The present chapter discusses Model V which is largely an extension of Model IV of Part II. Model V introduces the locational factor of agglomeration into the analytic schema of production of Part II. Like Model IV, this model considers both locations of production and levels of operation of the different productive and trading activities to be continuous decision variables of the social planners of the economy. The main point where it differs from Model IV is that it relaxes the assumption of constant returns to scale in the technology of production and that unlike Model IV it admits of the existence of economies of scale — both internal and external. The nonlinear technology of the following model accommodates scale effects of production along with externalities which account for the agglomeration factor of production. In all respects other than technology this model is very similar to the immediately preceding one of Part II. Our interest lies here in pointing out the role of non-pecuniary economies in cost that may arise owing to the locational closeness of one basic industry to another behind the choice of optimum locations of industries and of optimum allocation pattern of resources. Model V assumes that such economies can arise and only arise through externalities.
9.1. Variable Locations, Nonlinear Technology

Model V: Fixed Social Valuation of Final Goods

9.1. Assumptions and Stipulations:

(1) Assumptions on goods:

(1.1.b) All the reproducible goods (excepting transport) are imperfectly mobile over space.

(1.2.c) All the non-reproducible goods are imperfectly mobile over space.

(1.3) The transport good is a service that involves no problem of transportability of itself.

(1.4) All the non-reproducible goods have only input use in the different productive activities.

(1.5) Transport service is a purely intermediate good having input use only in the trading activities.

(1.6) Each of the reproducible goods (excepting transport) has got some final consumption use.

Assumptions (1.1.b) and (1.2.c) ensure homogeneity of all the goods (excepting transport) in respect of their mobility over space. All of them are imperfectly mobile and their movement would always require expenditure of transport input only in positive amount.

(2) Assumptions on the relation of the basic productive activities with the use of land:

(2.1) Any single point of location of production would be able to accommodate spatially any number of plants or establish-ments of any number of basic industries which may operate at any arbitrary levels.
(2.2) The transport industry has no locational problem.

The locations of the basic industrial activities are therefore represented by points (and not areas) of the geographic surface of our economy.

(3) Regularity assumptions on space:

(3.1.b) For any given degree of locational closeness among the different basic industries operating at given levels, the requirements of inputs by each of those industries would be independent of the actual location of the industrial complex.

(3.2) The amount of transport input required by any trading activity for shipping a given amount of any good over a given measure of linear distance would be the same irrespective of geographic positions of the terminal points of shipment.

(3.3) Transport service can be made available for facilitating shipment of goods between any two points of locations along a linear route joining them. The per unit real cost of providing such a service would be independent of actual geographic positions of the terminal points of shipment.

The technical coefficients of the basic industrial activities are now influenced by the values of the locational variables of the model. The nature of functional dependence of those technical coefficients on locational choice, is however assumed to be such that the coefficients of average input requirements in any given basic industry are dependent only on the measures of distance of its location from the locations of other basic industries and are independent of actual geographic location of the industrial complex. The
technical coefficients of the transport industry are in any case independent of all locational connotations. The coefficients of requirement of transport input in the different trading activities depend also on locational values only to the extent that they determine the measure of linear distance over which a shipment has to take place.

(4) Assumptions on the regional structure of the economy:

(4.1) The consumption of outputs of the basic industries can take place at point formed centres of location which are exogenously given to the model and are finite in number.

(4.2) The imperfectly mobile non-reproducible resources are initially localised at point formed centres of deposit which are exogenously given to the model and are finite in number.

(4.3.b) The location of any basic industrial activity is a continuous variable in the two dimensional plane of geographic surface of the economy.

(4.4.b) Production of any basic industry is locationally indivisible.

(4.5) The transport sector geographically links up the points of location (fixed or variable) of our economy by a network of linear routes as follows:

(i) Each of the locations of the basic industries is linked with the locations of consumption which import goods of final use from it.

(ii) Each of the locations of the basic industries is linked with the locations of resource deposit which export imperfectly mobile non-reproducible resources to it.
(iii) All the locations of the interindustrially related basic industries are linked among themselves.

(4.6) The fixed locations (of consumption and of resource deposit) of the economy are all distinct and no three of them are collinear.

The locations of basic industries are thus continuous variables while the locations of consumption and of deposit of primary resources are parameters in the present model. The precise map of transport network will, on the other hand, be conditional upon the locational choice of the basic industries.

(5) Assumptions on technology:

(5.1.c) For any given locational distribution the technology of the basic productive (or industrial) activities is nonlinear in character, so that it admits of the presence of agglomerative factors through externalities and scale effects of production.

(5.2.b) None of the basic productive (or industrial) activities admits of joint production.

(5.3) Each productive activity requires at least one input.

(5.4) All the intermediate inputs are currently produced.

(5.7) The technology allows for the possibility of intermediate input use of the reproducible goods in any basic productive (or industrial) activity.

(5.9) The basic industrial activities are connected among themselves through relations of interindustrial consumption.
For the movement of any imperfectly mobile good, the demand for transport input follows the functional rule of exact proportionality with respect to the weight carried and the distance covered. For our model the constant of proportionality is the same for all such goods and is assumed to be equal to unity.

The amounts of inputs required by the transport industry do not have to be transported.

The functional rule according to which agglomeration economies arise assumes that for given levels of operation of the different basic industrial activities, input requirements would decline in any of them when its distance from the location of at least one other gets shorter, while its distance from any of the remaining basic industries remains unchanged. The input requirements would, however, decrease in all the basic industries in the case of shortening of all the mutual distances between the locations of the basic industries.

The schema of production of the present model is nonlinear and it admits of scale effects of production and externalities which in their turn take the agglomerative factors properly into account (Assumption (5.1.c)). Assumption (5.12) specifies the functional rule according to which agglomeration arises through externalities. The possibility of joint production is however ruled out in our technology. The basic industrial activities of the model are interindustrially connected, There may exist, however only a one way
dependence of the transport sector on the basic industrial sector for intermediate input requirements. We now elucidate the meaning and implications of our assumptions (5.1.0) and (5.12) so that one can better appreciate the exact mathematical character of the technology of our model. In this connection we note the following points.

First of all, we should note that in our model of centralised planning for production and location there is no role of market organisation. Any economies of scale — internal or external — can arise in this model only because of the changes in real factors which would be consequence of changes in the scales of operation of the different industries. That is why such economies would be nonpecuniary in character in the present context and can be well represented in our model by a suitable assumption on mathematical forms of production functions of the different industries. The assumption on mathematical forms of the production functions should, in any case, be consistent with all the assumptions on technology stated above.

Secondly, internal economies of scale in any given industry of our model would, in the event of absence of all externalities, be a reality independent of locations of the different basic industries. But the case is not so because for external economies. External economies that arise of the behaviour of one of the basic industries of our model can only be internalised by the remaining other basic industries. The extent to which such external economies can be internalised by a basic industry depends inversely on the distance of its location from the location of the basic industrial activity that may be held responsible for those external effects.
Thirdly, we should note that the transport industry of our model has no locational specification. We neither assume the production function of this industry to be subject to any effects of externalities nor do we assume the production function of any other industry to be affected by the behaviour of this one. In any case, we like to make its production function subject to internal economies of scale.

Fourthly, the trading activities are in no way affected by any scale effect or any externalities. The functional rule according to which transport input is required by any trading activity remains typically Weberian in nature as it was in the preceding models.

Fifthly, the mathematical forms of production functions of the different industries should show that the economies of scale—internal or external—that can arise in the model are reversible and exhaustible only in asymptotic sense.

Finally, there remains one basic question which we shall have to answer. The question is: How do the economies of scale actually arise in our model? Our assumption in this regard is that the economies which are internal in nature, arise in an industry because of the advantages of specialisation and division of labour that can be realised when the scale of operation of the industry expands. The degree of specialisation can further be extended in some one of the basic industries if the scale of operation of some other basic industry expands and if that of its own remains stationary. The advantages of specialisation that arise because of such external reasons can again be realised better if the basic industrial activities are more closely located.
Knowledge can in fact better flow among the industries and there can develop better facilities of research which help specialisation, if the industries get locationally closer. Whenever any industrial activity derives any advantage of specialisation from variation in the scale of operation of its own or of others or from variation in the degree of closeness of its location to the location of some other industrial activity, it will in fact find it economic to switch over to a new technique or to a new organisational structure of production. Formally we here define a basic productive activity to represent a vertically integrated structure of production — the integration being taken over all the subsidiary operations involved in the process of production of a given reproducible good. Any input coefficient of our technical model should then represent the integral of requirements of an input in all these subsidiary functions of an industry. Since specialisation induces these component subsidiary functions of the production process of an industry to be organised and performed in an alternative way, the average input coefficients of any basic productive activity should then vary if the scale of operation of its own or of some other similar activity varies or if the degree of closeness of its location to the location of some other basic productive activity changes. In the case of the transport industry, the average input coefficients would be subject to variation only if the internal scale of operation changes. The latter coefficients would in fact be completely insensitive to all external factors.

9.1.B. Description of the Model:

Like Model IV, the present model assumes the number of goods of our economy to be four and adopts the notational rule of representing them by the
numbers 0, 1, 2, 3. Of these two are reproducible goods having some final consumption use and one is a purely intermediate item (this is the transport good), the remaining one being a non-reproducible resource having only input use. The 0th good is the non-reproducible item of our model, the 1st and the 2nd goods are reproducible ones having both intermediate input and final consumption use and the 3rd good is transport service which is purely an intermediate item. It is only the industries producing the goods with the numbers 1 and 2 that would involve locational problems.

The description of spatial distribution of consumers and of availability of the single primary factor of our present model is also similar to that of Model IV. The consumers are supposed to be all located only at two points of the economy which are parametrically given. The entire supply of the primary resource is on the other hand assumed to be lying initially localised at a single point of source which is also given as parameter. The location of production of any basic industry is assumed to be variable as it was in Model IV. The Assumption (4.4.b) however requires that all the production of a basic industry takes place at only one point of location in a single unit of production establishment. Accordingly we have only two variable points of location in the present model which would correspond to the sites of the two basic industries which produce the goods with the numbers 1 and 2.

