This chapter is mainly devoted to the consideration of a simple model of planning for production and location where the locational alternatives of production are given to the planners as parameters and where the technology of basic production is linear. It is assumed that all the reproducible goods of the model are perfectly mobile and that all of its non-reproducible goods (or primary factors) are perfectly immobile over the geographic surface of the economy. The transport problem is thus absent in this model. The perfect immobility of primary factors will, however, play a decisive role in determining the nature of spatial allocation of resources in such an economy. We shall be engaged in the present chapter in analysing specifically the implications of this perfect immobility of the non-reproducible (or primary) factors on—

(i) the nature and the conditions of regional interdependence of the economy;

(ii) the nature of the optimal allocation of resources and of the values of their efficiency prices at the different places of the economy.

This analysis has been made with the help of the techniques of linear programming. Towards the end of this chapter we set this problem of production and of its locational organisation in an alternative background of a
purely competitive economy and examine the nature of equilibrium solutions for allocation of resources and factor incomes of that altered model.

3.1 Fixed Locations Linear Technology Model I : Transport Cost — Zero or Infinity :

3.1.2 Assumptions and Stipulations :

(1) Assumptions on goods

(1.1.a) All the reproducible goods are perfectly mobile over space.

(1.2.a) All the non-reproducible goods are perfectly immobile over space.

(1.4) All the non-reproducible goods have only input use in the different productive activities.

(1.6) Each of the reproducible goods has got some final consumption use.

Assumptions (1.1.a) and (1.2.a) together imply that the transport cost is not variable in effective sense in the model. Accordingly neither the transport good, nor the activities producing it has been explicitly introduced in the model.

(2) Assumptions on the relation of the basic productive activities with the use of land :

(2.1) Any single point of location of production would be able to accommodate spatially any number of plants or establishments of any number of basic industries which may operate at any arbitrary levels.

Thus our locations of production would be represented by points (and not by areas) on the geographic surface of the economy.

1/ The reproducible goods that have been explicitly considered in the model do not include the transport item.
(3) **Regularity assumption on space**:  
(3.1.a.) Any given basket of inputs will produce the same amount of output of any given basic industry for all possible locations of it. 

This implies that the technological parameters of any basic productive activity of the model would be independent of all locational connotations.

(4) **Assumptions on the regional structure of the economy**:  
(4.1.) The consumption of outputs of the basic industries can take place at point formed centres of location which are exogenously given to our model and are finite in number.

(4.3.a.) For any basic industry of our model the locational alternatives of production in the entire economy are exogenously given and are finite in number.

(4.4.a.) Production of any basic industry is divisible only to a limited extent.

(4.6) The fixed locations (of consumption and production) of our economy are all distinct.

The locational alternatives of any activity of production or of consumption are all a priori fixed in the model.

(5) **Assumptions on technology**:  
(5.1.a.) For any given locational distribution the technology of the basic productive (or industrial) activities is general linear in character.
None of the basic productive (or industrial) activities admit of joint production.

Each productive activity requires at least one input.

All the intermediate inputs are currently produced.

Each of the non-reproducible immobile resources is indispensably required as input in the activities of at least one basic industry, although the ratio of its requirement with respect to other inputs may be different in the different activities of the same basic industry.

If some activities of a basic industry require a particular non-reproducible immobile good as input in positive amount, all the activities of that industry would require that good indispensably as input.

The technology allows for the possibility of intermediate input use of the reproducible goods in any basic productive (or industrial) activity.

The validity of assumptions (5.1.a), (5.2.b), (5.3) and (5.7) implies that our model assumes, in short, a generalised Leontief technology of production.

We would refer to the assumptions (5.5) and (5.6) as factor indispensability hypothesis throughout this chapter.

3.1.B. Description of the Model:

Our economy is multiregional in structure. It is divided into K number of regions, the criterion of regional demarcation being arbitrary from economic point of view. The local authorities of the regions enjoy autonomy of choosing
the sites of all production that would take place within their respective boundaries. We do not discuss here how the local authorities solve their respective choice problems, but simply assume that the locations of production are determined by some unspecified mechanism of choice. This choice is, however, in any case, supposed to be independent of the actual levels of production of the different industries and to be determined prior to the making of other decisions on production by any regional or central authority. Our economy, on the other hand, has got a Central Planning Authority (C.P.A. in abbreviation) which allocates resources over the industrial sectors of the different regions and dictates what amounts to be produced of each good in the different regions. The decisions on the choice of sites of production of every region are supplied as exogenously determined values to this Central Planning Authority which frames its problem of regional allocation on the basis of this information on locations. To the C.P.A., the locational alternatives of production of the different industries of the entire economy would then be given as parameters and would obviously be finite in number. The present model analyses only the nature of the problem of regional resource allocation of the C.P.A. It however supposes that the local authority of each region has chosen beforehand only one point of location to be the site of all production of all the industries. This ensures that production of all the different industries would take place only at a common point in each region. The C.P.A. considers each of these predetermined (i.e., a priori fixed) points of location

2/ This supposition has been made only for the sake of simplicity of analysis. The structure and the results of this model would be generalizable to a case where any regional authority chooses any finite number of points as possible locations of production of any given industry and where the sets of locational alternatives thus chosen are noncoinciding in any region for the different industries.
of the different regions as a possible locational alternative of production of any basic industry.

