \) which contradicts the
condition (a) of (I). Therefore \( f(x) = x \).

On the other hand suppose \( x \in \partial B(O,R) \), then two cases may arise:
(i) \[ ||f(x)|| \leq R \]

(ii) \[ ||f(x)|| > R \]

In the case (i) \( x = f(x) \), but in case (ii) we have \( x = R \cdot \frac{f(x)}{||f(x)||} \).

This implies \( f(x) = mx \) where \( m = \frac{||f(x)||}{R} > 1 \). Thus when \( x \in \delta B(0,R) \) no conclusion about the fixed point can be drawn in the case (ii).

Thus we obtain the following theorem:

**Theorem 4**: Let \( f : B(0,R) \rightarrow X \) be \( \alpha \)-contractive with constant \( \alpha \) and let \( f \) satisfy the following condition on \( \text{Int} B(0,R) \):

(a) \( \text{if} \ f(x) = mx \text{ for some} \ x \in \text{Int} \ B(0,R) \text{ then } m < 1 \), where \( X \) is a real Banach space, \( B(O,R) = \{ x \in X : ||x|| \leq R \} \), \( \delta B(O,R) = \{ x \in X : ||x|| = R \} \) and \( \text{Int} B(O,R) = \{ x \in X : ||x|| < R \} \).

Then there exists \( x \in B(O,R) \) such that \( f(x) = x \) only when \( x \in \text{Int} B(O,R) \).

Next we consider the mapping mentioned in (II).

Here we define as before a radial retraction \( r : X \rightarrow B(O,R) \) by

\[
 r(x) = \begin{cases} 
 x, & \text{if } ||x|| \leq R \\
 R \cdot \frac{x}{||x||}, & \text{if } ||x|| > R 
\end{cases}
\]

Since \( r \) is \( \alpha \)-non-expansive, the composite mapping \( r \circ f : B(O,R) \rightarrow B(O,R) \) is \( \alpha \)-contractive. Thus by Darbo's fixed point theorem there exists \( x \in B(O,R) \) such that \( r \circ f(x) = x \).

Thus \( x = r \circ f(x) = \begin{cases} f(x), & \text{if } ||f(x)|| \leq R \\
 R \cdot \frac{f(x)}{||f(x)||}, & \text{if } ||f(x)|| > R 
\end{cases} \)

If \( x \in \text{Int} B(O,R) \) then two cases may arise

(i) \[ ||f(x)|| \leq R \]

(ii) \[ ||f(x)|| > R \]
In the case (i) $x = f(x)$; but in case (ii) we have $x = R$. 

which implies $||x|| = R$, a contradiction to the assumption $||x|| < R$.

Again if $x \notin \partial B(O,R)$ we may consider the following two cases:

(i) $||f(x)|| \leq R$

(ii) $||f(x)|| > R$.

In the case (i) $x = f(x)$; but in case (ii) we have $x = R$. 

and this implies $f(x) = mx$, where $m = \frac{||f(x)||}{R} > 1$. Thus in this case no conclusion regarding the fixed point of $f$ can be drawn.

Thus we obtain the following theorem:

**Theorem 5:** Let $f : B(O,R) \rightarrow X$ be a $\alpha$-contractive with constant $k (0 < k < 1)$, and let $f$ satisfy the following condition on $\partial B(O,R)$:

(a) if $f(x) = mx$ for some $x \in \partial B(O,R)$ then $m > 1$ where $X, B(O,R), \partial B(O,R)$ and $\text{Int } B(O,R)$ have the same meaning as in theorem 4. Then there exists $x \in B(O,R)$ such that $f(x) = x$ only when $x \in \text{Int } B(O,R)$.

Finally we consider the mapping mentioned in (III). We define as before a radial retraction $r : X \rightarrow B(O,R)$ by

$$
r(x) = \begin{cases} 
  x, & \text{if } ||x|| \leq R \\
  R \frac{x}{||x||}, & \text{if } ||x|| > R.
\end{cases}
$$

Since $r$ is a nonexpansive, the composite mapping $r \circ f : B(O,R) \rightarrow B(O,R)$ is $\alpha$-contractive and hence by Darbo's fixed point there exists $x \in B(O,R)$ such that $r \circ f(x) = x$.

