CHAPTER FIVE

METHODS OF CONSTRUCTION OF SOME THREE OR 
FOUR ASSOCIATE DESIGNS OF DIFFERENT 
TYPES DESCRIBED IN CHAPTER TWO AND THREE

5.1 Introduction

Though cyclic designs i.e. designs with association schemes obtained by cyclical generalisations are special cases of partially balanced incomplete block designs, none of the P.B.I.B. designs mentioned by Bose and Nair (1939) are cyclic designs. These authors, however, have considered some methods for constructing three associate P.B.I.B. designs in general. Roy (1954a, 1954b) discussed some methods of constructing three associate P.B.I.B. designs from B.I.B. designs. Zelen (1954) discussed a method of constructing m-associate P.B.I.B. design from (m-1)-associate P.B.I.B. design by replacing each treatment in a block by a group of n treatments. Vartak (1955) constructed some three and higher associate designs by Kronecker product methods. Roy (1955, 1962) initiated the development of P.B.I.B. designs following a given association scheme by discussing various methods for constructing three and higher associate H.G.D. and O.G.D. designs. Raghavarao (1962) constructed some designs following H.G.D. association scheme by adding some blocks to the plans of a G.D. design. This method was utilised by Tharthare (1963) for constructing right angular designs. Raghavarao and Chandrasekhararao (1964) utilised Kronecker product method for constructing cubic designs.
But among all these, only some of the designs given by Roy (1954b, 1955, 1962) and the designs given by Zelen (1954) are found to be cyclic designs. In this chapter, we shall discuss some methods for constructing cyclic designs belonging to both the types. Unless otherwise stated, these methods are quite general and applicable to all the three associate (or four associate rectangular) association scheme of a particular type of generalisation.

5.2 Some useful lemmas

Before proceeding to discuss various methods of construction, we shall prove a few lemmas which will be useful in the sequel.

**Lemma 5.2.1.** The treatments of any three associate cyclic design of the first type of generalisation (having any association scheme) can be divided into \((m_2 + m_3 + 1)\) disjoint groups, \(G_1, G_2, \ldots, G_{m_2}\), of \(m_1\) treatments each such that (i) the treatments in the same group form a group of first associate among themselves; (ii) the groups \(G_2, \ldots, G_{m_2+1}\) contain treatments which are common second associates of any treatment of \(G_1\); and (iii) the groups \(G_{m_2+2}, \ldots, G_{m_2+m_3+1}\) consist of treatments which are common third associates of any treatment of \(G_1\).

**Proof:** Let us denote the treatment by \(g_{11}, g_{12}, \ldots, g_{m_1}\).
\(\varnothing_1, \varnothing_2 \cdots \varnothing_{m_1}, \varnothing_3, \cdots \varnothing_{(m_2+m_3+1)m_1}\). As \(n_1 = m_1-1\), so we may suppose without any loss of generality, that the first associates of \(\varnothing_{11}\) are \(\varnothing_{12}, \cdots \varnothing_{1m_1}\). Consider the treatments \(\varnothing_{11}\) and \(\varnothing_{12}\). As the relationship of first association between two treatments is commutative, so \(\varnothing_{11}\) is a first associate of \(\varnothing_{12}\). Since \(p_{11} = m_1-2\), \(\varnothing_{11}\) and \(\varnothing_{12}\) has \(m_1-2\) first associates in common. So these can be no other than \(\varnothing_{13}, \cdots \varnothing_{1m_1}\). Also, \(\varnothing_{12}\) has exactly \(m_1-1\) first associates. Hence, all the first associates of \(\varnothing_{12}\) are \(\varnothing_{11}, \varnothing_{13}, \cdots, \varnothing_{1m_1}\). This shows that any first associate of \(\varnothing_{11}\) (other than \(\varnothing_{12}\)) is a first associate of \(\varnothing_{12}\). These conditions are sufficient to ensure that the \(v\) treatments can be divided into \((m_2+m_3+1) - \text{groups}\) of \(m_1\) treatments, such that two treatments in the same group are first associates while any treatment can not have any first associate outside this group. Let us name these groups as \(G_1, G_2, \cdots, G_{m_2+m_3+1}\) where we suppose that the elements in the \(i\)-th group of treatments, \(G_i\) are \(\varnothing_{i1}, \varnothing_{i2}, \cdots, \varnothing_{im_1}\), \(i = 1, 2, \cdots, m_2+m_3+1\). This proves the first part of the lemma.

Let \(\varnothing_{11}\) be any treatment which is a second associate of \(\varnothing_{11}\). Now, the first associates of \(\varnothing_{11}\) are \(\varnothing_{12}, \cdots, \varnothing_{1m_1}\). As \(p_{12} = m_1-1\), so \(\varnothing_{11}\) has \(m_1-1\) second associates among \(\varnothing_{12}, \cdots, \varnothing_{1m_1}\) and hence all these treatments are second associates.
to $G_1$. Thus, if any two treatments are second associates of each other, then all the first associates of one including itself are second associates of the other. Thus, if any treatment of $G_i$ is a second associate of any treatment of $G_j$, then all the treatments of $G_j$ are second associates of any treatment of $G_i$ and conversely, $i = 1, 2, \ldots, m_2+m_3+1$. Let the second associates of any treatment $G_1$ be the treatments in the groups $G_2, G_3, \ldots, G_{m_2+1}$. As $p_{22} = n_2$, so all the treatments are second associates of any treatment of $G_1$. Similarly, the treatments in the groups $G_{m_2+2}, \ldots, G_{m_2+m_3+1}$ are the common third associates to any treatment of $G_1$. This proves the lemma.

