Chapter 2

Equitable Total Domination in Graphs
ABSTRACT

A subset $D$ of a vertex set $V(G)$ of a graph $G = (V, E)$ is called an equitable dominating set if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\text{deg}(u) - \text{deg}(v)| \leq 1$, where $\text{deg}(u)$ and $\text{deg}(v)$ are denoted as the degree of a vertex $u$ and $v$ respectively. The equitable domination number of a graph $\gamma^e(G)$ of $G$ is the minimum cardinality of an equitable dominating set of $G$. An equitable dominating set $D$ is said to be an equitable total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices. The equitable total domination number $\gamma_t^e(G)$ of $G$ is the minimum cardinality of an equitable total dominating set of $G$. In this chapter, we initiate a study on new domination parameter equitable total domination number of a graph, obtain some bounds on $\gamma_t^e(G)$, its exact values of standard class of graphs and its relationship with other domination parameters are established, characterize the equitable total domination and total domination parameters are equal and also discussed Northaus-Gaddum type results.
2.1 Introduction

All graphs considered here are finite, nontrivial, undirected, without loops or multiple edges or isolated vertices. The order and size of $G$ are denoted by $p$ and $q$ respectively. For undefined terms or notations in this chapter, may be found in Harary [24] and Haynes et al. [25].

The concept of domination arise from practical considerations. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form an association and move closely. Equitability among citizens in terms of wealth, health, status etc., is the goal of a democratic nation. In order to study this practical concept, a graph model is to be created. E. Sampathkumar is the first person to recognize the spirit and power of this concept and introduced various types of equitability in graphs like degree equitability, outward equitability, inward equitability, equitability in terms of number of equal degree neighbors, or in terms of number of strong degree neighbors, etc.

The concept of degree equitable domination in graphs was introduced by Venkatasubramanian Swaminathan and Kuppusamy Markandan Dhar-
malingam [55]. They have the following definition.

A subset $D$ of $V$ is called an equitable dominating set, if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\text{deg}(u) - \text{deg}(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma^e(G)$ and is called the equitable domination number of $G$. The concept of degree equitable domination in graphs has been studied by various authors, in [2, 47, 48, 53, 55, 57].

In general, if $G$ is a simple graph and $\phi : V \to N$ is a function, we may define equitability of vertices in terms of $\phi$ - values of the vertices and study of the equitability is defined by degree function.

Cockayne et al. [20], introduced the concept of total domination in graphs. A dominating set $D$ of $G$ is called a total dominating set, if $\langle D \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. The concept of total domination in graphs has been studied by several graph theorists in the world, for example, in [3, 4, 20, 27, 62].

In this chapter, we introduce the concept of equitable total domination in graphs.
2.2 Equitable Total Domination Number $\gamma_t^e(G)$

**Definition 2.2.1** A subset $D$ of $V$ is called an equitable total dominating set of $G$, if $D$ is an equitable dominating set and $\langle D \rangle$ has no isolated vertices. The minimum cardinality taken over all equitable total dominating sets is the equitable total domination number and is denoted by $\gamma_t^e(G)$.

For illustration, we consider the graph $G$ as shown in Figure 2.1.

![Figure 2.1](image)

In Figure 2.1, $V(G) = \{1, 2, 3, 4, 5, 6\}$.

The equitable total dominating sets are $D_1 = \{2, 3, 5, 6\}$, $D_2 = \{1, 3, 4, 6\}$, $D_3 = \{1, 2, 4\}$, and $D_4 = \{2, 4, 5\}$. Therefore, $\gamma_t^e(G) = 3$. 
2.3 Main Results on $\gamma_t^e(G)$

By the definition of an equitable total dominating set, the following result is obvious.

**Theorem 2.3.1** A total dominating set $D$ of $G$ is an equitable total dominating set if and only if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\text{deg}(u) - \text{deg}(v)| \leq 1$.

The following theorem will be useful in the proof of our result.

**Theorem 2.3.2** [4].

(i) For the paths $P_p$ and cycles $C_p$ on $p$ vertices,

$$
\gamma_t(P_p) = \gamma_t(C_p) = \begin{cases} 
\frac{p}{2} & \text{if } p \equiv 0 \text{ (mod 4)} \\
\left\lceil \frac{p}{2} \right\rceil + 1 & \text{otherwise.}
\end{cases}
$$

(ii) For the complete graph $K_p$ on $p \geq 2$ vertices, $\gamma_t(K_p) = 2$.