Hence $x = r \circ f(x) = \begin{cases} 
  f(x), & \text{if } ||f(x)|| \leq R \\
  R \frac{f(x)}{||f(x)||}, & \text{if } ||f(x)|| > R.
\end{cases}$

If $x \notin \text{Int } B(O,R)$, we shall show that $||f(x)|| \notin R$. If it does so, we then have
\[ x = R \frac{f(x)}{\|f(x)\|} \]
\[ \Rightarrow f(x) = R \frac{\|f(x)\|}{R} x \]
\[ \Rightarrow mx = \frac{m \|x\|}{R} x \]
\[ \Rightarrow x = R \frac{\|x\|}{R} x \] which is not possible since \( \|x\| < R \).

Hence \( f(x) = x \), i.e. \( f \) has a fixed point.

On the other hand, if \( x \notin \partial B(0, R) \), then we consider the two cases:

(i) \( \|f(x)\| < R \) and
(ii) \( \|f(x)\| > R \).

In the case (i) \( x = f(x) \), i.e. \( f \) has a fixed point; but in case (ii) we get no information about the fixed point of \( f \). Indeed, if \( \|f(x)\| > R \), then \( x = R \frac{\|f(x)\|}{R} \) would imply \( f(x) = mx \), where \( m = \frac{\|f(x)\|}{R} > 1 \).

Thus we have the following theorem:

**Theorem 6**: Let \( f : B(O, R) \rightarrow X \) be \( \alpha \)-contractive with constant \( k \) (\( 0 < k < 1 \)) and let \( f \) satisfy the following condition on \( \text{Int} \ B(O, R) \):

(a) if \( f(x) = mx \) for some \( x \notin \text{Int} \ B(O, R) \) then \( m > 1 \), where \( X, B(O, R), \partial B(O, R), \text{and} \text{Int} B(O, R) \) have the same meaning as in theorem 4. Then there exists \( x \in B(O, R) \) such that \( f(x) = x \) only when \( x \in \text{Int} B(O, R) \).

**Remark**: It may be noted that Theorems 3 to 6 can easily be proved for a densifying mapping \( f \).

**References**

SOME VARIANTS OF VIGNOLI'S THEOREM ON o-CONTRACTION

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1. Introduction. The following theorem of A. Vignoli [1] may be regarded as a nice extension of the well-known Darbo's fixed point theorem in the case of a ball. Throughout the paper we assume $X$ to be a real Banach space, $B(0, R)$ a ball with centre at 0 and radius $R$ and $\partial B(0, R)$ the boundary of the ball $B(0, R)$.

Theorem 1. Let $T : B(0, R) \to X$ be a-contractive with the constant $0 < k < 1$ and let $T$ satisfies the following condition on the boundary of $B(0, R)$:

(a) if $Tx = \beta x$ for some $x \in \partial B(0, R)$ then $\beta < \mu$ where $\mu$ is any real number satisfying the inequality

(b) $0 < k < 1 - |1 - \mu|$

Then there exists $x \in B(0, R)$ such that $T(x) = \mu x$.

Remark 1. When $\mu = 1$, Vignoli's theorem reduces to an extension of Darbo's fixed point theorem in the case of a ball, just like S. Reich's [2] extension of Brouwer's fixed point theorem.

As D. K. Bayen and S. K. Chatterjea [3] considered a nice variant of Reich's theorem and A. K. Sarkar and S. K. Chatterjea [4] have recently considered some other variants of Vignoli's theorem for $\mu = 1$, we intend to consider some new variants of Vignoli's theorem for any real number $\mu$ subject to a certain condition.
2. Some new variants of Vignoli's theorem

Theorem 2. Let \( T : B(0,R) \to X \) be \( \alpha \)-contractive with constant \( k, 0 < k < 1 \) and \( T \) satisfy the following conditions.

(a) if \( Tx = \beta x \) for some \( x \in B(0,R) \) then \( \beta \geq \mu \) where \( \mu \) is any real number satisfying the inequality

(b) \( 0 < k + |1 + \mu| < 1 \).

Then there exists \( z \in B(0,R) \) such that \( T(z) = -\mu z \).