The consumption activities of any region of our economy take place at a number of predetermined distinct places. The locations of consumption of the entire economy which are thus given to our model are supposed to have been determined by the geographic distribution of population and other sociological factors.

From the location of production of any region any of the reproducible goods may flow to any location of consumption or of production of the same or of any other region along a linear path joining the terminal points of the flow. No such movement would however involve any transport cost in the present model. The trading activities would not have therefore any real significance here except in a trivial sense. This explains the absence of the explicit consideration of the trading activities in the present model.

The number of reproducible goods of the model is assumed to be \( n \). All of these have got some final consumption use. The central planners' notion of welfare relations among the spatially separated consumers located at the different places induces the making of an assumption of a set of positive final use prices for these goods which is common for all the locations of consumption of all the regions. Let \( p \) be the row vector (or order \( 1 \times m \)) of these positive prices.

The predetermined location of production of every region is endowed with a vector of nonnegative fixed supplies of the different non-reproducible resources which are \( n_o \) in number and are all immobile. The information on the
supply situation of the perfectly immobile primary factors at the alternative
locations of production in the different regions is given to the C.P.A., which
would take it into account while framing its programme of production and
allocation. Let the column vector \( b_k \) (of order \( m \times 1 \)) be the supply vector of
the primary resources at the location of production of the \( k \)th region,
\( b_k \geq 0 \) for all \( k \). Let further \( y \) represent the row vector (of order \( 1 \times n \)) of
input prices of the reproducible goods for their intermediate input use and
\( y_k \) represent the row vector (of order \( 1 \times m \)) of the input prices of the non-
reproducible goods that would be prevailing at the location of production of
the \( k \)th region. Let the row vector \( y \) be \( \prod_{k=0}^{K} y_k \). ( \( \prod \) denotes the operation
of Cartesian multiplication over \( k \).) Thus \( y \) defines a system of imputed
input prices of all the different goods.

There are altogether \( n (\geq m) \) basic productive activities that describe
the technology of our economy. Each of these activities prevails at all the
a priori determined locations of production. Let \( A \) denote the matrix (of
order \( m \times n \)) of requirements of the reproducible goods as input in the different
activities. Similarly \( B \) denotes the matrix (of order \( n \times n \)) of input require­
ments of the primary factors. We derive matrices \( \Lambda \) and \( \overline{B} \) from \( A \) and \( B \) respec­
tively, where \( \Lambda \) and \( \overline{B} \) are net output matrices of the reproducible and of the
non-reproducible goods respectively. The columns of these matrices correspond
to the different basic productive activities and the rows correspond to the
goods or factors. Obviously \( \Lambda \geq 0 \) and \( B \geq 0 \) while \( \overline{B} \leq 0 \) and \( \Lambda \) is unrestricted
in sign. Any good enters a column of \( \Lambda \) or of \( \overline{B} \) with a positive (negative)
sign if it enters the corresponding basic productive activity as output (input)
in net sense. Let \( x_k \) denote the column vector (of order \( n \times 1 \)) of the intensity levels at which the different basic productive activities are operated at the location of production of the \( k \)th region. Finally, let the column vector \( \sum_{k=1}^{K} x_k \). Thus \( x \) defines a programme of industrial production of the entire economy.

We now like to ensure that our model of regional allocation and production is able to function meaningfully in a planned or in a freely competitive economy. In order to ensure this we make the following special assumptions on the technology and on the supply conditions of non-reproducible resources at the different places of production.

**Special Assumption 1:** It is possible for the economy to attain a positive bill of final goods. To put it otherwise, for our given \( \Lambda, \bar{B}, b \) \((k = 1, 2, \ldots, K)\) the inequalities \( \sum_{k=1}^{K} \Lambda x_k > 0 \), \( \bar{B} x_k > -b \), \( x_k > 0 \) for all \( k \) have a solution.

For the validity of Special Assumption 1, the necessary and sufficient conditions would be:

(i) there exists a submatrix \( \bar{\Lambda} \) (of order \( m \times m \)) of \( \Lambda \) such that \( \bar{\Lambda} \) is nonnegatively invertible,

(ii) for each \( i \), \( \bar{Q}_i \subseteq E \cup P_k \) at least for one \( k \), where \( \bar{Q}_i \) is the set of primary factors required in the production of the \( i \)th reproducible good, \( E \) is the empty set and where \( P_k \) represents the set of primary factors whose supplies are positive in magnitude at the predetermined location of production of the \( k \)th region.

\( \bar{Q}_i \) The set \( \bar{Q}_i \) is uniquely defined for all \( i \) because of the factor indispensability hypothesis. \( \bar{Q}_i \) may be empty for some \( i \).
The conditions (i) and (ii) may be verified for any empirical situation without much difficulty.4/  

**Special Assumption 2**: The processes of production are irreversible. In other words, the inequalities \( \sum_k \bar{a}_k = 0, \bar{b}_k = 0, x_k > 0 \) for all \( k \), \( x_k > 0 \) for at least one \( k \), would have no solution.