We begin our investigation of the equitable total domination number by computing its values for several well known class of graphs. In several instances, we shall use the integer functions (ceiling and floor) of a number.

In the following theorem, we proceed to compute $\gamma_t^e(G)$ for some standard graphs.
Theorem 2.3.3

(i) For any path $P_p$ with $p \geq 2$ vertices,

$$\gamma^e_t(P_p) = \begin{cases} \frac{p}{2} & \text{if } p \equiv 0 \text{(mod4)} \\ \lfloor \frac{p}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

(ii) For any cycle $C_p$ with $p \geq 3$ vertices,

$$\gamma^e_t(C_p) = \begin{cases} \frac{p}{2} & \text{if } p \equiv 0 \text{(mod4)} \\ \lfloor \frac{p}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

(iii) For any complete graph $K_p$ with $p \geq 2$ vertices,

$$\gamma^e_t(K_p) = 2.$$

(iv) For any wheel $W_p$ with $p \geq 4$ vertices,

$$\gamma^e_t(W_p) = \begin{cases} 2 & \text{if } p = 4, 5 \\ \lfloor \frac{p-1}{3} \rfloor + 1 & \text{otherwise.} \end{cases}$$

(v) For any complete bipartite graph $K_{m,n}$,

$$\gamma^e_t(K_{m,n}) = \begin{cases} 2 & \text{if } |m-n| \leq 1, 1 \leq m \leq n \\ m+n & \text{if } |m-n| \geq 2, m, n \geq 2. \end{cases}$$

(vi) For any star $K_{1,p-1}$ with $p \geq 2$ vertices,

$$\gamma^e_t(K_{1,p-1}) = p.$$
Proof.

(i) The degree of any vertex of path $P_p$ is either 1 or 2. Clearly, any total dominating set is equitable and by Theorem 2.3.2. Therefore $\gamma^e_t(P_p) = \gamma_t(P_p)$.

(ii) As $C_p$ is a connected 2-regular graph, any total dominating set is equitable and by Theorem 2.3.2. Hence $\gamma^e_t(C_p) = \gamma_t(C_p)$.

(iii) For any complete graph $K_p$ with $p \geq 2$, any two vertices of $K_p$ forms an equitable total dominating set of $K_p$.

Therefore $\gamma^e_t(K_p) = 2$.

(iv) Let $W_p$ be a wheel, such that $V(W_p) = \{u, v_1, v_2, \cdots, v_{p-1}\}$, where $u$ is the center vertex and each $v_i$ ; $i = 1, 2, \cdots, p-1$ is on the cycle.

Therefore the $deg_{W_p}(v_i) = 3$ ; $1 \leq i \leq p-1$ and $deg_{W_p}(u) = p - 1$, hence $p \geq 4$.

We consider the following cases:

**Case 1.** Let $p = 4$ or 5. Then $deg_{W_p}(u) = p - 1 \leq 4$ and $deg_{W_p}(v_i) = 3$, for all $i$, $1 \leq i \leq p - 1$. Therefore $D = \{u\} \cup \{v_i\}$, $1 \leq i \leq p - 1$, is an equitable total dominating set of $W_p$. Hence $\gamma^e_t(W_p) = |D|$
\[ \gamma^e_t(W_p) = |\{u\} \cup \{v_i\}| = 2. \]

**Case 2.** Let \( p \geq 6 \).

In this case \( \text{deg}_{W_p}(u) \geq 5 \), while \( \text{deg}_{W_p}(v_i) = 3 \) for all \( i \), where \( 1 \leq i \leq p - 1 \). However \( D = \{u\} \cup \{v_i\} \) is a total dominating set, but not equitable total dominating set.

If \( D = \begin{cases} 
\{u, v_1, v_4, \ldots v_{3k-2}\}, & \text{if } p - 1 = 3k \\
\{u, v_1, v_4, \ldots v_{3k-2}, v_{3k-1}\}, & \text{if } p - 1 = 3k + 1 \\
\{u, v_1, v_4, \ldots v_{3k-2}, v_{3k+1}\}, & \text{if } p - 1 = 3k + 2 
\end{cases} \)

then for any \( v_i \in V - D \), there exists \( v_{i-1} \) or \( v_{i+1} \in D \) such that, \( v_i v_{i-1} \) or \( v_i v_{i+1} \in E(G) \) and \( \text{deg}(v_i) = \text{deg}(v_{i-1}) = 3 \) or \( \text{deg}(v_i) = \text{deg}(v_{i+1}) = 3 \). Therefore \( D \) is an equitable total dominating set of \( W_p \).

