APPENDIX I

The classification of strongly interacting particles in the scheme of Unitary Symmetry has proved to be of great value not only in providing a "systematics of hadrons" but also in the understanding of a great amount of experimental data. In particular the SU(3) symmetry scheme, also known as The Eightfold Way, proposed by Gell-Mann and Neeman has been very successful. We here give the relevant features of the SU(3) symmetry of strong interactions. A collection of the fundamental papers on SU(3) along with explanatory discussion is contained in the book "The Eightfold Way" by Gell-Mann and Neeman (29).

The most general exact invariance that the theory of pions and nucleons, in the absence of electromagnetism, exhibits is the invariance under SU(2), the isospin group. The simplest non-trivial unitary representation of this group is the set of all 2 x 2 matrices of the form:

\[ \mathcal{U}(\lambda) = e^{i\lambda \cdot \tau} \]

where the \( \tau \) are the Pauli spin matrices with \( \tau_\alpha (\tau) = 0 \) and \( \lambda \) is a real 3 x 1 vector. This assures that the \( \mathcal{U}(\lambda) \) are unitary and unimodular i.e. \( \det[\mathcal{U}(\lambda)] = 1 \). If the strange particles are added to the picture and if the Lagrangian that characterizes their interactions with all the hadrons is to have
a higher symmetry than SU(2), then this symmetry must correspond to a larger group that contains SU(2) as a subgroup. This is the group SU(3) that can be characterized as the abstract group having the same multiplication table as the set of all 3 x 3 unitary, unimodular matrices, which, in fact, form its simplest non-trivial representation. The traceless 3 x 3 matrices (usually denoted by $\lambda_i$ and known as the generators of the group) that replace the three $\tau_i$ of SU(2) can be characterized as follows. Any 3 x 3 matrix can be written as a linear combination of nine fundamental matrices, for example:

\begin{align*}
(1 & 0 & 0) & (0 & 1 & 0) & \text{etc.}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{align*}

The $\lambda_i$, however, must be both traceless and Hermitian if $\sum_i \lambda_i$ is to be unitary and unimodular. A traceless 3 x 3 matrix has only eight independent components, since the trace condition eliminates one of the diagonal matrix elements in terms of the other two. Hence there are only eight independent matrices $\lambda_i$. Three of them, say, $\lambda_1$, $\lambda_2$, $\lambda_3$ must generate the isospin subgroup. Thus they can be written in terms of the 2 x 2 Pauli matrices $\tau_i$. Gell-Mann has given a standard choice for the other five. The eight $\lambda_i$'s are listed below:

\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
Clearly these $\lambda$'s are traceless and Hermitian. They are also "orthogonal" in the sense that:

$$\tau_\pi \left[ \lambda_i, \lambda_i \right] = 2 \delta_{ij} \quad \ldots \quad (1)$$

We note that $\lambda_3$ and $\lambda_8$ are diagonal and that

$$\left[ \lambda_3, \lambda_8 \right] = 0$$

which means that the SU(3) invariant theory admits two commuting "charges"; they are $I_3$ and $\gamma$ (hyper charge) which can be used to classify the hadrons in the SU(3) symmetry scheme. As with SU(2), for the $\sum \lambda_i \lambda_i$ to form a group, the commutator algebra of the $\lambda_i$ must close; that is, we must have:

$$\left[ \lambda_i, \lambda_j \right] = 2i \sum f_{ijk} \lambda_k$$

The $f_{ijk}$ are known as the "structure constants" of SU(3) and they generalize the totally antisymmetric $\epsilon_{ijk}$ structure constants of SU(2). They can be computed by using the identity

$$\tau_\pi \left[ \lambda_k \left[ \lambda_i, \lambda_i \right] \right] = 4i \sum f_{ijk} \lambda_k$$

which is a consequence of the trace orthogonality (eq. (1)) of $\lambda_i$. 

\[
\begin{align*}
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\]
The 3 x 3 representation of SU(3) also has the special algebraic property

\[ \{ \lambda_i, \lambda_j \} = \frac{4}{3} \delta_{ij} \mathbb{1} + 2 \varepsilon_{ijk} \lambda_k \]

where \( \varepsilon_{ijk} \) are totally symmetric in the indices \( i, j, k \) and these can be computed from the identity:

\[ \text{Tr} \{ \lambda_k \{ \lambda_i, \lambda_j \} \} = 4 \varepsilon_{ijk} \]