Proof. We consider the mapping \( F : B(0,R) \to X \) defined by

\[ F(x) = Tx + (1 + \mu)x \]

for \( x \in B(0,R) \) and \( \mu \) satisfies the condition (b) of the theorem. The mapping \( F \) is a \( \alpha \)-contractive. Indeed, let \( A \) be any subset of \( B(0,R) \). Then we have

\[ \alpha(F(A)) \leq \alpha(T(A)) + |1 + \mu| \alpha(A) \]

\[ = (k + |1 + \mu|) \alpha(A) \]

condition (b) of the theorem implies \( 0 < k + |1 + \mu| < 1 \), so that \( F \)

is \( \alpha \)-contractive with constant \( k + |1 + \mu| \).

Let \( r : X \to B(0,R) \) be a radial retraction. Since \( r \) is non-expansive, the composite mapping \( rF : B(0,R) \to B(0,R) \) is \( \alpha \)-contractive. Thus by Darbo's fixed point theorem there exists \( z \in B(0,R) \) such that \( z = rF(z) \).

Thus \( z = rF(z) = \begin{cases} F(z), & \text{if } \|F(z)\| < R \\ R - \frac{F(z)}{\|F(z)\|}, & \text{if } \|F(z)\| > R. \end{cases} \)

We shall now show that \( z \) is the fixed point of \( F \), i.e. \( F(z) = z \). If \( z \in \text{Int } B(0,R) \) then \( F(z) = z \), otherwise \( z = R - \frac{F(z)}{\|F(z)\|}, \) i.e.

\[ F(z) = \frac{\|F(z)\| - z}{R} \]

would imply that \( \|z\| = R \) which contradicts the assumption \( \|z\| < R \). Next suppose that \( z \in \partial B(0,R) \) and \( z \) is not a fixed point of \( F \). Then \( \|F(z)\| > R \) and
\[ z = R \frac{F(z)}{||F(z)||} \]

\[ \Rightarrow F(z) = R \frac{F(z)}{R} z \]

\[ \Rightarrow T(z) + (1+\mu)z = \frac{||F(z)||}{R} z \]

\[ \Rightarrow T(z) = \left( \frac{||F(z)||}{R} - 1 - \mu \right) z \]

\[ = \beta z, \text{ where } \beta = \frac{||F(z)||}{R} - 1 - \mu \]

Now \( \beta = \frac{||F(z)||}{R} - 1 - \mu \)

\[ = -\beta = \mu + (1 - \frac{||F(z)||}{R}) < \mu, \text{ which contradicts the condition (a) of the theorem. Thus } F(z) = z \text{ i.e. } z \text{ is a fixed point of } F. \]

Hence \( F(z) = z \) implies \( T(z) = -\mu z. \)

\textbf{Remark 2.} When \( \mu = -1, \) theorem 2 reduces to another extension of Banach's fixed point theorem for a set of a ball. Furthermore, condition (b) of the theorem implies \( \mu \in (-1,0). \)

\textbf{Theorem 3.} Let \( T : B(0,R) \rightarrow X \) be \( \alpha \)-contractive with the constant \( k, 0 < k < 1 \) and let \( T \) satisfy the following conditions:

(a) \( Tx = \beta x \) for some \( x \in \text{Int } B(0,R) \) then \( \beta > -\mu \) where \( \mu \) is any real number satisfying the inequality:

(b) \( 0 < k + |1+\mu| < 1. \)

Then there exists \( z \in B(0,R) \) such that \( T(z) = -\mu z. \)

\textbf{Proof.} Proceeding exactly in the same way as in the proof of theorem 2, we can prove that

\[ z = \rho F(z) = \begin{cases} F(z), & \text{if } ||F(z)|| \leq R \\ \frac{R}{||F(z)||} F(z) & \text{if } ||F(z)|| > R. \end{cases} \]

We shall now show that \( z \) is a fixed point of \( F \) i.e. \( z = F(z). \)
If \( z \in B(0,R) \) then two cases may arise.

(i) \( \| F(z) \| < R \) and

(ii) \( \| F(z) \| > R \).