Now, \( |D| = \begin{cases} 
k + 1, & \text{if } p = 3k \\
k + 2, & \text{if } p = 3k + 1 \text{ or } 3k + 2. 
\end{cases} \)

Also, when \( p - 1 = 3k \), \( \lceil \frac{p-1}{3} \rceil = k \), when \( p - 1 = 3k + 1 \) or \( 3k + 2 \), \( \lceil \frac{p-1}{3} \rceil = k + 1 \).

Hence, \( |D| = \lceil \frac{p-1}{3} \rceil + 1 \). Therefore \( \gamma^e_t(W_p) = \lceil \frac{p-1}{3} \rceil + 1 \).

**(v)** Let \( K_{m,n} \) be the complete bipartite graph with \( m \) vertices in one partition say \( V_1 \) and \( n \) vertices in another partition say \( V_2 \).
Then, \( \text{deg}_{K_m,n}(u) = \begin{cases} m, & \text{if } u \in V_1 \\ n, & \text{if } u \in V_2. \end{cases} \)

If \( |m - n| \leq 1 \), then for any vertex \( u \in V_1 \) and \( v \in V_2 \) constitute a dominating set which is an equitable total dominating set. Therefore, \( \gamma_t^e(K_m,n) = 2 \) for \( |m - n| \leq 1 \).

If \( |m - n| \geq 2 \) and \( |D| < m + n \), where \( D \) is a minimum equitable total dominating set of \( K_m,n \), then there exists \( u \in V - D \). Let \( u \in V_1 \).

Therefore \( \text{deg}_{K_m,n}(u) = n \). There is a vertex \( v \in D \), such that \( v \) is adjacent to \( u \in V_1 \) and \( |\text{deg}_{K_m,n}(v) - \text{deg}_{K_m,n}(u)| \leq 1 \). Since \( V_1 \) is independent, \( v \) must belongs to \( V_2 \). Therefore \( \text{deg}_{K_m,n}(v) = m \).

Hence \( |\text{deg}_{K_m,n}(v) - \text{deg}_{K_m,n}(u)| = |m - n| \geq 2 \), which is a contradiction. Therefore \( |D| = m + n \).

Hence \( \gamma_t^e(K_m,n) = m + n \) for \( |m - n| \geq 2 \) for all \( n, m \geq 2 \).

(vi) If \( G = K_{1,p-1}; p \geq 2 \), then clearly \( \gamma_t(G) = 2 \). By the definition of equitable total dominating set, \( D \) should contain all the vertices of \( G \). Therefore, \( \gamma_t^e(K_{1,p-1}) = p \); \( p \geq 2 \).

From the above results the following bound is immediate.

**Theorem 2.3.4** For any graph \( G \) with no isolates, \( 2 \leq \gamma_t^e(G) \leq p \).
Corollary 2.3.5 For any graph $G(\neq K_{m,n}; |m - n| \geq 2; m, n \geq 2)$ without isolated vertices, $\gamma'_t(G) \leq \frac{p}{2}$.

Definition 2.3.6 An equitable total dominating set is said to be minimal equitable total dominating set if no proper subset of $D$ is an equitable total dominating set.

In the following theorem, graphs with unique minimal equitable total dominating sets are characterized.

Theorem 2.3.7 For any graph $G$ without isolated vertices, an equitable total dominating set $D$ is minimal if and only if for every $u \in D$, one of the following two properties holds:

(i) There exists a vertex $v \in V - D$ such that $N(v) \cap D = \{u\}$, $|\deg(u) - \deg(v)| \leq 1$.

(ii) $\langle D - \{u\} \rangle$ contains no isolated vertices.

Proof. Assume that $D$ is a minimal equitable total dominating set and (i) and (ii) do not hold. Then for some $u \in D$, there exists $v \in V - D$ such that $|\deg(u) - \deg(v)| \leq 1$ and for every $v \in V - D$, either $N(v) \cap D \neq \{u\}$ or $|\deg(u) - \deg(v)| \geq 2$ or both. Therefore $\langle D - \{u\} \rangle$ contains an
isolated vertex, a contradiction to the minimality of $D$. Therefore (i) and (ii) holds.