**Special Assumption 3**: Something cannot be produced out of nothing. Formally speaking, the following inequalities have no solution:

(i) \( \sum_k \bar{a}_k > 0 \)

(ii) \( \bar{b}_k > 0 \) for all \( k \), at least one inequality in (i) or (ii) being inactive and,

(iii) \( x_k > 0 \) for all \( k \).

3.1.2. Let us now analyse the conditions of regional interdependence of our economy that has been described above. Let us begin with the following definitions.

**Definition 1**: Our model will be called interregionally reducible if there exists a proper subset of regions which can attain some semipositive bill of final goods independent of the rest of the economy. In other words, a model is interregionally reducible if there exists at least one proper nonempty subset \( R^* \) of the set of regional indices \( R = \{1, 2, \ldots, K\} \) such that the inequalities \( \sum_{k \in R^*} \bar{a}_k \geq 0, \bar{b}_k \geq b_k, x_k > 0 \) for all \( k \in R^* \) have a solution. If no such \( R^* \) exists, the model is interregionally irreducible.

4/ Because of the factor indispensability hypothesis the condition (ii) requires \( \sum_k b_k > 0 \). If for any empirical situation we get \( b_k > 0 \) for some \( k \) and get condition (i) to be valid, it will be possible for the \( k \)th region to attain a positive bill of final goods independent of others.
In the latter situation any region will have to depend on every other region directly or indirectly for carrying out any feasible programme of production.

Definition 1°: Our model will be called perfectly reducible in its interregional structure if it is possible to partition the set of all regions of the economy into nonempty subsets so that the regions in each partitioned subset can together attain some semipositive bill of final goods independent of the rest of the economy. In other words we should get perfect interregional reducibility if there exist subsects \( R_1, R_2, \ldots, R_r \), \( 1 < r \leq K \) such that

\[
\bigcup_{i=1}^{r} R_i = R, \quad \bigcap_{i,j=1, i \neq j}^{r} R_i \cap R_j = \emptyset \text{ for all } i, j, \text{ and for each } i, \text{ the inequalities } \sum_{k \in R_i} x_k \geq 0, \quad a_k x_k \geq -b_k, \quad x_k \geq 0 \text{ for all } k \in R_i \text{ have a solution, (}\emptyset \text{ denotes the empty set.)}
\]

We now try to formulate the necessary and sufficient conditions of interregional irreducibility which would in turn provide the empirical criteria for tests of such interregional connectedness. Since technology is the same everywhere, it appears from Definition 1 that the relative supply configurations of primary resources in the different regions would play a deciding role in determining the nature of regional interdependence. But a careful analysis would further reveal that the formulation of our required conditions of interregional irreducibility should also depend on the nature of interindustrial structure. For this reason we begin restating here the definition of indecomposability of interindustrial structure as it has been given by Gale \([5, p. 314]\).

We then try to formulate our required conditions.

5/ The terms 'indecomposability', 'irreducibility' and 'connectedness' are synonymously used in the context.
Definition 2: An interindustrial structure will be called **decomposable** if there exists at least one proper nonempty subset of goods such that none of the activities producing them (in net sense) would require any of the reproducible goods lying outside the subset as input for intermediate use. If no such proper subset of reproducible goods exists, the interindustrial structure will be called **indecomposable**.

We may alternatively define the same thing as follows:

Let \( M = \{1, 2, \ldots, n\} \) denote the set of indices of all reproducible goods of the model.

Let \( N = \{1, 2, \ldots, n\} \) represent the set of indices of all activities producing the goods in \( M \).

Let \( M^* \) be any subset of \( M \). \( M^* \) will be called an **independent** subset of \( M \) if there exists a subset \( N^* \subseteq N \) such that

(i) for each \( i \in M^* \), \( \bar{a}_{ij} > 0 \) for some \( j \in N^* \) and \( \bar{a}_{ij} = 0 \) for all \( j \in N - N^* \),

(ii) \( \bar{a}_{ij} = 0 \) for all \( i \in M - M^* \), all \( j \in N \), where \( \bar{a}_{ij} \) is the \( ij \)th column element of \( \bar{A} \).

Our model will be called interindustrially **decomposable** if any such \( M^* \) exists as an independent proper subset of \( M \). Otherwise, our model will be described as interindustrially **indecomposable**. The criteria for empirical verification of indecomposability of an interindustrial structure are readily obtained from the above definition.

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6/ The meaning of a set notation like \( N - N^* \) and similar other ones that have been used in the text of this Thesis has been elucidated in our notation.

7/ At this point we can incidentally suggest a criterion of empirical test of validity of Special Assumption 3. It is that the set of all activities producing the goods of any independent subset (proper or improper) like \( M^* \) would require some primary resources as input.
Let us now state and prove the necessary and sufficient conditions of interregional irreducibility of our economy.