In the case (1) \( z = F(z) \), i.e., \( z \) is the fixed point of \( F \). But in the case

(ii) we have \( z = R \frac{F(z)}{\| F(z) \|} \) which would imply \( F(z) = \frac{\| F(z) \|}{R} z \) and no conclusion regarding the fixed point of \( F \) can be drawn.

If \( z \in \text{Int} \ B(0,R) \) then \( z = F(z) \), otherwise \( z = R \frac{F(z)}{\| F(z) \|} \) which would imply \( F(z) = \frac{\| F(z) \|}{R} z \)

\[ \Rightarrow (1 + \mu + \beta)z = \frac{\| 1 + \mu + \beta \|}{R} \]

\[ = \frac{\| (1 + \mu + \beta) \|}{R} z \]

\[ \Rightarrow z = \frac{\| z \|}{R}, \text{ which contradicts the assumption } \| z \| < R. \]

Therefore \( F(z) = z \) and hence \( T(z) = -\mu z \).

Remark 3. When \( \mu = -1 \), theorem 3 reduces to another extension of Darbo's fixed point theorem in the case of a ball like theorem 2. Also condition (b) of theorem 3 implies that \( \mu \in (-2,0) \).

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REFERENCES

AN EXTENSION OF A THEOREM OF DIVICCARO, SESSA & FISHER SATISFYING A RATIONAL INEQUALITY

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Introduction: Let \((X, d)\) be a metric space and let \(S\) and \(T\) be mappings of \(X\) into itself. Now \(S\) and \(T\) are defined to be weakly commuting in the sense of Sessa (1982) iff \(d(STx, TSx) \leq d(Sx, Tx)\) for all \(x\) in \(X\). Using this sense of weakly commuting self-mappings M. L. Diviccaro, S. Sessa & B. Fisher (1986) have recently proved the following theorem:

**Theorem 1.** Let \(S, T\) and \(I\) be three mappings of a complete metric space \((X, d)\) such that for all \(x, y\) in \(X\) either

\[
d(Sx, Ty) \leq \frac{a d(Ix, Sx) d(Iy, Ty) + b d(Ix, Ty) d(Iy, Sx)}{d(Ix, Sx) + d(Iy, Ty)}
\]

if \(d(Ix, Sx) + d(Iy, Ty) \neq 0\), where \(1 < a < 2\) and \(b \geq 0\),

or \(d(Sx, Ty) = 0\) if \(d(Ix, Sx) + d(Iy, Ty) = 0\).

If the range of \(I\) contains the ranges of \(S\) and \(T\), if either \(I\) is continuous and weakly commuting with either \(S\) or \(T\), or if \(S\) is continuous and weakly commuting with \(I\), or if \(T\) is continuous and weakly commuting with \(I\), then \(S, T\) and \(I\) have a unique common fixed point \(z\). Further, \(z\) is the unique common fixed point of \(S\) and \(I\) and of \(T\) and \(I\).

Now it may be of interest to remark that the above theorem 1 can be extended further as follows:

**Theorem 2.** Let \(S, T, P\) be three mappings of a complete metric space \((X, d)\) such that for all \(x, y\) in \(X\) either

\[
d(Sx, Ty) \leq \frac{a d(Px, Sx) d(Py, Ty) + b d(Px, Ty) d(Py, Sx)}{d(Px, Sx) + d(Py, Ty)}
\]

\[+ \frac{c d(Px, Sx) d(Py, Sx) + e d(Py, Ty) d(Px, Ty)}{d(Px, Ty) + d(Py, Sx)} \]

\[\ldots(1)\]

if \(d(Ix, Sx) + d(Py, Ty) \neq 0\) and \(d(Px, Ty) + d(Py, Sx) \neq 0\) where \(a, b, c, e \geq 0\) with \(a + c + e < 2\),

or \(d(Sx, Ty) = 0\) if \(d(Px, Sx) + d(Py, Ty) = 0\) \[\ldots(1')\]

If the range of \(P\) contains the ranges of \(S\) and \(T\), if either \(P\) is continuous and weakly commuting with either \(S\) or \(T\), or if \(S\) is continuous and weakly commuting with \(P\), or if \(T\)
is continuous and weakly commuting with P, then S, T and P have a unique common fixed point * in X. Further, * is the unique common fixed point of S and P and of T and P.