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space. We now suppose that \( \mathbb{R}^2 \) well represents the geographic surface of our economy. Let \( \mathbf{r}_k \in \mathbb{R}^2 \) \((k = 1, 2)\) and \( \mathbf{r} \in \mathbb{R}^2 \) be the given locations of the \( k \)-th centre of consumption and of the only deposit of the primary factor respectively. Let \( \mathbf{x} \in \mathbb{R}^2 \) \((i = 1, 2)\)
Denote the variable location of the jth industry. The two element vectors $r_k (k = 1, 2)$ and $l_j (j = 1, 2)$ are all row vectors. Let $L$ denote the row vector $(l_1, l_2) \in \mathbb{R}^2$. $L$ would, therefore, represent the locational distribution of all the basic industrial activities. We should further note that if $q$ and $q^*$ be any two points of location in the two dimensional plane of geographic surface, $\delta_{qq^*}$ will represent the linear distance between them. That is

$$\delta_{qq^*} = \text{modulus of } \sqrt{(q - q^*)^2 (q - q^*)^2}$$

which is positive whenever $q \neq q^*$. ($q$ and $q^*$ being both row vectors).

Let us suppose that $\gamma$ (positive) is the maximum available amount of the primary factor at the fixed locational point $r$. The allocation of this inelastically given supply of the primary factor in the different industries (including transport) will be a part of the real problem that the central planners of production of our model will have to solve.

Like Model IV, the present model assumes a fixed social valuation of final goods at the different consumption locations. The notion of welfare relations among the spatially separated consumers as held by the social planners of our location-allocation model determines the fixed prices of the final goods at the different locations of consumption. Let $\pi_{jk}$ be the given final use price of the jth good at the kth location of consumption ($j = 1, 2; k = 1, 2$). We assume all these prices to be positive and take the row vector $p$ to represent these prices. Thus $p = (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) \in \mathbb{R}^4$. The central planners of production of the present model will choose the optimum programme of production and location in the light of this social valuation of final goods at alternative locations of consumption. Besides this, it should be noted that all the goods of our model — reproducible and non-reproducible — would have also some input
price since each of them would be required as input in some of the processes of technical or spatial transformations.

Let us now spell out the technical coefficients of our model. In the following formal presentation of our technical model we redefine (as we did while describing Model IV) an industrial activity to represent the vertical integration of the basic productive activity producing the good concerned and the trading activities that gather the inputs required in the process of basic production of the industry at its site of production. In other words, in the following presentation the input coefficients of any industrial activity would represent the total requirements of all the different goods in the integrated process of technical transformation of that industry, the industry being operated at unit level, and of the spatial transformations required for gathering the necessary inputs (primary and intermediate) at the locations of production. The trading activities that distribute the final goods in the locations of consumption would however exist independently in our formal model and would not be integrated with the process of basic production. In fact, an industrial activity would here broadly correspond to the activity of organising the entire process of production of a reproducible good. The trading activities would on the other hand correspond to the activities of distribution of the different final goods produced among the consumers at the different places.

The mathematical forms that we assume for production functions of the different industries presume that for any given locational distribution of the basic industries and for any given levels of outputs of the different industries the average requirements of inputs in any industry would be fixed and unambiguously defined. There will in fact remain no technical possibility
of substitution of one input for another in any industry when the levels of 
the 
operation and/locations of the different basic industries are given. Because 
of this nonsubstitution hypothesis the number of reproducible goods and the 
number of industrial activities will be equal. But the average requirement of 
an input (per unit of output of a good) may change if the activity or output 
levels change and/or if the locations change.

Let the levels of operation of the activities of the different indus-
tries be represented by the column vector \( x = (y_1, y_2, y_3) \in \mathbb{R}^3 \). We so define 
the unit levels of operation of these activities that \( y_j \) may alternatively be 
interpreted to represent the level of gross output of the \( j \)th good. Similar 
the column vector \( z = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}) \in \mathbb{R}^4 \) represents the levels of 
operation of the different trading activities that distribute the final goods 
in the locations of consumption. The unit levels of operation of these activi-
ties have also been so chosen that \( \beta_{jk} \) denotes the amount of the \( k \)th good 
that moves from \( j \) to \( k \) for its consumption use.

Of the technical coefficients \( a_{ij}(L, x') \) denotes the requirement of 
the \( i \)th good as input in per unit production of the \( j \)th good when the levels 
of outputs of the different reproducible goods are those given by \( x \) and the 
locational distribution of the basic industries is the one given by \( L^{1/2} \). We 
assume for the sake of simplicity of analysis, \( a_{ij} \)-coefficients to be of the 
following mathematical form:

1. The parenthesis in the notation \( a_{ij}(L, x') \) denotes functional dependence 
on \( L \) and \( x' \), i.e., transpose of \( x \).
\[ \alpha_{ij}(L, x') = \alpha_{ij}^0 \left[ \gamma_j + \sigma_{ij}(L)g_k \right] - \rho_j \delta_{1j}^x \quad \text{for } i = 0, 1, 2; \ j = 1, 2, 3; \ k = 1, 2; \ j \neq k \]

\[ = \omega_i \alpha_{ij}^0 \left[ \gamma_j + \sigma_{ij}(L)g_k \right] - \rho_j \delta_{1j}^x \]

\[ + \omega_k \alpha_{1j}^0 \left[ \gamma_j + \sigma_{1j}(L)g_k \right] - \rho_j \delta_{1j}^1 \delta_{2j}^k \]

\[ \quad \text{for } i = 3; \ j = 1, 2; k = 1, 2; j \neq k \]

\[ = 0 \quad \text{for } i = 3, \ j = 3. \]

where,

(i) \( \omega_i \) (i = 0, 1, 2) is a positive scalar representing the weight of one unit of the \( i \)th good. (We assume all the goods (excepting transport) of the model to have positive weight).

(ii) \( \alpha_{ij}^0 \) (i = 0, 1, 2; j = 1, 2, 3) and \( \rho_j \) (j = 1, 2, 3) are non-negative constant real numbers satisfying the constraints:

\[ \alpha_{12} > 0, \quad \alpha_{21} > 0 \quad \text{as required by assumption (5.9)}, \]

\[ 0 < \rho_j < 1 \quad (j = 1, 2, 3). \]

(iii) \( \sigma_{ij}(L) \)'s (i = 0, 1, 2; j = 1, 2, 3) are non-negative continuous scalar functions of \( L \) which are alternatively expressible as \( \sigma_{ij}(\delta_{11,2}) \) (i = 0, 1, 2; j = 1, 2, 3). It is further stipulated that \( \sigma_{ij} \)'s (i = 0, 1, 2; j = 1, 2) are all non-negative continuous monotonic declining function of their only argument \( \delta_{11,2} \) satisfying the inequality conditions \( \sigma_{ij}(c) > 0 \) and \( \sigma_{ij}(\infty) = 0 \) and that \( \sigma_{ij} \) is zero for all values of \( L \) or \( \delta_{11,2} \) whenever \( i = 0, 1, 2 \) and \( j = 3. \)
We should note that the elasticity of \( a_{ij} \) with respect to \( \gamma_j \) or \( \gamma_k \) (for all \( i, j, k \)) would be a homogeneous function of degree zero in \( \gamma_j \).

Let \( \mu_{ik}^{jk} \) (\( i = 0, 1, 2, 3 \) \( j = 1, 2 \) \( k = 1, 2 \)) denote the amount of the \( i \)-th good that is required as input in the distributive trading activity that ships the \( j \)-th good from its location of production to the \( k \)-th location of consumption, when the activity is operated at unit level. The definition of the trading activities ensures that

\[
\mu_{0i}^{jk} = 0 \quad \text{for} \quad i = 0, 1, 2 \quad j = 1, 2, 3, \quad k = 1, 2
\]

\[
\mu_{ji}^{jk} = 1 \quad \text{for} \quad j = 1, 2 \quad k = 1, 2 \quad \text{and} \quad \mu_{3j}^{jk} = c_j \delta_{j1} \delta_{k1} \geq 0
\]

for \( j = 1, 2 \) \( k = 1, 2 \).

We may summarise the above information about the technology of our economy by the following matrix notations.

Let \( A(L, x') \) denote the matrix of average interindustrial input requirements in the different industries. The typical \( i \)-th row \( j \)-th column element of \( A(L, x') \) is \( a_{ij}(L, x') \) (\( i = 1, 2, 3 \) \( j = 1, 2, 3 \)). Thus \( A(L, x') \) is a square matrix of order 3.

Let \( D(L) = [\mu_{ik}^{jk}] \) (\( i = 1, 2, 3 \) \( j = 1, 2 \) \( k = 1, 2 \)) denote on the other hand the matrix of requirements of the different reproducible goods as input in the trading activities distributing the final outputs in the locations of consumption when they are operated at unit levels. The order of this matrix is therefore 3x4. The trading activities are so ordered columnwise in the matrix that the \( i \)-th row is found to be the vector \( (\mu_{i1}^{12}, \mu_{i1}^{12}, \mu_{i1}^{12}, \mu_{i2}^{22}) \).

Let further \( a_o(L, x') \) denote the vector of requirements of the primary factor as input in per unit production of the different industrial goods when the levels of output of the different industries are those given by \( x \) and the
locational distribution of the basic industries is the one given by \( L \). Thus 
\( a_0(L, x) \) is a row vector of order 3, the \( j \)th element of this vector being 
\( a_{0j}(L, x') \) \( (j = 1, 2, 3) \). We should note that the trading activities do not directly require any primary input.

I would always represent an identity matrix of order 3 in the following discussion.

Before presenting the location-allocation problem of the central planners of our economy we introduce the following special assumptions in order to ensure workability of our model.