**Theorem 3.1:** If the interindustrial structure of our model of production be indecomposable, the necessary and sufficient condition of its interregional irreducibility would be \( \sum_k b_k - b_{k^0} \neq 0 \) for all \( k^0 \).

**Proof:**

**Sufficiency:** Suppose \( \sum_k b_k - b_{k^0} \neq 0 \) for all \( k^0 \).

Let \( G_i \) be the set of reproducible goods for whose production the \( i \)th non-reproducible good would be required indispensably as input. \( G_i \) is nonempty for all \( i \) because of the factor indispensability hypothesis. As a result we get that, for any \( k^0 \), the subeconomy excluding the \( k^0 \)th region is not able to produce all the reproducible goods of the model. Since the interregional structure of our model is indecomposable, the subset of reproducible goods which may all be produced in positive amounts by any such subeconomy cannot contain any subset which would itself be an independent proper subset of \( M_i \). Thus for any \( k^0 \), the subeconomy consisting of the regions in \( \mathbb{R} - k^0 \) will have to depend on the \( k^0 \)th region for carrying out any feasible programme of production. Hence the sufficiency of the condition of our theorem.

**Necessity:** Suppose we have on the contrary to the assertion of the theorem, the validity of the inequality \( \sum_k b_k - b_{k^0} > 0 \) for some \( k^0 \) to be consistent with interregional irreducibility. Because of the validity of Special Assumption 1 and of the prevalence of identical technology everywhere the subeconomy consisting of the regions in \( \mathbb{R} - k^0 \) would then be able to attain some positive bill of final goods. Thus by Definition 1, we get the model to interregionally reducible and end up with a contradiction. Hence the necessity of the condition of our theorem.
Remark 1: If for our model the hypothesis of Theorem 3.1 be valid, we should get -

(i) interregional reducibility if and only if there exists at least one k₀ such that \( \sum_{k} b_{k} - b_{k^0} > 0 \).

(ii) perfect interregional reducibility if and only if it is possible to partition the set \( \mathcal{R} \) into subsets \( \mathcal{R}_i \) (\( i = 1, 2, \ldots, \bar{r} \)) where \( 1 < \bar{r} < K \) so that \( \sum_{k \in \mathcal{R}_i} b_{k} > 0 \) for all \( i \).

Notations: Let \( M_i \) (\( i = 1, 2, \ldots, \bar{m} \)) be disjoint nonempty independent subsets of \( M \), none of these subsets containing any proper independent subset of reproducible goods in turn. If our model be interindustrially decomposable, \( M_i \) should form a proper subset of \( M \) for all \( i \).

Let \( N_i \) denote the subset of basic productive activities that produce the reproducible goods in \( M_i \).

Let \( B_{i} \) and \( A_{i} \) be the submatrices of \( B \) and \( A \) respectively which contain only the activities that are in \( N_i \).

Let \( x^i \) be a column vector of levels of the activities that are in \( N_i \).

Let \( u^i \) be a sum vector whose order is the same as that of \( x^i \).

Let \( I(M_i) \) denote the set of indices of non-reproducible goods for which the vector \( B_{i} u^i \) contains only negative elements. Because of the validity of Special Assumption 3, this set should be nonempty for all \( i \).

Let \( F(x^i) \) denote the set of indices of reproducible goods for which \( B_{i} x^i \) contains only negative elements.

8/ A sum vector is one whose every element is equal to unity.
Let $X$ represent on the other hand the set of all feasible solutions of the inequalities: $x^i > 0$, $\sum x^i > 0$.

Finally, let $F^i = \bigcup_{x^i \in X^i} F(x^i)$.

Because of the validity of Special Assumption 3, $F(x^i)$ is nonempty for all $x^i \in X^i$ and so also would be $F^i$. Again the indecomposability of the interindustrial structure of the goods in $M_i$ and the validity of the factor indispensability hypothesis would ensure for any $x^i \in X^i$ the set $F(x^i)$ to be equivalent to the set $I(M_i)$. Thus we get, for all $x^i \in X^i$, $F(x^i) = F^i = I(M_i)$. This would obviously be true for all $i$.

**Theorem 3.2**: If the interindustrial structure of our model of production be decomposable, the necessary and sufficient condition of its interregional irreducibility would be that for each $i$, $I(M_i) \subseteq \bigcup_{k \in R \cap k^0} P_k$ for all $k^0$.

**Proof**: Since by hypothesis our model is interindustrially decomposable, all $M_i$'s of the model would be proper subsets of $M$.