The known mesons and baryons can be classified in accordance with their isospin and hypercharge, into multiplets whose states can be taken to correspond to those in the irreducible representations of the SU(3) symmetry group. The well-known pseudoscalar \((J^P = 0^-)\) octet, the vector meson \((1^-)\) octet, the baryon \((1/2^+)\) octet and the baryon \((3/2^+)\) decuplet have been firmly established. If we represent the eight baryon fields and the eight meson fields in the octets by \( \psi_i \) and \( \phi_i \), respectively, then the 3 x 3 traceless matrices \( B \) and \( M \) defined by:

\[
B = \frac{1}{\sqrt{2}} \sum_{i=1}^{8} \lambda_i \psi_i
\]

\[
M = \frac{1}{\sqrt{2}} \sum_{i=1}^{8} \lambda_i \phi_i
\]

allow us to study the interactions of the hadrons belonging to these octets, in the SU(3) symmetry scheme. We give below explicitly the matrices \( B \) and \( M \) derived from the pseudoscalar...
The conjugate matrix \( B \) represents the anti-baryons and is same as \( B \) with the particles replaced by their anti-particles.

Now, to every unitary \( SU(3) \) transformation, on fields in the octet, there corresponds a \( 3 \times 3 \) unitary, unitary transformation \( u \) on the matrices \( B, B' \) and \( M \) such that:

\[
B' = u B u^T.
\]

We can use this correspondence to find combinations of the fields that are invariant under \( SU(3) \), and thus to construct...
SU(3) invariant Lagrangians. The SU(3) invariant Yukawa interactions involve $M$, linearly, and $B$ and $\bar{B}$ bilinearly. Since $M$ and $B$ considered as $3 \times 3$ matrices do not commute with each other, we have in fact two such SU(3) invariant Yukawa forms which can be written in terms of two invariants (which property follows directly from eqs. (2))

$$\tau_\lambda (\mathcal{B} M B) \quad \text{and} \quad \tau_\lambda (\mathcal{B} \bar{B} M).$$

The two linear combinations

$$\sqrt{2} \tau_\lambda \left\{ (\mathcal{B} M B) + (\mathcal{B} \bar{B} M) \right\}$$

$$\sqrt{2} \tau_\lambda \left\{ (\mathcal{B} M B) - (\mathcal{B} \bar{B} M) \right\}$$

are referred to as pure D-type and pure F-type Yukawa interactions. These can be written down explicitly by using the matrix forms for the $B$ and $M$ (and for $\bar{B}$) given earlier. The interactions that are relevant for our discussion are given below (we suppress the $\gamma_5$).

**D-type coupling**:

$$\pi^0 \bar{\tau} \rho + \kappa^- \left\{ -\frac{1}{\sqrt{3}} \pi^0 \bar{\rho} + \pi^0 \bar{\rho} + \sqrt{2} \bar{\pi} \bar{\pi} - \frac{1}{\sqrt{3}} \pi^0 \bar{\pi} \right\}$$

**F-type coupling**:

$$\pi^0 \bar{\tau} \rho + \kappa^- \left\{ -\sqrt{3} \pi^0 \bar{\rho} - \bar{\pi} \bar{\rho} - \sqrt{2} \bar{\pi} \bar{\pi} + \sqrt{3} \bar{\pi} \bar{\pi} \right\}$$
The coefficients in these expressions can be regarded as the coupling constants for specific interactions, when the coupling constant is normalized to one. Thus we see that in the limit of pure SU(3) we can write

\[ g_\pi^\pm = g_\pi^0 \mp \frac{1}{\sqrt{3}} g_\pi^\mp = D \mp F \]

\[ g_\pi^+ = \frac{1}{\sqrt{3}} D - \sqrt{3} F \]

\[ g_\pi^- = D - F \]

By defining \( \alpha = F/(F + D) \) the \( \pi^+ \pi^- \) and \( \pi^0 \pi^- \) coupling constants can be written in terms of the \( \pi^-N \) coupling constant \( g_\pi \) as:

\[ g_\pi^+ = (1 + 2\alpha) \frac{g_\pi}{\sqrt{3}} \]

\[ g_\pi^- = (1 - 2\alpha) g_\pi \]

In the limit of pure SU(3), the particles belonging to the same multiplet would have the same mass. The physical situation, however, is that the particles belonging to a given multiplet do have different masses though not widely differing from each other; this suggests that the symmetry as it is realized in nature is "broken". In analogy with the violation of the SU(2) symmetry by electromagnetic interactions (which results in particles belonging to a given isospin multiplet having different masses) - where the degeneracy of the \( 2I + 1 \) particles in an isospin multiplet (with isospin \( I \)) is assumed to arise from an \( I_3 \) dependent
interaction, it is assumed that the violation of SU(3) symmetry is caused by a "medium-strong" interaction that is dependent upon the diagonal generator $\lambda_8$; the only other diagonal generator, $\lambda_3$, is responsible for the electromagnetic mass differences).