Proof of Theorem 2: It follows from (1) that

\[
\begin{align*}
\frac{d(Ty, Sx)}{d(Py, Ty)} & \leq \frac{ad(Py, Ty) + bd(Py, Sx)}{d(Py, Ty)} \\
& \quad + \frac{cd(Py, Ty) + ed(Py, Sx)}{d(Py, Ty)} \\
& \quad + \frac{ed(Py, Sx)}{d(Py, Ty)}
\end{align*}
\]  

(2)

By virtue of the symmetric property of metric, we obtain from (1) & (2) that

\[
\begin{align*}
\frac{d(Sx, Ty)}{d(Px, Sx)} & \leq \frac{ad(Px, Sx) + bd(Px, Ty)}{d(Px, Sx)} \\
& \quad + \frac{cd(Px, Sx) + ed(Px, Ty)}{d(Px, Sx)} \\
& \quad + \frac{ed(Px, Sx)}{d(Px, Sx)}
\end{align*}
\]  

(3)

Let \(x_0\) be an arbitrary point in \(X\) and we define

\[
\begin{align*}
Sx_{2n} &= Px_{2n+1}, \quad n = 0, 1, 2, \ldots \\
Tx_{2n+1} &= Px_{2n+2}, \quad n = 1, 2, 3, \ldots
\end{align*}
\]

Now using the inequality (3) we thus obtain,

\[
\begin{align*}
\frac{d(Sx_{2n}, Tx_{2n+1})}{d(Px_{2n}, Sx_{2n})} & \leq \frac{ad(Px_{2n}, Sx_{2n}) + bd(Px_{2n+1}, Sx_{2n})}{d(Px_{2n}, Sx_{2n})} \\
& \quad + \frac{cd(Px_{2n}, Sx_{2n}) + ed(Px_{2n+1}, Sx_{2n})}{d(Px_{2n}, Sx_{2n})} \\
& \quad + \frac{ed(Px_{2n}, Sx_{2n})}{d(Px_{2n}, Sx_{2n})}
\end{align*}
\]  

(4)

Let \(d(Sx_{2n}, Tx_{2n+1}) \leq \frac{ad(Sx_{2n}, Tx_{2n}) + bd(Sx_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n})}
\]

(5)

\[
\begin{align*}
1 - \frac{c + e}{2} & \leq \frac{d(Sx_{2n}, Tx_{2n})}{d(Sx_{2n}, Tx_{2n})}
\end{align*}
\]  

(6)
or, \[ d(Sx_{2n}, Tx_{2n+1}) \leq \frac{2a+c+e-2}{2-c-e} \cdot d(Sx_{2n}, Tx_{2n-1}) \]

\[ = Kd(Sx_{2n}, Tx_{2n-1}), \] where \( K = \frac{2a+c+e-2}{2-c-e} < 1 \).

Similarly by considering \( d(Tx_{2n-1}, Sx_{2n}) \) and using (3) we obtain
\[ d(Tx_{2n-1}, Sx_{2n}) \leq K \cdot d(Tx_{2n-2}, Sx_{2n-2}). \] Thus the sequence \([Sx_0, Tx_1, ..., Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, ...]\) is a Cauchy sequence in the complete metric space \( X \) and so it has a limit \( w \) in \( X \).

Hence the sequences \([Sx_{2n-1}] = [Px_{2n-1}] \) and \([Tx_{2n-1}] = [Px_{2n}]\) converge to the point \( w \), since they are subsequences of the above sequence in \( X \). First of all we suppose that \( P \) is continuous. Then the sequences \([Px_{2n}]\) and \([PSx_{2n}]\) converge to the point \( Pw \). Again if \( P \) weakly commutes with \( S \) we then have
\[ d(SPx_{2n}, Pw) \leq d(SPx_{2n}, Psx_{2n}) + d(Psx_{2n}, Pw) \]
\[ \leq d(Sx_{2n}, P_{x2n}) + d(Psx_{2n}, Pw) \]
which implies, on letting \( n \) tend to infinity, that the sequence \([SPx_{2n}]\) also converges to \( Pw \). We now claim that \( Tw = Pw \).