**Sufficiency**: Suppose for each $i$, $I(M_i) \subseteq \bigcup_{k \in R \cap k} P_k$ for all $k^0$.

No proper subset of $R$ can then attain any semipositive bill of reproducible goods with positive elements for some of the goods belonging to the set $\bigcup_i M_i$, because of nonavailability of the requisite primary resources. The problem that remains is to check whether it is possible for any proper subeconomy to produce a semipositive bill of reproducible goods with positive elements for only some of the goods belonging to the set $M - \bigcup_i M_i$. If this set be empty, the sufficiency of the condition is immediate. But even if it is found to be nonempty, that possibility would be ruled out because of the fact that $M - \bigcup_i M_i$ is only a nonindependent proper subset of $M$, by definition. Hence the sufficiency of the condition of the theorem.
Necessity : Suppose we have on the contrary to the assertion of the theorem, the validity of the relation \( I(M_i) \subseteq \bigcup_{k \in \mathbb{R} - k^0} P_k \) for one \( i \) and one \( k^0 \) to be consistent with interregional irreducibility. It is then possible for the subeconomy consisting of the regions in \( \mathbb{R} - k^0 \) to produce some semipositive bill of reproducible goods with positive elements only for some of the goods belonging to the subset \( M_i \). By Definition 1, we should get our model to be interregionally reducible and thus end up with a contradiction. Hence the necessity of the condition of the theorem.

Remark 2 : If for our model the hypothesis of Theorem 3.2 be valid, we should get -

(i) interregional reducibility if and only if \( I(M_i) \subseteq \bigcup_{k \in \mathbb{R} - k^0} P_k \) at least for one \( i \) and one \( k^0 \).

(ii) perfect interregional reducibility if and only if it is possible to partition the set \( \mathbb{R} \) into subsets \( R_j \) (\( j = 1, 2, \ldots, \bar{r} \) where \( 1 < \bar{r} < K \)) so that for each \( j, I(M_i) \subseteq \bigcup_{k \in R_j} P_k \) at least for one \( i \).

3.1.D. The Problem:

We now pose the problem of the Central Planning Authority of our economy. The problem is: 'What to produce at the different locations of production? How to produce the different goods at the different places?' The C. P. A. solves this problem, and dictates to the regional authorities the actual programmes of production that are to be carried out in their respective regions and regulates the interindustrial and the interregional flows of the reproducible goods accordingly. The fundamental decision variables of the
C. P. A. are the intensity levels at which the different basic productive activities are to be operated at the various locations of production of the different regions. The values of these variables that solve the following linear programme (P) would answer all the questions mentioned above.

\[(P) \text{ Maximise } \sum_k p \Lambda_k x_k \text{ subject to } \sum_k \Gamma_k x_k = 0, \sum_k \Sigma_k x_k = a, x_k \geq 0 \text{ for all } k.\]

The optimal solutions of activity levels should maximise the final use value of net output of the entire economy subject to the different balance constraints. The dual of the linear programme (P) would yield the efficiency prices of the different intermediate and primary inputs that should prevail at the different locations of production in order to support any optimal solution of (P). The dual programme is the following one.

\[(D) \text{ Minimise } \sum_k y_k b_k \text{ subject to } -y_o \Lambda - y_k \Sigma = p \Lambda, y_o \geq 0, y_k \geq 0 \text{ for all } k.\]

Any optimal solution of imputed prices as given by (D) would obviously minimise the total input cost of the economy without violating the nonnegativity condition of prices and the condition that no net profit should be possible anywhere in the economy.

We discuss the solvability of the linear programmes (P) and (D) and analyse the nature of their optimal solutions in the subsections of the following Section 3.2. We record our observations on the implications of interregional irreducibility on the nature of the optimal solutions wherever it is relevant.
3.2. A. Solvability of (P) and (D):

Let X and Y be the sets of optimal solutions of (P) and (D) respectively. By the theorems of linear programming, it can be easily proved that the validity of Special Assumptions 1-3 is sufficient to ensure the non-emptiness of X and Y and the existence of only nontrivial solutions in each of these sets. This is why we referred to Special Assumptions 1-3 as a set of sufficient conditions for ensuring the meaningful functioning of our model.

3.2. B. The natures of the solutions of (P) and (D):

If we assume our model to be interregionally irreducible, we would get $x_k$ to be semipositive for any $x \in X$ and any $k$. $x$ would in fact indicate a distribution of the basic productive activities over regions such that for each $k$, $x_k$ contains zero values for all the basic productive activities which require as input the primary resources which are not available at the predetermined location of production of the $k$th region. In an optimal situation the $k$th region will have therefore to import from the rest of the economy some of those reproducible goods for the production of which those primary resources are indispensably required. On the other hand, for each $k$, the $x_k$ of an optimal $x$ would require the $k$th region to specialise in the production of some of the reproducible goods which will be exported to the rest of the economy in positive amounts. We should note that these features of an optimal regional distribution of the basic productive activities and of optimal interregional flows of goods follow directly from the making of a stipulation of interregional
irreducibility in our model and from the feasibility requirements of our programme (P). They are in fact independent of the precise value of $p$ (positive) although $p$ can affect $x$.

Let us now analyse the nature of the elements of the set $Y$. For any $y \in Y$, the imputed efficiency input prices of the reproducible goods would be equalised over all the regions because of the very nature of construction of the model. Secondly, by the equilibrium theorem of linear programming [2, p.13] the total imputed income to the reproducible goods for their intermediate input use would be zero for any $y \in Y$. As a result the total gross value produced in an optimal situation in the entire economy would be equal to the total imputed income going to the primary factors according to any $y \in Y$.