**Special Assumption 1**: There exist some \( L \), some non-negative \( x \) and some semipositive \( z \) such that the inequalities 
\[
\begin{align*}
\begin{bmatrix} I - A(L, x') \end{bmatrix} x - D(L) z & > 0, \\
\begin{bmatrix} L x' \end{bmatrix} x & < \overline{\gamma}
\end{align*}
\]
hold good. This is, in other words, the postulate of attainability of some semipositive programme of final consumption.

**Remark 1**: If the value of \( \overline{\gamma} \) be sufficiently large, our economy can attain any given \( z \geq 0 \) for any given locational distribution of the basic industries \( L \). Proof of this is immediate from the functional nature of each element of the vector \( \lambda \begin{bmatrix} I - A(L^0, \lambda x^0') \end{bmatrix} x^0 \) (obtained from \( -A(L, x') \) by the substitutions \( L = L^0 \) and \( x = \lambda x^0 \)) for any given \( L^0 \) and any given \( x^0 > 0 \). (We should note that for any feasible programme of production for which \( x \geq 0 \), our \( x \) should be positive because of the assumptions \((4,6)\) and \((6,9)\).) Each element of this vector is a continuous convex function of \( \lambda \) over the domain of all non-negative real numbers, monotonically increasing over the subdomain of all \( \lambda \geq \lambda^0 \) (where \( \lambda^0 > 0 \) is a critical value) and tending to assume indefinitely large positive value as \( \lambda \) tends to +\( \infty \). For the attainability
of any $z^o > 0$ for any given $L^o$, what we now require is the solvability of the following inequalities in $x$:

$$\begin{bmatrix} 1 - A(L^o, x') \end{bmatrix} x - D(L^o) z^o \geq 0, \quad a_0(L^o, x') x \leq \eta^o, \quad x \geq 0.$$ 

From the functional nature of the element functions of the vector $\lambda \mid I - A(L^o, \lambda x^o') \mid x^o$, where $x^o > 0$ is given, it follows that there exist a critical positive value $\lambda^* > 0$ such that any $x = \lambda x^o$ where $\lambda > \lambda^*$, satisfies the inequalities

$$\begin{bmatrix} 1 - A(L^o, x') \end{bmatrix} x > D(L^o) z^o, \quad x \geq 0$$

The only problem that remains regarding attainability of $z^o$ is whether there exists at least one such $x$ that would be consistent with the inequality of primary resource constraint $a_0(L^o, x') x \leq \eta^o$. It is then quite immediate that if the value of $\eta^o$ be sufficiently large, we should find no difficulty in ensuring the existence of at least one $x$ which would be consistent with all the relevant inequalities as stated above, so that $z^o$ may be attainable for the locational distribution of the basic industries as given by $L^o$.

It now follows, therefore, as a corollary to this result, that the interrelationship among the values of the different $a^o_{ij}$ (all $i, j$ $i \neq 0$), $\rho_j$ (all $j$), and the values of $\sigma_{ij}(L)$ (all $i, j$ $i \neq 0$), $\delta_{1j}(L)$ (all $j$), $\delta_{1jk}$ (all $j, k$), $\delta_{12}$ for any given $L$ would not as such be playing, independent of $\eta^o$, any determining role in ensuring the validity of our attainability postulate. What will in fact be important for the validity of our Special Assumption 1 is how the value of $\eta^o$ stands relatively among the values of the different technical parameters and of the relevant measures of distances for alternative values of $L$. 
Special Assumption 1°: In our economy the maximum available amount of the primary factor \( t_j \) is so large that for any \( L \) for which \( l_j \) for each \( j (j = 1, 2) \) lies within the convex hull generated by the fixed locations \( r, m_1 \) and \( m_2 \), some semipositive \( z \) is attainable, i.e., the inequalities
\[
[I - A(L, x')] x - D(L)z \geq 0, \quad a_0(L, x')x \leq \theta_j, \quad x \geq 0, \quad z \geq 0
\]
have a solution.

Definitions and Notations: Let \( S[L] \) be defined to be the set of all \( L \) for each of which some \( z \geq 0 \) would be attainable. To put it formally, \( S[L] \) is the set of all \( L \) for each of which the inequalities
\[
[I - A(L, x')] x - D(L)z \geq 0, \quad a_0(L, x')x \leq \theta_j, \quad x \geq 0, \quad z \geq 0
\]
have some solution in \( x \) and \( z \).

Special Assumption 1 thus reduces to the requirement that \( S[L] \) is nonempty.

Let \( T[L] \) be the set of all \( L \) for each of which \( l_j \) for each \( j (j=1,2) \) lies in the convex hull generated by the points of fixed location \( r, m_1 \) and \( m_2 \). In other words, \( T[L] \) is the set of all \( L \) for each of which the inequalities
\[
l_j = \lambda_{ij} r + \lambda_{ij} m_1 + \lambda_{ij} m_2 \quad j = 1, 2, \quad 0 \leq \lambda_{ij} \leq 1 \text{ for } i = 0, 1, 2;
\]
j = 1, 2 and \( \sum_{i=0}^{2} \lambda_{ij} = 1 \) for \( j = 1, 2 \) would have a solution in \( \lambda_{ij} \) (all \( i, j \)).

\( T[L] \) is obviously a nonempty compact set of \( L \). Special Assumption 1° will now be equivalent to that \( T[L] \subseteq S[L] \).

Special Assumption 2: For any choice of \( L \), any of our industrial activities would require directly only a positive amount of the primary factor whenever it is operated at positive level, while other industrial activities operate at nonnegative ones. In other words, for any given \( L \) and any given \( j \),
semipositivity of $x$ along with the positivity of $y_j$ would imply the holding of the inequality $a_{0j} \left[ y_j + c_{0j}(L) y_k \right] - \sum_{j=1}^{3} y_j > 0$.

Remark 2: In view of the assumptions on the values of $f_j^3 (j = 1, 2, 3)$ and $c_{0j}(L)$ ($j = 1, 2, 3$), the validity of Special Assumption 2 can be ensured if only if $a_{0j} > 0$ for $j = 1, 2, 3$. The proof of this assertion is so obvious that it need not be stated here in details.

We should also note that our Special Assumption 2 rules out the possibility of getting something out of nothing. In order to establish the postulate of impossibility of cockaigne in our model it would suffice if we can show that for any given $L$ and for any solution of the inequalities $A(L, x') x - D(L) z \geq 0$, $x \geq 0$, $z \geq 0$, we should get $a_{0j}(L, x') x > 0$.

The validity of this sufficient condition in our model follows in fact quite immediately from our Special Assumption 2. The proof of this assertion is quite easy and is accordingly omitted here.

9.1.6: The Problem:

Let us now pose the central problem of Model V. The problem is one of centralised planning for production and location and it is essentially the same as the central problem of the preceding model of Chapters Seven - Eight. The decision making variables of the central planners of production of our model are $L$, $x$ and $z$. The planners consider a programme of production and distribution $(x', z')$ and a locational organisation of production $L$ to be optimal if $(x', z')$ and $L$ together maximise the value of total sales proceeds of the final goods in the locations of consumption, without violating any of the balance constraints of the primary and the intermediate inputs and...
any of the relevant nonnegativity conditions. The problem of the planners is to determine the values of \( L, x \) and \( z \). Under the new assumptions on our technology in the present model, the problem of location-allocation would then assume the following formal structure:

\[
\begin{align*}
\text{PI} & : \text{Find } x, z \text{ and } L \text{ which} \\
& \text{maximise } p_z \\
& \text{subject to } \begin{cases} \\
1 + \Delta(L, x) x - D(L) z \geq 0 \\
\theta(L, x) x \leq \eta \\
x \geq 0, \quad z \geq 0 
\end{cases}
\end{align*}
\]

A \( z \) yielded by any optimal solution of the nonlinear programme PI will describe a point on the consumption possibility frontier of our economy. To any such point would however be associated a specific locational choice of the basic industrial activities. The efficient (or optimal) values of \( L \) associated with the alternative efficient (or optimal) values of \( z \), i.e., with the alternative points on the consumption possibility frontier may however be different. We discuss the problem of solvability of PI in the immediately following section.

9.2. In view of Special Assumption 1°, the problem PI has got some non-trivial feasible solution. We shall prove shortly the existence of some non-trivial optimal solution of PI.

Before engaging ourselves in the problem of establishing solvability of PI we should note that if the value of \( L \) is fixed, the location-allocation problem reduces to one of allocation of resource which involves only \( x \) and \( z \) as variables.
Let us denote such a derived problem of allocation by $P_t(L)$, for any given $L$, would however be a nonlinear program while in the preceding model the similar allocation problem was obtained linear.

First of all, we prove solvability of $P_t(L)$ and then with its help try to derive the proof of solvability of $P_t$.

**Notations:** For any $L \in S[L_j]$ let $V_L \left( x', z' \right)$ be the set of all feasible solutions of the allocation programme $P_t(L)$.

**Lemma 9.1:** For any given $L \in S[L_j]$, the feasible solution set of $P_t(L)$, i.e., the set $V_L$ is nonempty and compact containing at least one nontrivial solution.

**Proof:** The definition of $S[L_j]$ immediately gives that the set $V_L$ is nonempty whenever $L \in S[L_j]$ and that for any $L \in S[L_j]$, $V_L$ contains at least one feasible $(x', z')$ of $P_t(L)$, for which $z \geq 0$.

Since all the functions which are elements of the vectors $(I - A(L, x')) x - D(L) z$ and the function $\gamma(L, x') x$ are continuous over the domain of all nonnegative $(x', z')$ and since none of the constraints of the programme $P_t(L)$ is given in strict inequality form, the set $V_L \left( x', z' \right)$ is closed, whenever $L \in S[L_j]$.