If \( Tw \neq Pw \), then \( d(Tw, Pw) > 0 \) and using (3) we have
\[ d(SPx_{2n}, Tw) \leq \frac{ad(Px_{2n}, SPx_{2n}) d(Pw, Tw) + bd(Px_{2n}, Tw) d(Pw, SPx_{2n})}{d(Px_{2n}, SPx_{2n}) + d(Pw, Tw)} \]
\[ + \frac{(c+e)}{2} d(Px_{2n}, SPx_{2n}) d(Pw, Tw) + d(Pw, Tw), \]
\[ \leq d(Pw, Sw) + d(Tw, Sw) \]
\[ \leq c + e, \]
\[a contradiction, since c + e < 2.\] Thus \( Tw = Pw \).

Next we claim that \( Sw = Tw \). If not then \( Sw \neq Tw = Pw \) and \( d(Sw, Tw) > 0 \). Now from (3) we have
\[ d(Sw, Tw) \leq \frac{ad(Pw, Sw) d(Pw, Tw) + bd(Pw, Sw) d(Pw, Sw)}{d(Pw, Sw) + d(Pw, Tw)} \]
\[ + \frac{c+e}{2} \left( d(Pw, Sw) d(Pw, Tw) + d(Pw, Tw) \right), \]
\[ = \frac{c+e}{2} \cdot d(Pw, Sw) \]
\[ \leq \frac{c+e}{2} \cdot d(Tw, Sw), \]
i.e., \( 1 - \frac{c+e}{2} \cdot d(Sw, Tw) \leq 0 \), again a contradiction, since \( c + e < 2 \).
Thus $Sw = Tw$. Hence $Sw = Tw = Pw$.

Similarly if $P$ is continuous and weakly commutes with $T$, we can prove $Sw = Tw = Pw$.

Next suppose that $S$ is continuous instead of $P$. Then the sequences $\{S^nx_{2n}\}$ and $\{SPx_{2n}\}$ converge to the point $Sw$. Since $S$ weakly commutes with $P$ we have

$$d(PSx_{2n}, Sw) \leq d(PSx_{2n}, SPx_{2n}) + d(SPx_{2n}, Sw)$$

which implies, on letting $n$ tend to infinity, that the sequence $\{PSx_{2n}\}$ also converges to $Sw$. Again since the range of $P$ contains the range of $S$, there exists a point $u$ in $X$ such that $Sw = Pu$. We claim $Tu = Sw$.

If $Tu \neq Sw = Pu$, then $d(Tu, Sw) > 0$ and from (3) we obtain

$$d(S^nx_{2n}, Tu) \leq \frac{ad(PSx_{2m}, S^nx_{2n})}{d(PSx_{2m}, S^nx_{2n})} + \frac{bd(PSx_{2m}, Tu)}{d(PSx_{2m}, Tu)} + \frac{c+e}{2} \cdot \left( d(Pu, Sw) + d(Sw, Tu) \right)$$

and letting $n$ tend to infinity, it follows that

$$d(Sw, Tu) \leq \frac{c+e}{2} \cdot d(Pu, Sw) + d(Pu, Sw) \leq \frac{c+e}{2} \cdot d(Pu, Sw) + d(Sw, Tu)$$

i.e. $(1 - \frac{c+e}{2})d(Sw, Tu) \leq 0$, a contradiction, since $c+e < 2$.

Now we shall show that $Su = Tu$. If $Su \neq Tu = Pu$, we have from (3)

$$d(Su, Tu) \leq \frac{ad(Pu, Su) + bd(Pu, Tu) + bd(Pu, Su)}{d(Pu, Su) + d(Pu, Tu) + d(Pu, Su)} + \frac{c+e}{2} \cdot \left( d(Pu, Su) + d(Su, Tu) \right)$$

i.e. $(1 - \frac{c+e}{2})d(Su, Tu) \leq 0$, again a contradiction and hence $Su = Tu = Pu$.

A similar conclusion is achieved if we assume that $T$ is continuous and weakly commutes with $P$.

Therefore we come to the conclusion that there exists a point $w$ in $X$ such that $Sw = Tw = Pw$. If $P$ weakly commutes with $S$, we have $d(SPw, PSw) \leq d(Pw, Sw) = 0$ which implies that $SPw = PSw$. Thus $SSw = SPw = PSw = Pw$.