These results are valid independent of the nature of interregional connectedness of the model. For the prices of the second group of inputs, i.e., of the non-reproducible resources, the picture of interregional comparison is a bit different. The reality or the unreality of interregional irreducibility would matter to some extent in determining whether the imputed rewards to the non-reproducible goods should be equalised over the regions or not. We summarise our observations on this problem in the following theorem which gives a set of sufficient conditions for getting the interregional equalisation of the efficiency prices of primary factors as a necessity in our model. The set of sufficient conditions of Theorem 3.3 assumes among other things one condition of perfect interregional reducibility along with that $R_i$ contains only one regional index for each $i$. The construction of the theorem also reveals what else would be required as conditions for ensuring the conclusion of primary
factor price equalisation as a necessity. The only usefulness of posing such a
theorem is that it may act as a pointer to the reasons why the imputed rewards
of primary factors are likely (or not likely) to be equalised over the regions
of a planned economy in a given situation. It should be mentioned at this
point that the following theorem makes use of the analytic approaches which
were adopted by Lionel McKenzie in his paper [17].

For each k, let us consider the following pair of primal and dual
linear programmes.

\[ P_k(y_0) : \text{Maximise } (p_x + y_o \lambda) x_k \text{ subject to } B x_k - s_k = -b_k, \]
\[ x_k > 0, s_k > 0, \text{where } x \text{ is a column vector of order } n. \]

\[ D_k(y_0) : \text{Minimise } y_k \text{ subject to } -y_k B > p_s + y_o \lambda, \gamma_k > 0. \]

**Lemma 3.1**: For any \( x^o \in X, y^o \in Y \), there exist uniquely defined
vectors \( x^o_k, y^o_k \) such that \( x^o_k \) and \( y^o_k \) solve \( P_k(y^o) \) for all \( k \).

Further, \( y^o_k \) solves \( D_k(y^o) \) for all \( k \).

**Proof**: The proof is immediate from the theorems of linear programming.
(Theorem 3.5 of Gale [6] may be consulted.)

**Notations**: For any \( x^o \in X, y^o \in Y \), let \( B(x^o, y^o) \) denote the matrix
which consists of only those column vectors of the augmented matrix \( [B -I] \)
on which the optimal solution \( (x^o_k, x^o_k) \) of \( P_k(y^o) \) depends positively.
(Note that I denotes here an identity matrix of order \( n_o \).)

**Definition 3**: When \( x^o \) and \( y^o \) solve the linear programmes \( (P) \) and \( (D) \)
respectively and when the rank of \( B(x^o, y^o) \) is \( n_o \) a pair of regions \( i \) and \( j \)
will be described as connected by a chain provided there exists a finite sequence
of regional indices \( r_1, r_2, \ldots, r_K \) (where \( K \leq K' \) and \( K \) depends on \( i \) and \( j \)) such that the following conditions (i) and (ii) hold.

(i) \( r_1 = i, \ r_K = j \)

(ii) for each \( l \) (\( l = 1, 2, \ldots, K \)), there exists a matrix \( B_{r_1} \) (of order \( m_0 \times m_0 \)) which consists of some \( m_0 \) independent column vectors of \( B_{r_1} (x^0, y^0) \), such that any two successive cones of the sequence \( C_{r_1}, C_{r_2}, \ldots, C_{r_K} \) have got interior intersection; \( C_{r_1} \) representing the convex polyhedral cone generated by the vectors of the matrix \( B_{r_1} \).

**Theorem 3.3**: If there exist \( x^0 \in X \) and \( y^0 \in Y \) and if for each \( k \), \( B_k (x^0, y^0) \) contains a submatrix \( B_k \) (of order \( m_0 \times m_0 \)) consisting of \( m_0 \) independent vectors so that any two regions of the economy are connected by a chain, then the primary factor price equalisation would be necessary for any optimal solution of (D).

**Proof**: We have \( x^0 \in X \), \( y^0 \in Y \) and \( B_k (x^0, y^0) \) which contains a submatrix \( B_k \) of rank \( m_0 \). We are further given that for \( x^0, y^0 \) and \( B_k \) (all \( k \)) any two regions of the economy are connected by a chain.

We now suppose on the contrary to the assertion of the theorem that for our \( y^0, y_{i^0}^0 \neq y_{j^0}^0 \) for some regional indices \( i^0 \) and \( j^0 \), where \( i^0 \neq j^0 \).