Let $\mathcal{X}$ be the set of all nonnegative $x$ satisfying the normalising condition $||x|| = 1$. $\mathcal{X}$ is thus a nonempty compact set. Let $x^0$ be an element of the set $\mathcal{X}$. $x^0$ is obviously semipositive. The substitution $x = \lambda x^0$ in the function $a_0(L, x') x$ makes it reducible to the functional form:

$$\lambda \lambda_0(L, \lambda x^0) x^0.$$ Since $0 < \sum_j \lambda_j < 1$ (j = 1, 2, 3), $x^0 \geq 0$, $a_{0j} > 0$ (j = 1, 2, 3) (by Special Assumption 2) and $a_{0j} (L) > 0$ (j = 1, 2, 3) for all $L \in S[L_j]$, we shall get for any given $L \in S[L_j]$, $a_0(L, \lambda x^0) x^0$ to be a nonnegative
real valued continuous monotonically increasing concave function of $\lambda$ over the domain of all nonnegative real numbers. This function would also tend to assume indefinitely large positive value as $\lambda$ tend to $+\infty$. There should therefore exist for any given $L \in S[L]$, a critical positive value $\lambda^0 = \lambda^0(x^0)$ such that $\lambda a_0(L, \lambda x^0) x^0 > L$ according as $\lambda > \lambda^0(x^0)$.

Thus we can ensure for any choice of $L$ from $S[L]$ the existence of a positive real valued function $\lambda^0(x)$ over the domain of $X$ such that for any $x \in X$ we shall obtain $\lambda a_0(L, \lambda x) x > L$ according as $\lambda > \lambda^0(x)$. The values of the parameters of the function $\lambda^0(x)$ will depend obviously among other things upon $L$.

Let us now consider any Cauchy sequence $\{x_u, \lambda_u\}$ for which $x_u \in X$ for all $u$ and for which $x_u$ and $\lambda_u$ for all $u$ are consistent with the equality $	ilde{\gamma} - \lambda a_0(L, \lambda x) x = 0$, where $L$ is given and is an element of $S[L]$. Let $x_\ast$ and $\lambda_\ast$ be the limiting values of $x$ and $\lambda$ of this sequence respectively.

Since $X$ is closed, $x_\ast \in X$. Again from the definition of the sequence $\{x_u, \lambda_u\}$ we should get $\lambda_u = \lambda^0(x_u)$ for all $u$ and get all the terms of the corresponding sequence $\{\tilde{\gamma} - \lambda a_0(L, \lambda x_u) x_u\}$ equal to zero. It is then immediate that limit $\tilde{\gamma} - \lambda a_0(L, \lambda x_\ast) x_\ast = 0$ as $u \to \infty$.

Since $\tilde{\gamma} - \lambda a_0(L, \lambda x) x$ is on the other hand a jointly continuous function in $\lambda$ and $x$ over the domain of all $x \geq 0$ and $\lambda \geq 0$, when $L$ is given, we can also have limit $\tilde{\gamma} - \lambda a_0(L, \lambda x_\ast) x_\ast = \tilde{\gamma} - \lambda^0(x) a_0(L, \lambda^0(x) x \ast) x_\ast = 0$ as $\lambda \to \lambda^0(x)$ implying $\lambda_\ast = \lambda^0(x)$. Hence the continuity of $\lambda^0(x)$ over the domain of $X$.

Since $X$ is compact and $\lambda^0(x)$ is continuous over it, Weierstrass's theorem would ensure the existence of a maximum of $\lambda^0(x)$ over $X$ whenever
the L on which parameters of $\lambda^0(x)$ depend belongs to $S[L]$. Let $\lambda^*$ be the value of this constrained maximum. Since the parameters of the function $\lambda^0(x)$ would be depending on L, the value of $\lambda^*$ should also be conditional upon the choice of L. We shall accordingly write $\lambda^*$ often as $\lambda^*(L)$.

By the definition of $V_L$, we already have the result that the inequalities $\omega(L, x') x < \gamma, x \geq 0$ are valid for any $(x', z') \in V_L$ when $L \in S[L]$. For an $L \in S[L]$, any element of the set $V_L$ will therefore yield only an $x$ which satisfies the inequalities $x \geq 0, || x || \leq \lambda^*(L)$.

Let $X^*$ be the set of all x which satisfy the inequalities $x \geq 0, || x || \leq \lambda^*(L)$. Obviously the set of all $x$ yielded by the different feasible solutions of the problem $P_L(x)$ for any given $L \in S[L]$, would be contained in $X^*$ and therefore be bounded.

Again it should be noted that for any given $L \in S[L]$ any element of the set $V_L$ would satisfy the inequalities $x \geq 0$.

$$D(L)x \leq [I - A(L, x')] x$$

The basic assumptions on technology and the mathematical form of the vector function $[I - A(L, x')]_x$ suggest that for any given $L \in S[L]$ each element of the vector $[I - A(L, x')]_x$ would be a continuous function over the domain of all nonnegative x. Since for any given $L \in S[L]$ $X^*$ is by construction compact, the ith element of the vector $[I - A(L, x')]_x$ would attain a maximum value, say $\hat{c}_i$, over the domain of $X^*$ (Weierstrass's Theorem).

Let, further, $\hat{c}^*$ denote the maximum of $\hat{c}_i$ over all i (i = 1, 2, 3), $\hat{c}^*$ obviously exists. (The value of $\hat{c}^*$ would, however, be conditional upon the choice of L.)
Since $X^*$ contains the set of all $x$ yielded by the different elements of the set $V_L$ where $L \in \mathcal{S}_L$, $X^*$ must contain at least one $x$ for which $[I - A(L, x')]x$ is semipositive. This ensures that $\mathbf{9}_L \geq 0$ for all $i$ and $\mathbf{9}_L > 0$ at least for one $i$. Thus, $\mathbf{9}_L$ should be positive.

Let $t$ be the positive column vector $(\mathbf{9}^*, \mathbf{9}^*, \mathbf{9}^*)$.

From the observations mentioned above it is now immediate that for any given $L \in \mathcal{S}_L$ any element of $V_L$ would give only a $z$ for which the inequalities $z \geq 0$, $D(L)z \leq t$ remain valid. On the other hand for any $L$ we have $D(L)$ to be a matrix with all columns semipositive. Then $D(L)s$ would be semipositive for any semipositive $z$. Since we have the vector $t$ to be positive too, for any given $L \in \mathcal{S}_L$ all solutions of the inequalities $z \geq 0, D(L)z \leq t$ that would form a bounded set. It then follows immediately the set of all $z$ yielded by the different elements of the set $V_L$ where $L \in \mathcal{S}_L$, would be bounded.

We have shown that the set of all $x$ and the set of all $z$ given by the different elements of the set $V_L$, where $L \in \mathcal{S}_L$, are bounded. Hence the boundedness of $V_L$ for any $L \in \mathcal{S}_L$.

Hence the compactness of $V_L$ for any $L \in \mathcal{S}_L$ and the validity of our lemma.

**Corollary to Lemma 9.1**: For any $L \in \mathcal{S}_L$, the allocation programme $\Pi(L)$ is solvable in nontrivial sense.

**Proof**: By Lemma 9.1, the set of all feasible solutions of $\Pi(L)$ for any $L \in \mathcal{S}_L$ is nonempty and compact, containing at least one $(x', z')$ for which $z \geq 0$. Since the objective function $p_z$ of the programme $\Pi(L)$ is a linear continuous function of $z$, it will attain a maximum over this set of
all feasible solutions. Again the maximising nature of the objective function
of the programme and the positivity of the price vector \( p \) further ensure that
the optimal solution of \( \pi(L) \) for any \( L \in S[L] \), will be some \((z', z')\) for
which \( z \geq 0 \) and the optimal value of the programme is positive. Hence the
solvability of \( \pi(L) \) in nontrivial sense whenever \( L \in S[L] \).

Definitions and Notations: Let \( j \) and \( e_j \) \((j = 1, 2, 3)\) denote the
maximum and the minimum values of the function \( \sigma_{o,j}(L) \) over the domain \( T[L] \).
Since \( \sigma_{o1}(L) \), \( \sigma_{o2}(L) \) and \( \sigma_{o3}(L) \) are functions of \( L \) jointly continuous over
the domain of the compact set \( T[L] \), \( j \) and \( e_j \) should exist for all \( j \) \((j=1,2,3)\).
(Weierstrass's Theorem.) The definition of \( j \) and \( e_j \) thus implies the validity
of the inequality \( e_j \leq \sigma_{o,j}(L) \) \( \leq j \) for all \( j \) and all \( L \in T[L] \). Since \( \delta_{12} \)
has also a positive maximum value over the domain of the set \( T[L] \) and since
for each \( j \), \( \delta_{o,j}(\delta_{12}) \) is a nonnegative continuous monotonically declining
function of its argument \( \delta_{12} \), satisfying the inequalities \( \delta_{o,j}(0) > 0 \) and
\( \delta_{o,j}(\infty) = 0 \), we should get \( \sigma_{o,j}(L) \) to be positive for all \( j \) and all \( L \in T[L] \).
This implies that \( j \) and \( e_j \) should both be positive for all \( j \).

Let \( e = \min \, e_j \) and let \( \mu = \max \, j \). Obviously \( e \) and \( \mu \) exist
and are both positive. Let \( \kappa = \max (\frac{1}{\mu}, \frac{1}{e}) \). Obviously \( \kappa \) exists and is
positive in value.

Let further \( \bar{p} = \max \sum_{j} p^j \). \( \bar{p} \) also exists and satisfies the
inequality condition \( 0 < \bar{p} < 1 \).

Since \( \sigma_{o,j} > 0 \) \((j = 1, 2, 3)\) (by Special Assumption 2) on the one
hand and \( \kappa > 0 \), \( \mu > 0 \) and \( 0 < \bar{p} < 1 \) on the other, there should always
exist some positive real number \( \gamma \) such that \( \sigma_{o,1} > \gamma \) and \( \sigma_{o,2} (\frac{\bar{p}}{\mu}) \) > \( \kappa \gamma \).
Let us finally define $H$ to be the set of all $x$ satisfying the inequalities

$$
\gamma_1 \gamma_2 + \gamma_3 > 0,
\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0.
$$

Lemma 9.2: The set $H$ is nonempty and compact.