Conversely, if for every vertex $u \in D$, the statement (i) or (ii) holds and $D$ is not minimal. Then there exists $u \in D$ such that $D - \{u\}$ is an equitable total dominating set. Therefore there exists $v \in D - \{u\}$ such that a vertex $v$ equitably dominates a vertex $u$. That is, $v \in N(u)$ and $|\text{deg}(u) - \text{deg}(v)| \leq 1$. Hence $u$ does not satisfy (i). Then $u$ must satisfy (ii) and there exists $v \in V - D$ such that $N(v) \cap D = \{u\}$ and $|\text{deg}(u) - \text{deg}(v)| \leq 1$. And also there exists $w \in D - \{u\}$ such that $w$ is adjacent to $v$. Therefore $w \in N(v) \cap D$, $|\text{deg}(w) - \text{deg}(v)| \leq 1$ and $w \neq u$, a contradiction to $N(v) \cap D = \{u\}$. Hence $D$ is a minimal equitable total dominating set.

\textbf{Proposition 2.3.8} For any graph $G$ without isolated vertices,
\[ \gamma_t(G) \leq \gamma^e_t(G). \]
Further, the equality holds in the following theorem.

\textbf{Proof.} Every equitable total dominating set is a total dominating set. Thus $\gamma_t(G) \leq \gamma^e_t(G)$. 

In the next result, we characterize $\gamma^e_t(G)$ and $\gamma_t(G)$ are equal.

\textbf{Theorem 2.3.9} If $G$ is a $r$-regular for $r \geq 1$ or $(k, k+1)$ bi-regular for any positive integer $k$, then $\gamma^e_t(G) = \gamma_t(G)$. 

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Proof. Suppose $G$ is a regular graph. Then every vertex of $G$ is of same degree say $k$. Let $D$ be the minimal total dominating set of $G$, then $|D| = \gamma_t(G)$. If $u \in V - D$, then $D$ is a total dominating set, then there exists $v \in D$ and $uv \in E(G)$, also $\text{deg}(u) = \text{deg}(v) = k$. Therefore $|\text{deg}(u) - \text{deg}(v)| = 0 \leq 1$. Hence $D$ is an equitable total dominating set of $G$, such that $\gamma^e_t(G) \leq |D| = \gamma_t(G)$. And by Proposition 2.3.8, we have $\gamma_t(G) \leq \gamma^e_t(G)$. Therefore $\gamma_t(G) = \gamma^e_t(G)$.

Now, suppose $G$ is a $(k, k+1)$ bi-regular graph. Then the degree of each vertex in $G$ is either $k$ or $k+1$, where $k$ is a positive integer. Let $D$ be a minimal total dominating set of $G$, i.e. $|D| = \gamma_t(G)$ and $u \in V - D$, then there exists $v \in D$ such that $uv \in E(G)$ and one of the vertex $u$ or $v$ is with degree $k$ and other is with degree $k+1$. This implies $|\text{deg}(u) - \text{deg}(v)| = 1$. Therefore $D$ is an equitable total dominating set of $G$. Hence $\gamma^e_t(G) \leq |D| = \gamma_t(G)$. And by Proposition 2.3.8, we have that $\gamma_t(G) \leq \gamma^e_t(G)$. Therefore, $\gamma^e_t(G) = \gamma_t(G)$.  

In the following theorems, we give the upper bound for $\gamma_t^e(G)$ in terms of order, maximum degree and girth of $G$. 

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Theorem 2.3.10 For any connected graph $G(\neq K_m,n; |m-n| \geq 2; m,n \geq 2)$ contains no isolated vertices and $\Delta(G) < p - 1$, then $\gamma_t^c(G) \leq p - \Delta(G)$.

Proof. Let $v$ be a vertex with maximum degree $\Delta(G)$ in $G$ and $X = V - N[v]$. If $X = \phi$, then $\Delta(G) = p - 1$ and $\gamma_t^c(G) = p - \Delta(G) + 1 = 2$, but $\Delta(G) < p - 1$, therefore $X \neq \phi$. Let $x \in X$ is adjacent to $y \in N(v)$. Let $V'$, $\Delta'$ be the vertex set and maximum degree of the component of $G[X]$ which contains $x$. The component of $G[V']$ has a total equitable dominating set $Y$ of cardinality at most $|V'| - \Delta' + 1$.