Since the regions \( i^0 \) and \( j^0 \) are connected by a chain, we can ensure the existence of a sequence of regional indices \( r_1, r_2, \ldots, r_{K'} \) (where \( K' \leq K \)) such that (i) \( r_1 = i^0, \ r_K = j^0 \) and that (ii) \( C_{r_1}, C_{r_{k+1}} \) contains some vector \( z_k \) as an interior point of both the convex polyhedral cones \( C_{r_1} \) and \( C_{r_{k+1}} \) (\( k = 1, 2, \ldots, K' - 1 \)), \( C_{r_1} \) representing the convex polyhedral cone generated by the vectors of \( B_{r_1} \).
Since for any $1 \ (l = 1, 2, \ldots, K)$, $y^o_{r1}$ should solve the problem $P_k(y^o)$ for $k = r_1$ (by Lemma 3.1), and since $\hat{B}_{r1}$ of rank $m_0$ is a submatrix of $\hat{B}_{r1}$ ($x^o, y^o$) which is in turn contained in the matrix $[\hat{B} - I]$ for any $1$, we get by the equilibrium theorem of linear programming [6, p. 19],

$$1 - y^o_{r1} \begin{bmatrix} \hat{B}_{r1} \\ r_1 \end{bmatrix} = (p \hat{A} + y^o \hat{A}, 0)_{r1}$$

for all $1 \ (1 = 1, 2, \ldots, K)$,

$$p \hat{A} + y^o \hat{A}, 0)_{r1}$$

being the relevant subvector (or order $m_0$) of the row vector $(p \hat{A} + y^o \hat{A}, 0)$ whose order is $n + m_0$.

$$(2) - y^o_{r1} \begin{bmatrix} \hat{B}_{r1} \\ r_1 \end{bmatrix} \geq (p \hat{A} + y^o \hat{A}, 0)_{r1}$$

for all $h \ (1 = 1, 2, \ldots, K)$, $h = 1, 2, \ldots, K; \ 1 \neq h$.

From (1) and (2) it follows immediately that for any $l (1 = 1, 2, \ldots, K - 1)$.

$$(3) \ (y^o_{r1} - y^o_{r1+1}) \begin{bmatrix} \hat{B}_{r1} \\ r_1 \end{bmatrix} > 0$$

$$(4) \ (y^o_{r1} - y^o_{r1+1}) \begin{bmatrix} \hat{B}_{r1} \\ r_1 \end{bmatrix} < 0$$

Since for any $1 \ (1 = 1, 2, \ldots, K - 1)$ $z_{r1}$ belongs to the interior of both $C_{r1}$ and $C_{r1+1}$, and since the number of extreme half lines generating $C_{r1}$ for any $1$ is the same as its dimensionality (the extreme half lines being the column vectors of the matrix $\hat{B}_{r1}$), there exist unique vectors $\hat{x}_{r1} > 0$ and $\hat{x}_{r1+1} > 0$ such that for any $l \ (1 = 1, 2, \ldots, K - 1)$,

$$(5) \ z_{r1} = \begin{bmatrix} \hat{B}_{r1} \\ r_1 \end{bmatrix} \hat{x}_{r1} = \hat{B}_{r1+1} \hat{x}_{r1+1}$$

Since by our supposition $y^o_{r1} = y^o_{r1}$, we should have, $y^o_{r1} \neq y^o_{r1+1}$ for some $l$ where $1 \leq l < K - 1$. Since further, $\hat{B}_{r1}$ is a
nonsingular matrix, we cannot have all the inequalities of (3) and of (4') as holding with equalities for all \( 1 \) at the same time. Then for some \( 1 \) \( (1 < 1 < S - 1) \) we should get,

\[
(6) \quad (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{B}_{r_1} \mathbf{x}_{r_1} > 0 \text{ and, }
\]

\[
(7) \quad (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{B}_{r_{1+1}} \mathbf{x}_{r_{1+1}} \leq 0
\]

Now multiplying both sides of (6) and (7) by the vectors \( \mathbf{x}_{r_1} > 0 \) and \( \mathbf{x}_{r_{1+1}} > 0 \) respectively, we arrive at the following conclusion:

For any \( 1 \) for which \( y_{r_1}^o \neq y_{r_{1+1}}^o \), the inequalities

\[
(8) \quad (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{B}_{r_1} \mathbf{x}_{r_1} = (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{z}_{r_1} > 0 \text{ and }
\]

\[
(9) \quad (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{B}_{r_{1+1}} \mathbf{x}_{r_{1+1}} = (y_{r_1}^o - y_{r_{1+1}}^o) \mathbf{z}_{r_{1+1}} < 0
\]

hold at the same time. But (8) and (9) obviously contradict each other. Thus for the given \( y^o \), there cannot exist any \( 1 \) \( (1 < 1 < S - 1) \) for which \( y_{r_1}^o \) and \( y_{r_{1+1}}^o \) should be unequal. In other words, we cannot have any regional indices \( i^o \) and \( j^o \) such that \( y_{i^o}^o \neq y_{j^o}^o \). We should then necessarily obtain the equality

\[
y_{1}^o = y_{2}^o = \ldots = y_{K}^o.
\]

If \( Y \) be a single element set, our theorem has been proved. If \( Y \) contains more than one element, there remains the problem of ruling out the possibility of existence of any \( y \) (other than \( y^o \)) in the set \( Y \) which would indicate a nonequalisation of prices of the primary factors over the regions.