Proof: Since $\gamma > 0$, $\lambda > 0$, $\alpha_0 > 0$, $0 < \rho < 1$ and $\rho < \rho_3 < 1$, we should get $\gamma (\gamma_1 + \gamma_2) - \rho + \alpha_0 (\gamma_1 - \rho_3)$ to be a nonnegative continuous function over the domain of all nonnegative $x = (\gamma_1, \gamma_2, \gamma_3)$. Since $\gamma$ is also positive, and since $H$ defines the set of all solutions of the inequalities

$$
\gamma_1 \gamma_2 + \gamma_3 > 0, \gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0,
$$

we should get the set $H$ to be nonempty and closed.

Consider again the set $X$ of all nonnegative $x$ that satisfy the normalising condition $||x|| = 1$. Obviously $X$ is nonempty. Let $x^0$ be any arbitrary element of the set $X$. Obviously $x^0 > 0$. If we substitute $x = \lambda x^0$, the function

$$
\gamma (\gamma_1 + \lambda \gamma_2) - \rho + \alpha_0 (\gamma_1 - \rho_3)
$$

reduces to the expression

$$
\lambda^1 - \rho \gamma (\gamma_1 + \lambda \gamma_2) - \rho + \lambda^1 - \rho_3 \alpha_0 (\gamma_3^1 - \rho_3)
$$

which turns out to be a nonnegative continuous concave monotonically increasing function of $\lambda$ over the domain of all nonnegative real numbers and which tends to assume indefinitely large positive value as $\lambda$ tends to $+\infty$. (This follows from the facts that $x^0 > 0$, $\alpha_0 > 0$, $\gamma > 0$, $\lambda > 0$, $0 < \rho < 1$, $0 < \rho_3 < 1$.) There should therefore exist a critical value $\lambda^0 = \lambda^0(x^0)$ such that

$$
\lambda^1 - \rho \gamma (\gamma_1 + \lambda \gamma_2) - \rho + \lambda^1 - \rho_3 \alpha_0 (\gamma_3^1 - \rho_3) \gtrless \frac{\gamma}{\lambda^0(x)}
$$

as $\lambda \gtrless \lambda^0(x)$. Thus we can ensure the existence of a positive real valued function $\lambda^0(x)$ over the domain of the compact set $X$ such that for any $x \in X$, the inequality $\lambda a_0(L, \lambda x) x \gtrless \gamma \lambda^0(x)$ is obtained according as $\lambda \gtrless \lambda^0(x)$. 
Consider any Cauchy sequence \( \{ x'_j, \lambda'_j \} \) for which \( x'_j \in \bar{X} \) for all \( j \) and for which \( x'_j \) and \( \lambda'_j \) for all \( j \) are consistent with the equality
\[
\lambda^1 - \bar{p} \Gamma (\gamma_1 + | k | \gamma_2) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3) 1 - \bar{p}^3 = \eta^i.
\]
Let \( x^* \) and \( \lambda^* \) be the limiting values of \( x \) and \( \lambda \) of this sequence. Since \( \bar{X} \) is closed, \( x^* \in \bar{X} \). Again from the definition of the sequence \( \{ x'_j, \lambda'_j \} \), it follows that \( \lambda^j = \lambda^0 (x'_j) \) for all \( j \) and that all terms of the sequence
\[
\{ \eta^i - \lambda^1 - \bar{p} \Gamma (\gamma_1 + | k | \gamma_2) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3) 1 - \bar{p}^3 \}
\]
are equal to the value zero. (\( \gamma_1 \) is the \( i \)th element of the vector \( x'_j \)). It is then immediate that
\[
\lim_{j \to \infty} \eta^i - \lambda^1 - \bar{p} \Gamma (\gamma_1 + | k | \gamma_2) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3) 1 - \bar{p}^3 = 0
\]
Since \( \eta^i - \lambda^1 - \bar{p} \Gamma (\gamma_1 + | k | \gamma_2) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3) 1 - \bar{p}^3 \) is a function continuous in \( \gamma_1 \)'s and \( \lambda \), i.e., in \( x \) and \( \lambda \), we should also have the equality
\[
\lim_{j \to \infty} \eta^i - \lambda^1 - \bar{p} \Gamma (\gamma_1 + | k | \gamma_2) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3) 1 - \bar{p}^3 = \\eta^i - \lambda^1 - \bar{p} \Gamma (\gamma_1^* + | k | \gamma_2^*) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3^*) 1 - \bar{p}^3
\]
where \( \gamma_1^* \) is the \( i \)th element of the vector \( x^* \). Thus we finally get,
\[
\eta^i = \lambda^1 - \bar{p} \Gamma (\gamma_1^* + | k | \gamma_2^*) 1 - \bar{p} - \lambda^1 - \bar{p}^3 \alpha_0^0 (\gamma_3^*) 1 - \bar{p}^3
\]
implying \( \lambda^* = \lambda^0 (x^*) \). Thus \( \lambda^0 (x) \) is a continuous function over the domain of \( \bar{X} \).

Since \( \bar{X} \) is a compact set and \( \lambda^0 (x) \) is a function continuous over the domain of \( \bar{X} \), Weierstrass's theorem would ensure the existence of a maximum of \( \lambda^0 (x) \) over the domain of \( \bar{X} \). Let \( \lambda^* \) be the value of this constrained maximum.
The definition of the set \( H \) ensures that the inequalities

\[
\eta > \gamma (\gamma_1 + k \gamma_2)^{1 - \frac{\mu}{2}} + \alpha_0 (\gamma_0)^{1 - \frac{\mu}{3}} \quad \gamma_i > 0 \quad (i = 1, 2, 3)
\]

data valid for any \( x \in H \). It is therefore immediate that any element \( x \) of set \( H \)

satisfies the inequality \( || x || \leq \lambda^x \). Thus \( H \) should be bounded.

Hence the compactness of \( H \) which has already been shown to be non-empty and the validity of our lemmas.

Definitions and Notations : Let \( V^0 \left[ (x', z') \right] \) be the set of all

\( (x', z') \) which is the union of \( V_L \left[ (x', z') \right] \) over all \( L \in T \left[ L \right] \). To put

it formally, \( V^0 = \bigcup_{L \in T \left[ L \right]} V_L \). Since \( V_L \) contains at least one \( (x', z') \)

for which \( z \geq 0 \) and since \( T \left[ L \right] \) is non-empty and is contained in \( S \left[ L \right] \),

the set \( V^0 \left[ (x', z') \right] \) would also be non-empty and would contain at least

one non-trivial \( (x', z') \) for which \( z \geq 0 \).

Let us now define \( V \) to be a point-to-set valued mapping from \( T \left[ L \right] \)

into \( V^0 \left[ (x', z') \right] \) where \( V_L \) is the image set of \( \omega \) \( L \) under the mapping.

This mapping is also 'onto' \( V^0 \), because of the very construction of \( V^0 \).

Lemma 9.3 : There exists a compact set \( V \left[ (x', z') \right] \) such that

\( V \subseteq V^0 \).

Proof : We should first of all note that \( \alpha_0 > 0 \) for \( j = 1, 2, 3 \),

\( 0 < 2^j < 1 \) for \( j = 1, 2, 3 \) and \( \bar{z} < \alpha_0 (L) < \bar{z} \) for any \( L \in T \left[ L \right] \)

and for any \( j = 1, 2, 3 \), where \( \bar{z} > 0 \) and \( \bar{z} > 0 \). For any \( L \in T \left[ L \right] \) and for \( \bar{z} \) given by

any element of the corresponding set \( V_L \), the validity of the following

inequalities would, therefore, follow immediately ;
\[ x = (\gamma_1, \gamma_2, \gamma_3) \geq 0 \]

\[ \gamma_1 \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( V_L \))

\[ \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \gamma \) and the positivity of \( \sigma_{o2}(L) \) for any \( L \in T \))

\[ \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \gamma \) and the inequality \( 0 < \rho < 1 \))

\[ \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \kappa \))

\[ \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \kappa \))

\[ \geq \alpha_{o1} (\gamma_1 + \sigma_{o1}(L) \gamma_2) - \rho_1 \gamma_1 + \alpha_{o2} (\gamma_2 + \sigma_{o2}(L) \gamma_1) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \gamma \))

\[ \geq \gamma (\gamma_1 + \kappa \gamma_2) - \rho_1 \gamma_1 + \kappa \gamma (\gamma_1 + \kappa \gamma_2) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

(by the definition of \( \gamma \))

\[ \geq \gamma (\gamma_1 + \kappa \gamma_2) - \rho_1 \gamma_1 + \kappa \gamma (\gamma_1 + \kappa \gamma_2) - \rho_2 \gamma_2 + \alpha_{o3} \gamma_3 \leq \beta_3 \]

Thus all \( x \) given by the different elements of \( V^0 = \bigcup \{ V_L \} \) would also enter the set \( H \). (by the definition of \( H \)).

Let us define \( K \) to be the Cartesian product of the sets \( \bigcup \{ L \} \)

and \( \bigcup \{ L \} \). \( K \) is so constructed that all its elements are non-vectors, since
$T_{[L]}$ is a nonempty compact set (by definition) and since $K$ is nonempty compact (by Lemma 0.2), so also should be the set $X$.

Let $[I - A(L, x')]_j$ and $D(L)_j$ denote the $j$th rows of the matrices $[I - A(L, x')]$ and $D(L)$ respectively. Let $C_{[L, x']}_j$ be the matrix of order 2x3 whose $j$th row is $[I - A(L, x')]_j$; $(j = 1, 2)$. Similarly $D_{[L]}_j$ denotes the matrix of order 2x4 whose $j$th row is $D(L)_j$; $(j = 1, 2)$. In other words, $C_{[L, x']}$ and $D_{[L]}$ consist of the first two rows of the matrices $[I - A(L, x')]_j$ and $D(L)$ respectively. Obviously, $D(L)$ is independent of $L$ and each of its columns is semi-positive. As a result, the vector $D(L)x$ would be semi-positive for any $x$ semi-positive.