Thus writing for the mass of an SU(3) state as $M(I, Y)$ one can write:

$$M(I, Y) = M_0 + \Delta M$$

Using the fact that $\sum_i \lambda_i^2 = \text{constant}$ we can write:

$$\Delta M = \langle I, Y | A \lambda_8 + B \sum_{i,k} d_{ijk} \lambda_i \lambda_k | I, Y \rangle$$

which leads to the Gell-Mann-Okubo mass formula (23):

$$M = M_0 \left\{ 1 + \alpha \gamma + \beta \left[ I(I+1) - \frac{1}{4} Y^2 \right] \right\}$$ (3)

This leads to the following relations for the baryon (1/2$^+$) octet and meson octet (for the meson families, CPT invariance implies that $a = 0$):

$$3 \, m_{A^+} + m_{\Sigma} = 2 \, m_{\Xi} + 2 \, m_{\Lambda}$$

$$3 \, m_{\Xi} + m_{\Xi'} = 4 \, m_{\Sigma}$$

where in the second relation the subscripts refer to the isospin values and squares of the masses are customarily used (as was suggested by Feynman (30)). For the baryon octet the mass formula
The pseudoscalar meson octet has been satisfied to within 1%. The mass formula is satisfied to within 3%. In the case of the vector meson (1^−) octet, however, the formula requires an I = 0 meson with m_o ≈ 928 Mev whereas neither of Ω and Φ mesons has this mass which lies midway between them. The way out of this 'cul de sac' was given by Sakurai who, following the idea of Gell-Mann, suggested that the medium strong interaction which transforms like Λ and which cannot transfer the quantum numbers of Ω and Φ (both having the same I = 0, J = 0) can cause transitions between these states; and from this point of view it is essentially arbitrary which of the two Φ or Ω one calls the SU(3) singlet and which one identifies with the I = 0, J = 0 member of the octet. Thus the "physical particles" Φ and Ω are a linear combination of the pure SU(3) singlet state (say Ω) and the I = 0, J = 0 state (Φ) of octet. Thus if the states are normalized one can write:

$$\omega^\circ = \omega \cos \theta - \phi \sin \theta$$

$$\phi^\circ = \omega \sin \theta + \phi \cos \theta$$

The 'mixing angle' θ can be calculated by using the observed masses which, up to a sign, is:

$$\theta \approx 40^\circ$$

From the mass formula (3) it is clear that M_o, the "common" mass of the vector meson octet is same as m_o, the mass of the I = 0, J = 0 member \(\sim 928\) Mev.
The isospin subgroup $SU(2)$ of $SU(3)$ can be characterized by the isospin operators identified with the $SU(3)$ generators as:

$$I_\pm = \lambda_1 \pm i \lambda_2$$

$$I_3 = \lambda_3$$

In a similar manner, the other two $SU(2)$ subgroups can be characterized by the "rising" and "lowering" operators $U_+$ and $V_+$:

$$U_\pm = \lambda_6 \pm i \lambda_7$$

$$V_\pm = \lambda_4 \pm i \lambda_5$$

With the identification of hypercharge $Y$

$$Y = \frac{2}{\sqrt{3}} \lambda_8$$

it is easy to derive the commutation relations among these operators. We note in particular:

$$[U_+, U_-] = 4 U_3$$

$$[V_+, V_-] = 4 V_3$$

It is clear that the $U = 1$, $U_3 = 0$ and $V = 1$, $V_3 = 0$ states can be expressed as linear combination of pure isospin states. With respect to the vector meson octet, these may be written as:

$$U_3 = -\frac{1}{2} \phi^0 + \frac{\sqrt{3}}{2} \phi^\circ$$

$$V_3 = \frac{1}{2} \phi^0 + \frac{\sqrt{3}}{2} \phi^\circ$$