If $\Delta' = 1$ then $|V'| = 2$ and the set $\{v, y, x\} \cup (X - V')$ is equitable total dominating in $G$ and $\gamma_t^c(G) \leq 3 + (p - \Delta(G) - 1) - 2$

$$= p - \Delta(G).$$

If $\Delta'(G) > 1$, then the set $\{v, y\} \cup Y \cup (X - V')$ is equitable total dominating in $G$ and $\gamma_t^c(G) \leq 2 + (|V'| - \Delta' + 1) + p - \Delta(G) - 1 - |V'|$

$$= p - \Delta(G) + (2 - \Delta')$$

$$\leq p - \Delta(G).$$

Therefore $\gamma_t^c(G) \leq p - \Delta(G)$.

Theorem 2.3.11 If $G(\neq K_m,n; |m-n| \geq 2; m,n \geq 2)$ be a graph with $\text{diam}(G) = 2$, then $\gamma_t^c(G) \leq \delta(G) + 1$ and this bound is sharp.
Proof. Let \( v \in V(G) \) and \( \deg(v) = \delta(G) \). Since \( \text{diam}(G) = 2 \), \( N(v) \) is a dominating set in \( G \), \( D = N(v) \cup \{v\} \) is an equitable total dominating set of \( G \) and \( |D| = \delta(G) + 1 \). Hence \( \gamma^e_t(G) \leq \delta(G) + 1 \).

We know that \( \gamma^e_t(C_5) = 3 \), \( \delta(C_5) = 2 \), and \( \text{diam}(C_5) = 2 \), then \( \gamma^e_t(C_5) = \delta(C_5) + 1 \).

\[ \text{Theorem 2.3.12} \quad \text{For any connected graph } G \text{ with girth } g(G) \geq 5, \text{ and } \delta(G) \geq 2, \gamma^e_t(G) \leq p - \left\lceil \frac{g(G)}{2} \right\rceil + 1. \]

Proof. Let \( G \) be a connected graph with girth \( g(G) \geq 5 \) and \( \delta(G) \geq 2 \). Now let us remove the cycle \( C_p \) of shortest length from \( G \) to form \( G' \). Suppose an arbitrary vertex \( v \in V(G') \), then \( v \) has at least two neighbors say \( x \) and \( y \). If \( x, y \in C_p \) and \( d(x, y) \geq 3 \), then by replacing the path from \( x \) to \( y \) on \( C_p \) with the path \( xvy \) which reduces the girth of \( G \), a contradiction. If \( d(x, y) \leq 2 \), then \( x, y, v \) are on either \( C_3 \) or \( C_4 \) in \( G \), contradiction to the hypothesis that \( g(G) \geq 5 \). Hence no vertex in \( G' \) has two or more neighbors on \( C_p \). Therefore, \( \gamma^e_t(G') \leq p - \left\lceil \frac{g(G)}{2} \right\rceil + 1 \).

\[ \text{Theorem 2.3.13} \quad \text{[51]. For any graph } G \text{ without isolated vertices,} \]

\[ \gamma(G) \geq \left\lceil \frac{p}{\Delta(G)+1} \right\rceil \]
Next theorem gives the lower bound for $\gamma^e_t(G)$ in terms of order and maximum degree of $G$.

**Theorem 2.3.14** For any nontrivial connected graph $G$,

$$\left[\frac{p}{1 + \Delta(G)}\right] + 1 \leq \gamma^e_t(G).$$

Nordhaus - Gaddum provided the best possible bound on the sum of the chromatic numbers of a graph and its complement. A corresponding result for the domination number was presented by Jaeger and Payan[32]. If $G$ is a graph of order $p \geq 2$, then $3 \leq \gamma(G) + \gamma(G) \leq p + 1$.

We now give the best possible bounds on the sum and product of the equitable total domination number of a graph and its complement.

**Theorem 2.3.15** If $G(\neq K_{m,n}; |m - n| \geq 2 ; m, n \geq 2)$ has $p$ vertices, no isolated vertices and $\Delta(G) < p - 1$, then

(i) $\gamma^e_t(G) + \gamma^e_t(G) \leq 2\left[\frac{p}{2}\right]$

(ii) $\gamma^e_t(G) \cdot \gamma^e_t(G) \leq (\left[\frac{p}{2}\right])^2$.

Further, the equality holds for $G = C_4$ and for any self-complementary graphs and these bounds are sharp.