The basic assumption on technology and the mathematical form of the vector function $C_{[L, x']}x$ that each element of the vector $C_{[L, x']}x$ is a continuous function of $(L, x')$ over the domain of the compact set $K$. Weierstrass's theorem further ensures that each element of the vector $C_{[L, x']}x$ would attain a maximum over the set $X$. Let $j$ $(j = 1, 2)$ be the maximum value that the $j$th element of the vector $C_{[L, x']}x$ can attain over $X$. Let $*$ = maximum $(j)$. Obviously $*$ exists. Since $H$ contains the set of all $x$ given by the different elements of the set $V^0$ which is the union of $V_L$ over all $L \in T_{[L]} \subseteq S_{[L]}$, there exists some $(L, x')$ in the set $X$, for which $C_{[L, x']}x \geq 0$. This implies that $j \geq 0$ for $j = 1, 2, 3$ and $j > 0$ at least for one $j$ $(j = 1, 2, 3)$. Thus $*$ should be positive.

Let $\vec{z}$ be the positive column vector $(*, *)$ whose order is 2.

Let $Z$ be the set of all $z$ satisfying the following inequalities

$$D(L)z \leq \vec{z}, \quad z \geq 0.$$

Since for any given $L \in T_{[L]}$ and for any $(x', z')$ of the corresponding set $V_L$, we should have $D(L)z \leq C_{[L, x']}x \leq \vec{z}$, $z \geq 0$, it immediately
follows that all \( z \) given by the different elements of the set \( V^0 = \bigcup_{L \in T|_L} V_L \) would form a set which is contained in \( Z \).

Again \( Z \) is a closed set since the inequalities \( \tilde{D}(L)z \leq \tilde{t} \) and \( s > 0 \) are all independent of \( L \) and linear and none of them is given in strict inequality form. The boundedness of \( Z \) also follows from the facts that \( \tilde{D}(L)z \geq 0 \) whenever \( z \geq 0 \) and that \( \tilde{t} > 0 \). Thus \( Z \) is compact. Thus the set of all \( z \) given by the different elements of the set \( V^0 \) will enter the compact set \( Z \).

Let us finally define \( \overline{V} \) to be Cartesian product of the sets \( H \) and \( Z \). Since \( H \) and \( Z \) are both compact, so also should be \( \overline{V} \). Since any \( x \) equal to the \( x \)-component of any element of \( V^0 \) should belong to \( H \) and since any \( z \) equal to the \( z \)-component of any element of \( V^0 \) should belong to \( Z \), we should finally get \( V^0 \subseteq \overline{V} \).

Hence the lemma.

Remark 3: From Lemma 9.3 it follows that the mapping \( \overline{V} \) : from \( T|_L \) onto \( V^0 \) is also a mapping from \( T|_L \) into \( \overline{V} \). \( \overline{V} \) is nonempty since \( V^0 \) is so.

Lemma 9.4: The set-valued mapping \( \hat{V} \) : from \( T|_L \) into \( \overline{V} \left[ (x', z')' \right] \) is a closed one.

Proof: Let \( \{ L_{\nu} \} \) be any Cauchy sequence contained in the compact set \( T|_L \). Let \( L_{\nu} \) be the limit of this sequence. Since \( T|_L \) is closed, \( L_{\nu} \in T|_L \).

Let \( \{(x_{\nu}', z_{\nu}')\} \) be a sequence of points in the set \( T \) such that \( (x_{\nu}', z_{\nu}') \in V_{L_{\nu}} \) for all \( \nu \) and that \( \lim_{\nu \to \infty} x_{\nu} = x_0 \) and \( \lim_{\nu \to \infty} z_{\nu} = z_0 \).
Since \((x^*, z^*) \in V_{L*}\), we should have \(x_* \geq 0, \ z_* \geq 0\) for all \(u\).

Since \(\lim_{u \to \infty} x_u = x_*\) and \(\lim_{u \to \infty} z_u = z_*\), we should also get \(x_*\) and \(z_*\) to be nonnegative. Again \((x^*, z^*) \in V_{L*}\) implies

\[
\sum I - \Delta(L, x^*)]x_0 - D(L, x^*)x_u \geq 0 \quad \text{and} \quad \sum \gamma - a_o(L, x^*)x_u \geq 0 \quad \text{for all } u.
\]

It is therefore immediate that

\[
\lim_{u \to \infty} \sum I - \Delta(L, x^*)]x_0 - D(L, x^*)x_u = 0
\]

and

\[
\lim_{u \to \infty} \sum \gamma - a_o(L, x^*)x_u = 0.
\]

We should also note that all the elements of the vector expression

\[
\sum I - \Delta(L, x^*)]x - D(L, x)\]

and the scalar expression \(\gamma - a_o(L, x^*)x\) are continuous functions of \((L, x^\prime, z^\prime)\) over the domain of the Cartesian product of all \(L \in T[x]\), all \(x \geq 0\) and all \(z \geq 0\). This implies

\[
\lim_{u \to \infty} \sum I - \Delta(L_u, x_u^*)]x_0 - D(L_u, x_u^*)x_u = \sum I - \Delta(L_u^*, x_u^*)]x_u^* - D(L_u^*)x_u^*
\]

and

\[
\lim_{u \to \infty} \sum \gamma - a_o(L_u, x_u^*)x_u = \sum \gamma - a_o(L_u^*, x_u^*)x_u^*.
\]

All these finally give

\[
\sum I - \Delta(L_u^*, x_u^*)]x_u^* - D(L_u^*)x_u^* \geq 0, \quad \sum \gamma - a_o(L_u^*, x_u^*)x_u^* \geq 0,
\]

\(x_u^*\) and \(z_u^*\) being already nonnegative.

Thus \(x_u^*\) and \(z_u^*\) should be consistent with the feasibility of the program \(P(L_u)\). In other words, \((x^*, z^*) \in V_{L*}\) where \(L_u \in T[x]\).

Obviously \(V_{L*} \subseteq V\).

Hence the closedness of the set-valued mapping \(V\) from \(T[x]\) into \(V\) and the validity of the lemma.
Lemma 9.5: The set-valued mapping $V$ from $T[L]$ into $\overline{V}$ is upper semicontinuous.

Proof: The set $\overline{V}$ is compact (by Lemma 9.3). The mapping $V$ from $T[L]$ into $\overline{V}$ is closed (by Lemma 9.4). Then by Lemma 4.4 of Minicco, H [19, p. 66] the mapping $V$ is upper semicontinuous.

Hence the lemma.

Definitions and Notations: We now define a new location-allocation problem $P_1^C$ whose structure is more restrictive than that of $P_1$.

$$P_1^C: \text{maximise } p_z$$
subject to
$$[I - A(L, x')]x - B(L)z \geq 0$$
$$a_o(L, x')x \leq \gamma$$
$$z \geq 0, \quad z \geq 0$$
$$L \in T[L].$$

Let $F_L$ be the Cartesian product of any given $L \in T[L]$ and the set of all $(x', z')$ belonging to the image set $V_L$ of $L$ under the mapping $V$. Let $F$ denote the union of $F_L$ over all $L \in T[L]$. Obviously $F$ defines the set of all feasible $(L, x', z')$ of the programme $P_1^C$.

Lemma 9.6: $F$ is nonempty and compact.

Proof: $F$ is nonempty since $T[L]$ is nonempty and since $V_L$ is nonempty for any $L \in T[L]$.

$F$ is compact if $T[L]$ and $\bigcup_{L \in T[L]} V_L = \overline{V}$ are compact. $T[L]$ is compact by its construction. We have also observed that

(a) each image set $V_L$ of the set-valued mapping $V$ from $T[L]$ into $\overline{V}$ is nonempty and compact (by the relation $T[L] \subseteq C[L]$ and the Lemma 9.1), where $T[L]$ is already nonempty and compact.
(b) the set valued mapping \( V; \) from \( T[L] \) into \( \bar{V} \) is upper semicontinuous (by Lemma 9.5).

From (a) and (b) we get the set \( \bigcup_{L \in T[L]} V_L = \bar{V} \) to be compact (by Lemma 4.5 of Nikaido [13, p. 97]).

Hence the compactness of the nonempty set \( F \) and the validity of our Lemma.

**Corollary to Lemma 9.6:** The nonlinear problem \( P_{T^0} \) is solvable in nontrivial sense.

**Proof:** The feasible solution set of \( P_{T^0} \) as represented by \( F \) is nonempty and compact (by Lemma 9.6). The objective function of the problem is \( p_z \) which is continuous over the entire domain of \( F \). Weierstrass's theorem would then ensure solvability of \( P_{T^0} \). Since, further, \( T[L] \subseteq S[L] \), the validity of the Corollary to Lemma 9.1 would ensure the existence of a nontrivial solution of \( P_{T^0} \), for which \( z \geq 0 \).

Hence the validity of the Corollary to Lemma 9.6.

**Lemma 9.7:** The vector \((L^*, x^*, z^*)\) for which \( z^* > 0 \), solves the problem \( P_{T^0} \) if and only if solves the programme \( P_{T^0} \).