Now $d(PSw, SSw) + d(Pw, Tw) = 0$ and using the condition (1) we get $d(SSw, Tw) = 0$ which gives $SSw = Tw$. Therefore from (4), we have $SPw = SSw = Tw = Pw$ and consequently
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$Pw-z$ is a fixed point of $S$. Further (4) implies that $Pz=PPw=SPw=z$. and using (3) and the assumption that $Tz \neq z$, we have

$$d(z, Tz) - d(Sz, Tz) \leq \frac{a d(Pz, Sz) d(Pz, Tz) + bd(Pz, S) d(Pz, Tz)}{d(Pz, Sz) + d(Pz, Tz)}$$

$$+ \frac{c+e}{2} \frac{d(Pz, Sz) d(Pz, Sz) + d(Pz, Tz) d(Pz, Tz)}{d(Pz, Tz) + d(Pz, Sz)}$$

$$= \frac{c+e}{2} d(z, Tz).$$

i.e. $\left(1 - \frac{c+e}{2}\right) d(z, Tz) \leq 0$, a contradiction, since $c+e<2$ & therefore $Tz=z$. Hence $Sz = Tz = Pz = z$, i.e. $z$ is a common fixed point of $S, T$ & $P$. Again if $P$ weakly commutes with $T$, we can similarly prove that $z$ is also a common fixed point of $S$, $T$ and $P$. Now suppose that $z'$ is a second common fixed point of $S$ & $P$. Then $d(Pz', Sz') + d(Pz, Tz) = 0$ and condition (1') implies that $d(Sz', Tz') = 0$, i.e. $Sz'=Tz'$. Therefore $z' = Sz' = Tz = z$. We can similarly prove that $z$ is the unique common fixed point of $P$ and $T$. Indeed, if $z'$ be a second common fixed point of $P$ and $T$, then $d(Pz, Sz) + d(Pz', Tz') = 0$ and from the condition (1') we get $d(Sz, Tz') = 0$, i.e. $Sz=Tz'$ and hence $z'=Tz'=Sz=z$. This completes the proof of the theorem.

REMARK (i) If we put $c-e=0$ in (1), then we get theorem 1 of Diviccare, Sessa & Fisher (1986).

(ii) If we take $P$ as the identity map on $X$ and $c-e=0$ in (1), then above theorem 2 reduces to theorem 3 of B. Fisher (1978).

The following corollaries are also interesting:

COROLLARY 1 Let $S$ and $T$ be two mappings of a complete metric space $(X, d)$ into itself such that for all $x, y$ in $X$ either

$$d(Sx, Ty) \leq \frac{ad(x, Sx) d(y, Ty) + bd(x, Ty) d(y, Sx) + cd(y, Ty) d(x, Ty)}{d(x, Sx) + d(y, Ty)}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ and $d(x, Ty) + d(y, Sx) \neq 0$, where $a, b, c \geq 0$ with $a+c+e<2$

or, $d(Sx, Ty) = 0$ if $d(x, Sx) + d(y, Ty) = 0$.

Then $S$ and $T$ have a unique common fixed point $z$ in $X$. Further $z$ is the unique fixed point of $S$ and $T$. 

Corollary 2. Let $S$ and $P$ are two mappings of a complete metric space $(X, d)$ into itself such that for all $x, y$ in $X$ either

$$d(Sx, Sy) \leq \frac{ad(Px, Sx) + bd(Py, Sy) + cd(Px, Sx) + ed(Py, Sx)}{d(Px, Sx) + d(Py, Sy)} + \frac{cd(Px, Sx) + ed(Py, Sx)}{d(Px, Sx) + d(Py, Sx)}$$

If $d(Px, Sy) + d(Py, Sx) \neq 0$ and $d(Px, Sy) + d(Py, Sx) \neq 0$, where $a, b, c, e \geq 0$ with $a + c + e < 2$, or $d(Sx, Sy) = 0$ if $d(Px, Sx) + d(Py, Sy) = 0$.

If $P$ weakly commutes with $S$ or $P$ is continuous, then $S$ and $P$ have a unique fixed point in $X$.

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