Let \( y^* \) be an element of \( Y \) which is distinct from \( y^o \). By Lemma 3.1, \( x^o \) and \( y^* \) should satisfy the condition that for each \( k \), \( (x_{k}^{o'}, z_{k}^{o'}) \) solves
the problem $P_k (y^*_0)$. As a result, we have $\tilde{R}_k (x^0, y^*) = \tilde{R}_k (x^0, y^c)$ for all $k$ and have any two regions of the economy to be connected by a chain for $x^0$ and $y^*$. Proceeding exactly in the same way as we did in the case of $y^0$, we should get for $y^*$ the equality $y^*_1 = y^*_2 = \ldots = y^*_K$ to be necessarily valid. Hence the theorem.

Remark 3: We should note that the validity of the hypothesis of Theorem 3.3 requires $b_k$ to be a strictly positive vector for all $k$ and therefore requires the interregional structure of our model of production to be perfectly reducible with $R_i$ containing only one regional index for each $i$. If our model turns out to be irreducible in its interregional structure, then for each $k$, $b_k$ would contain at least one zero element so that the rank of $\tilde{R}_k (x^0, y^0) < m_0$ for any $x^0 \in X$ and any $y^0 \in Y$. In that case we cannot obviously ensure that for $y^0$, the equality $y^*_1 = y^*_2 = \ldots = y^*_K$ should be true. It may, however, so happen in a case of interregional irreducibility that for some, but not for all $y$ belonging to the set $Y$, the efficiency prices of the primary factors are equalised everywhere. In that case the primary factor price equalisation may be possible, but it would not be necessary.

3.3 Before concluding the present chapter we like to set the problem of regional allocation of resources of Section 3.1.D. in an alternative background of a purely competitive economy and examine some of the results of the altered model. We retain in the new setting of the problem all the assumptions on the goods, the relation of the basic productive activities with the use of land, the regularity of space, the regional structure and the technology as mentioned above in Section 3.1. The only change we make is that instead of
posing the problem as one of fully centralised planning, we view it as one of equilibrium allocation in space under conditions of perfect competition everywhere. The description of this altered model may be summarised as follows.

We may now consider a problem of general equilibrium of production into which duly takes account the problem of allocation of resources in the alternative industries at the different locations. The analysis is not however complete since we do not consider the consumer demand side of an equilibrium analysis. We begin with assuming an arbitrary set of positive final use prices of the reproducible goods common for all the locations of consumption of the different regions. Let $p$ (positive) represent this price vector. Similarly, all other notations that have been introduced in the above sections should preserve their respective meanings in the following context. The only point of significance is that the vector of activity levels $x$ and the vector of input prices $y$ are now determined not by any centralised decision making authority, but by the decentralised mechanism of production and distribution of a competitive system.

We may now define a competitive equilibrium as follows.

**Definition 4**: A system of activity levels $x$ and a system of input prices $y$ are said to yield a competitive equilibrium if -

(i) $x$ satisfies all the balance constraints of input requirements,

(ii) for each $k$, $x_k$ helps the $k$th region to maximise its net revenue (net of intermediate input cost in terms of $y$) subject to the local availability constraints of the immobile nonreproducible resources at the location of production of the region,
(iii) for each $k$, $y_k$ helps the $k$th region to minimise its total primary factor cost subject to the no profit condition in all the basic productive activities that are at the disposal of the producers of the region, and if

(iv) $y_o$ be nonnegative indicating the input price of any reproducible good to be zero whenever it is produced in the economy in net positive amount.

Thus $x$ and $y$ would yield a competitive equilibrium if they satisfy the following mathematical conditions:

(a) $\sum_k \alpha_k > 0$, $-b_k < \sum_k x_k < 0$ for all $k$.

(b) for each $k$, $(p_x + y_o) x_k$ is maximum subject to $b_k x_k = 0$, $x_k > 0$.

(c) for each $k$, $y_k b_k$ is minimum subject to $-b_k y_k = p_x + y_o b_k$, $y_k > 0$.

(d) $y_o > 0$, $y_u u_j = 0$ whenever $\sum_j A_k x_k > 0$ where $u_j$ is the $j$th unit vector.

By theorems of linear programming it can be shown that any $x$ and any $y$ satisfy (a) - (d) if and only if

(i) $x$ maximises the total gross value produced in the economy without violating the balance constraints of (a) and,

(ii) $y$ minimises the total primary input cost of the economy without violating the nonnegativity condition of prices and the competitive condition that no profit should be possible anywhere in the economy.

Thus the necessary and sufficient condition for $x$ and $y$ yielding a competitive equilibrium is that they should solve the linear programmes (P) and (D) of Section 3.1.D, respectively.

A $j$th unit vector is one whose $j$th co-ordinate element is equal to unity and whose other elements are equal to zero.
The arguments given in Section 3.2.A. would ensure the existence of a competitive equilibrium in the altered model. The nature and the properties of the equilibrium solutions of activity levels and input prices will remain exactly the same as those of the optimal solutions of the planning problem of Section 3.1.D. All the arguments and assertions of Section 3.2 will therefore preserve their validity in the context of an equilibrium analysis.