**Proof:** Necessity: Suppose \( L^* \), \( x^* \) and \( z^* \) solve \( P_{T^0} \) and \( z^* > 0 \). Then the column vector \((x^*, z^*)\) is an optimal solution of the conditional allocation programme \( P_{T^0} \). Since, further, \( z^* > 0 \), we get \( L^* \in S[L] \). By Special Assumption 1, we already have, \( T[L] \subseteq S[L] \). Now if the set \( S[L] - T[L] \) which is a subset of \( S[L] \) and is complementary to \( T[L] \), be empty, then \( L^* \in T[L] \) and \((L^*, x^*, z^*)\) solves \( P_{T^0} \). If, on the other hand, \( S[L] - T[L] \) be found to be nonempty, we can also show \( L^* \in T[L] \) as follows:
Suppose, on the contrary, \( L^* \in S[L] - T[L] \). Because of the assumptions (5.6) and (5.3), special assumption 2 and of mathematical forms of production functions of the different industries, it will then always be possible to choose an \( L^0 \in T[L] \) such that

\[
\delta_{11}^{(j)} \leq \delta_{12}^{(j)}, \quad \delta_{11}^{(j)} \leq \delta_{12}^{(j)} \quad (j = 1, 2), \quad \delta_{13}^{(j)} \leq \delta_{13}^{(j)} \quad (j = 1, 2, \ldots, \delta_{13}^{(j)} = 1, 2),
\]

at least one inequality of these being so nonbinding that \((x^*, z^*) \in V_{L^0}\) and that the optimal value of the programme \( P(L^0) \) is strictly greater than \( P(L^*) \). Since by hypothesis, the optimal value of \( P(L) \) for any \( L \in S[L] \), we end up with a contradiction. Thus \( L^* \not\in S[L] - T[L] \) implying further \( L^* \in T[L] \), even if \( S[L] - T[L] \) be nonempty. Thus \((L^*, x^*, z^*)\) should be a feasible solution of \( P(L) \).

Since the set of all feasible solutions \((L, x, z)\) of \( P(L) \) should also belong to the set of all feasible solutions of \( P(L) \), any optimal solution of \( P(L) \) would also be an optimal solution of \( P(L) \) whenever it is feasible for \( P(L) \). Since \((L^*, x^*, z^*)\) is an optimal solution of \( P(L) \) and a feasible solution of \( P(L) \), it would also be an optimal solution of \( P(L) \).

**Sufficiency**: Let us suppose \( L^*, x^* \) and \( x^* \) solve \( P(L) \) and \( x^* \geq 0 \).

Since any feasible solution of \( P(L) \) is also feasible for \( P(L) \) and since any optimal solution of \( P(L) \) with \( z \geq 0 \), solves \( P(L) \) (as observed in the necessity part of proof of the present lemma), it is immediate that all nontrivial optimal solutions of \( P(L) \) would enter the set of nontrivial optimal solutions of \( P(L) \). Since the vector \((L^*, x^*, z^*)\) for which \( z^* \geq 0 \), is an optimal solution of \( P(L) \), it would solve \( P(L) \) in a nontrivial sense.

Hence the lemma.
Theoren 9.1.: The location-allocation problem $PI$ is solvable in nontrivial sense.

Proof: From Lemma 9.7 we get that the set of nontrivial optimal solutions of $PI$ is equivalent to that of the programme $PI^C$. Corollary to Lemma 9.6 already shows that $PI^C$ is solvable in nontrivial sense. These together imply that $PI$ is solvable and that for any of its optimal solution $z \geq 0$.

Hence the theorem.

Remark 4: The arguments given in the proof of solvability of the location-allocation problem $PI$ show that for any optimal locational organization of production, $l_j$ ($j = 1, 2$) should lie in the convex polyhedron generated by the points of fixed location $r$, $n_1$ and $n_2$. The basic assumptions of indecomposability of interindustrial consumption structure of the basic industries (assumption (5.9)), noncoincidence of any location of deposit of the primary factor with any other fixed point of location of the model (assumption (4.6)) and of indispensability of primary factor in all the basic industries would necessitate $x$ to a positive vector for any feasible solution of $PI$ for which $z$ is semipositive. Since $PI$ of our present model should have a nontrivial optimal solution, we finally get that for any optimal $(L, x, z')$ of $PI$, $L \leq T_1 [L], x > 0$ and $z \geq 0$.

We have proved the solvability of our location-allocation problem $PI$.

There remains a problem of existence of a system of supporting efficiency prices to a solution of production and distribution of outputs that is given by some optimal solution of $PI$. I could not derive any decisive result on this problem. With the given assumptions and special assumptions of the
present model it would not be ensured that for any optimal \( (L^*, x^*, z^*) \) of \( P \), there would necessarily exist a system of nonnegative shadow input prices of the reproducible and non-reproducible goods which would satisfy the following conditions even if the left hand side functional expressions of the constraints of the program \( P(L^*) \) were differentiable at \( (x^0, z^0) \):

(i) the marginal value of net return being nonpositive in each activity (basic productive or trading) when levels of operation of the different activities are marginally increased from their respective values given by the vectors \( x^0 \) and \( z^0 \);

(ii) marginal value of net return being exactly zero for any activity which is operated at positive level in the optimal situation described by \( (x^0, z^0) \);

(iii) shadow price of any intermediate or primary input being exactly zero whenever it is in excess supply after meeting the requirements of all the basic productive and trading activities in the optimal situation described by \( (x^0, z^0) \).

I suspect that strengthening of some of the assumptions of our model or making of some new assumptions may be necessary in order to ensure the existence of a set of nonnegative values of the lagrange multipliers associated with the optimal solution \( (x^0, z^0) \) of the nonlinear program \( P^*(L^0) \) so that the existence of a shadow input price vector satisfying (i) - (iii) may be guaranteed.
9.3. This section attempts to clarify the role of externality in determining location-allocation solutions of the problem $PI$ of our model. In the present model we have assumed positivity of $\sigma_{ij}(L) = \varphi_{ij}(5_{12})$ for any $L (i = 0, 1, 2; j = 1, 2)$ in order to ensure the existence of some definite effect of externalities on economies of scale in the different basic industries. The factors of externalities $\varphi_{ij}$'s ($i = 0, 1, 2; j = 1, 2$) are all nonnegative continuous monotonically decreasing functions of their only argument $5_{12}$ over the domain of all nonnegative real numbers. Moreover we have assumed $\varphi_{ij}(0) > 0$ ($i = 0, 1, 2; j = 1, 2$) and $\varphi_{ij}(\infty) = 0$ ($i = 0, 1, 2; j = 1, 2$).

Besides these, $\varphi_{ij}(L) = 0$ for any $L$ and for $i = 0, 1, 2$. All these stipulations on the functional nature of $\varphi_{ij}$'s imply that the extent to which externalities generated in the system are internalized by the $j$th basic industrial activity depends positively on the degree of closeness of its location to that of the $k$th one ($j = 1, 2; k = 1, 2; j \neq k$). We now take the following illustrative mathematical form for the function $\varphi_{ij}(5_{12})$ which remains consistent with the above mentioned stipulations and analyze how $\varphi_{ij}$'s play their role in determining the optimal solutions of $PI$ for this illustrative situation.

We assume $\sigma_{ij}(L) = \varphi_{ij}(5_{12}) = \frac{\xi_{ij}5_{12}}{\xi_{ij}5_{12}}$ for $i = 0, 1, 2; j = 1, 2$.

where $\xi_{ij}$ and $X_{ij}$ ($i = 0, 1, 2; j = 1, 2$) are positive constants independent of $L$. Obviously it is the positivity of $\xi_{ij}$ and $X_{ij}$ ($i = 0, 1, 2; j = 1, 2$) through which externalities would make their present felt in the model. Moreover, for any $i$, $j$, the ratio of $\xi_{ij}$ to $\xi_{ij}(0)$, the coefficient $\xi_{ij}$ in the denominator of the expression of $\sigma_{ij}$, on the other hand, primarily determines the effect of
variations in the degree of locational concentration of the basic industries on the external economies that any particular basic industry can internalise. It is however the sensitivity of the optimal solutions of PI with respect changes in $X_{ij}$'s that is important in indicating the importance of agglomerative factors in determining the optimal location-allocation pattern of our problem. We now analyse how and why optimal solutions of PI should change at the same time when $X_{ij}$ changes only for some $i, j$ for some reason and other parameters remain unchanged.

Suppose $X_{i0j0}$ ($i^0, j^0$ being some given values of the indices $i, j$ respectively) goes down because of technical progress. The optimal $(L, x, z')$ of PI will change. Let $v$ and $\delta$ denote henceforth the values of the objective function of PI for any optimal solution of PI and of $\delta_{112}$ for any $L$. The values of $v$ and of $\delta$ for an optimal $L$ change when the optimal solution of PI changes because of variation in $X_{i0j0}$. Change in the optimal value of $\delta$ will measure the extent to which locational concentration of the basic industries goes up because of fall in the value of $X_{i0j0}$. The magnitude of this change in optimal $\delta$ relative to that in $X_{i0j0}$ will therefore serve as an indicator of the significance of the role of the coefficient of externality $X_{i0j0}$ in determining the extent of aerial agglomeration of the basic industries.

In some situation it may however so happen that the magnitude of this change in optimal $\delta$ is zero. Even in that case our optimal choice of $L, x, z$, and the values of optimal $x$, $z$, and $v$ may change consequently. The magnitudes of values of all these net changes in $L, x$ and in $z$ would then indicate the

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2 We use the term 'aerial agglomeration' in the same sense in which it was used by Lösch. See [16, pp. 10-12].
sensitivity of location-allocation solution of PI with respect to a forward shift in the coefficient of externality $X_{i,j}$, for which the optimal degree of locational closeness of the two basic industries remains unchanged. At present our interest however lies only in analytically decomposing the net changes in the optimal values of $L$, $x$ and $z$ and in attributing the change in $X_{i,j}$ to the different underlying forces that are generated by any change in $X_{i,j}$.

In Model III or Model IV locational variables influenced directly only the requirements of the transport good for gathering inputs of the different basic industries at their respective locations of production. Accordingly any optimal locational solution of Model III or of Model IV may be described as a transport-oriented one. In the present model, on the other hand, locational variables influence all the average input coefficients of the basic industries, $\alpha_{i,j}$ (i = 0, 1, 2, 3, j = 1, 2). It is however only through the value of $\delta$ that $L$ influences $\alpha_{i,j}$'s (i = 0, 1, 2, 3, j = 1, 2). Since any variation in $\delta$ causes change both in the requirement of transport input in a basic industry for gathering inputs at its site of production and in the level of agglomeration economies that arise because of externalities and are determined by any basic industry, the optimal locational solution of the problem PI will be determined by both agglomerative and transport factors.

We now derive the following programme $PI^*$ from the original location-allocation problem PI by imposing an additional constraint that $\delta_{1,2}$ should be equal to an arbitrary value $\delta^0$. $PI^*$ is thus a location-allocation programme conditional upon a given measure of locational closeness of the basic industries. The mathematical structure of $PI^*$ is obviously more restrictive than that of PI.
\[ \text{PI}^* : \text{Find } L, x \text{ and } z \text{ that} \]
\[ \text{maximise } p_z \]
\[ \text{subject to } - \Delta(L, x) x - D(L) z \geq 0 \]
\[ a_0(L, x) x \leq \gamma \]
\[ x \geq 0, z \geq 0, \delta = \delta_{11} = \delta^0 \]

where \( \delta^0 \) (nonnegative) is a priori fixed.

If we now consider any parametric change in the value of \( \delta^0 \) of \( \text{PI}^* \), the consequent change in the value and in the solution of \( \text{PI}^* \) may then be interpreted to be reflecting the effects of variation in the stipulated measure of locational closeness of the two relevant basic industrial activities, while the basic parameters of externalities are held constant.

Since \( \delta_{11} \) is a continuous function of \( L \) over the domain of the compact set \( [L] \), there exist a maximum and a minimum of the function \( \delta_{11} \) over this domain (Weierstrass's theorem). Let this constrained maximum of \( \delta \)-function be denoted by \( \delta > 0 \). The minimum value of \( \delta \) of this constrained problem is obviously zero. For any value of \( \delta^c \) lying in the closed interval \((0, \delta)\), the solvability of \( \text{PI}^c \) is ensured. The proof of this assertion consists of the following two logical steps.

First, we have to show that the set of all optimal solutions of \( \text{PI}^* \) for any given \( \delta^0 \) lying in the closed interval \((0, \delta)\), is equivalent to that of the following problem \( \text{PI}^{**} \).
PI**: Find \( L, x \) and \( z \) that

maximise \( pz \)

subject to

\[
\begin{align*}
\mathbb{1} - A(L, x') \leq D(L) z \geq 0 \\
\alpha_0 (L, x') x & \leq T
\end{align*}
\]

\[x \geq 0, \ z \geq 0, \ L \in T \cap T^0\]

where \( T^0 \) is the set of all \( L \) for which \( \delta_1 = \delta^0, \delta^0 \)

being a priori fixed and belonging to the closed interval

\((0, 3)\).

We should note that \( T^0 \) is closed and \( \mathbb{1} \) is compact. Thus \( T \cap T^0 \) should also be compact. The logic explaining equivalence of the set of optimal solutions of PI* for any choice of value of \( \delta^0 \) from the closed interval \((0, 3)\), to that of PI** would however be exactly analogous to that given in the preceding section for establishing the equivalence between the two optimal solutions/respective programmes PI and PI°.

Secondly, we shall have to establish the solvability of PI**. The logic-mathematical approach of proving this is also exactly similar to that adopted for establishing the solvability of the programme PI° in section 9.3.

The details of the proof of our proposition that PI** is solvable whenever \( \delta^0 \) lies in the closed interval \((0, 3)\) will therefore sound repetitive and is accordingly omitted here.

Let \( v(\delta^0) \) be the optimal value of the programme PI**. Consider now the possibility of parametric variation in \( \delta^0 \). Since PI** is solvable for any \( \delta^0 \) belonging to the closed interval \((0, 3)\), \( v(\delta^0) \) will be a function of \( \delta^0 \).
unambiguously defined over the domain of this interval. The function \( v(\delta^0) \) should also attain a maximum over the closed and bounded interval \((2, 5)\) so that our original location-allocation problem \( P_I \) may be solvable and that we may not end up with any contradiction with Theorem 1. Let \( \delta^0 \) be the value of \( \delta^0 \) that maximises \( v(\delta^0) \) over the interval \((0, 5)\) (closed and bounded).

Let the problem \( P^{*}_I \) with \( \delta^0 = \delta^0 \) give \((L^0, X^0, Z^0)\) as its optimal location-allocation solution. By the requirements of consistency it follows that \((L^0, X^0, Z^0)\) solves our original problem \( P_I \) as well. The following figure 9.1 illustrates this situation graphically. (We draw here the graphs of \( v(\delta^0) \) for the different values of the parameter \( X_{i,j} \) as continuous ones. We do not however require continuity of the function \( v(\delta^0) \) for our present discussions. It is the existence of a maximum of \( v(\delta^0) \) over the closed interval \((0, 5)\) that would suffice for establishing our contentions of the present theoretical discussion on Model V.)

![Figure 1.](image-url)
If the curve levelled I gives the graph of the function $v(\delta^0)$, then the point $A$ would describe an optimum situation. For each point on a curve like I there exists a location-allocation solution which would solve the associated nonlinear programme $P_{I^*}$. For a point like $A$ which gives the maximum of $v(\delta^0)$ over the closed interval $(0, \delta)$, the associated programme $P_{I^*}$ has an optimal solution $(L, x', z')$ which would consistently solve our original problem $P_I$ too. The different curves of Figure 0.1 correspond to the different values of parameter $X_{ij0}^0$ of our model. If the parametric values of our model be such that curve II of Figure 1 gives the relevant graph of our function $v(\delta^0)$ over the domain $(0, \delta)$ (closed interval) would be equal to zero and we would get complete geographic conglomeration of the basic industrial activities in the optimal situation.

We consider here change only in the value of the parameter $X_{ij0}^0$. Suppose for the initial value of $X_{ij0}^0$ we get curve I as the graph of $v(\delta^0)$. Suppose, the value of $X_{ij0}^0$ goes $\succeq$. The mathematical forms of the functions $\alpha_{ij}(L, x')$ and $\alpha_{ij}(L)$ (for $i = 0, 1, 2, 3$ and $j = 1, 2$) would necessitate an $\succeq$ shift of the graph of $v(\delta^0)$ for any upward movement in the value of $X_{ij0}^0$. This follows from the fact that for any given value of $\delta^0$, the initial optimal solution of $P_{I^*}$ remains feasible even after the decrease in value of the parameter $X_{ij0}^0$. Let curve II of Figure 1 represent the graph of the function $v(\delta^0)$ when the parametric change has already taken place in the form of a fall in the value of $X_{ij0}^0$ from its initial level. The initial optimal solution was obtained at $A$. 


while the new one would be obtained at C in the figure. Let $V_A$, $\delta_A^O$, $L_A^O$, $x_A^O$, $z_A^O$ and $v_A$, $\delta_A^O$, $L_A^O$, $x_A^O$, $z_A^O$ denote the optimal values of the variables $v$, $\delta^O$, $L$, $x$, and $z$ associated with the points $A$ and $C$ respectively. The values of $V_C - V_A$, $\delta_C^O - \delta_A^O$, $L_C^O - L_A^O$, $x_C^O - x_A^O$, and $z_C^O - z_A^O$ would obviously measure the magnitude and direction of net changes in the optimal values of all the significant variables of our model that would be caused by the decrease in the value of the parameter $X_{1.0}$. We now decompose this net change in the values of the different variables into two component parts and attribute them to two factors: one part of the change is due to movement from $A$ to $B$ and the other part of the change is due to movement from $B$ to $C$ in Figure 1. Movement from $A$ to $B$ causes the differences $v_B - v_A$, $\delta_B^O - \delta_A^O$, $L_B - L_A$, $x_B - x_A$, and $z_B - z_A$ which measure the effect of the decrease in $X_{1.0}$ that is conditional upon the stipulation $\delta_B^O = \delta_A^O$. These measures of change in the values of the variables would in fact reflect only the effect of greater economies that fall in the value of directly result from the shift in $X_{1.0}$, and do not operate through the realisation of any greater cost advantages that may be derived from a greater locational concentration of the basic industries that may in its turn be downward induced by the shift in $X_{1.0}$. Since $\delta_C^O + \delta_A^O$ may be possible, a change in the measure of locational concentration (i.e., $\delta_1^O$) of the basic industries downward would be induced by the movement in the value of $X_{1.0}$. It is the consequence of this induced change in the value of $\delta^O$ that the differences $v_C - v_B$, $\delta_C^O - \delta_B^O$, $L_C - L_B$, $x_C - x_B$, and $z_C - z_B$ measure. These measures of the values of the variables would therefore reflect the effect of the difference in cost advantages that the basic industrial activities experience while internalising the benefits of external economies, because of the.
greater mutual closeness of their locations when the basic parameters of externalities $X_{ij}$'s are held constant at the new values.

We may however analyse the effect of change in the value of $X_{i0\,j0}$ on the optimal solution of PI in an alternative way. This alternative is to view the movement from A to C in Figure 1 as one of, first, from A to D and then from D to C. In that case the movement from A to D would reflect the effect of the change in the value of $\delta^0$ that would be induced by a shift in $X_{i0\,j0}$, while the basic parameters of externalities are held constant at the initial values. The movement from D to C, on the other hand, reflects the effect of all the value of $X_{i0\,j0}$ that operates directly through the greater advantages of external economies that have been caused by this technical progress, while $\delta^0$ remains fixed at the value given by the new optimal solution of our problem PI (which is zero in the graphical situation described by Figure 1). The choice of the way of decomposing and of interpreting the movement from A to C in Figure 1 is a matter of judgement. What we like to emphasise in conclusion is that the discussion made above can offer a method of disentangling analytically the forces through which any change in $X_{ij}$ for any $i$, $j$ would make its effect felt on the optimal location-allocation solution of problem PI of our present model. The distinction between the effects of the movement among curves and that along a curve (whose resultant is the effect of a movement like one from A to C) in Figure 1 is in fact a pointer not only of what role externalities play in our model, but also of how the role is played in it.