Lemma 5.2.2: The treatments of a four associate class cyclic rectangular design belonging to the first type of generalisation can be divided into $(m_2+m_3+m_2m_3+1)$ disjoint groups, $G_1, G_2, \ldots, G_{m_2+m_3+m_2m_3+1}$, of $m_1$ treatments each such that (i) the treatments in the same group form a group of first associate among themselves; (ii) the groups $G_2, G_3, \ldots, G_{m_2+1}$ contain treatments which are common second associates of any treatment of $G_1$; (iii) the groups $G_{m_2+2}, \ldots, G_{m_2+m_3+1}$ consist of treatments which are common third associates of any treatment of $G_1$; and (iv) the groups $G_{m_2+m_3+2}, \ldots, G_{m_2+m_3+m_2m_3+3}$ consist of treatments which are common fourth associates of any treatment of $G_1$. 
Lemma 5.2.3. For a three associate cyclic design (belonging to any association scheme) of the first type of generalisation, if \( \lambda_2 = 0 \) and \( p_{33}^2 = 0 \), then the design is disconnected.

Proof: Consider any treatment \( \theta \). As \( \lambda_2 = 0 \), so \( \theta \) does not occur with any of its second associates. Let \( \phi \) be a second associate of \( \theta \). Let \( G_1(\psi) \) denote the group of treatments (including \( \psi \)) which form the first associate group of \( \psi \). As \( \phi \) is a second associate of \( \psi \), so all the first associates of \( \psi \) are second associates of \( \theta \). So, \( \theta \) and \( \psi \) can not be connected through a first associate of \( \psi \). Hence the only possibility is that they should be connected through a third associate of \( \psi \). Let \( G_{11}(\phi), G_{12}(\phi), \ldots, G_{1m_3}(\phi) \) denote the \( m_3 \) groups of \( m_1 \) treatments each, which comprise all the third associates of \( \psi \). But as \( p_{33}^2 = 0 \), so \( \theta \) and \( \psi \) can not have any third associates in common. Hence \( G_{11}(\phi), G_{12}(\phi), \ldots, G_{1m_3}(\phi) \) must be second associates of \( \theta \). So \( \theta \) can not occur with any treatment in these groups. Thus, \( \theta \) and \( \psi \) can not be connected.

Note: Though disconnected designs are also important for the purpose of constructing new connected designs (because by adjoining these designs with suitable designs having the same block sizes, some new connected designs can be obtained) we shall confine ourselves to the construction of connected designs only.

Lemma 5.2.4: For the designs under consideration, (three associate, second type of generalisation) if any two treatments are
third associates of each other then all the first and second associates of one are third associates of the other.

Let $\theta$ be any treatment. Let $\phi$ be a third associate of $\theta$. Suppose the $m_2$ first associates of $\theta$ are $\theta_{11}, \theta_{12}, \ldots, \theta_{1m_2}$; $m_3$ second associates of $\theta$ are $\theta_{21}, \theta_{22}, \ldots, \theta_{2m_3}$. As $p_{11}^3 = 0$, so none of $\theta_{12}$, $i = 1, 2, \ldots, m_2$ is a first associate of $\phi$; as $p_{12}^3 = 0$ none of them can also be a second associate of $\phi$.

Hence, all of them are third associates of $\phi$. Similarly, as $p_{21}^3 = 0$, $p_{22}^3 = 0$, so all of $\theta_{21}, \theta_{22}, \ldots, \theta_{2m_3}$ must be third associates of $\phi$. Thus the set of $(1 + m_2 + m_3)$ treatments $(\theta; \theta_{11} \ldots \theta_{1m_2}; \theta_{21}, \theta_{2m_3})$ form a group of treatments $G_1$; say, such that any two treatments within this group are either first or second associates; but any treatment outside $G_1$ is a third associate of any treatment of $G_1$. We may denote these groups as $G_1, G_2, \ldots, G_{m_1}$.

5.3 Method of symmetrically repeated ratios

Method of symmetrically repeated ratios is the same as the method of symmetrically repeated differences which was introduced by Bose and Nair (1939) for P.B.I.B. designs in general. But when the association scheme is known in advance, we may state our theorems in a simpler manner. Thus

Theorem 5.3.1: (First type of generalisation). Let $G_1$ and $G_2$
be two abelian groups formed by \( G_1 \): \( 1, c, c^2, \ldots, c^{m_1-1} \) when \( c^{m_1-1} = 1 \) and \( G_2 \): \( 1, a, a^2, \ldots, a^{m_2+m_3+1} \) where \( a^{m_2+m_3+1} = 1 \). Let it be possible to divide the non-unit elements of \( G_2 \) into two subsets \( G_2 - 1 = A \cup B; \ A \equiv A^{-1} \), where the \( m_2 \) elements of \( A \) are such that among the \( m_2(m_2+1) \) ratios arising out of its elements, the elements of \( A \) appear \( \alpha \) times and the elements of \( B \) appear \( \beta \) times.

Obviously, \( m_2\alpha + m_3\beta = m_2(m_2-1) \),

Consider the group of elements \( G \equiv G_1(\overline{X})G_2 \). With each element of \( G \) let us associate a treatment. Let it be possible to find a set of \( t \) blocks \( B_1, B_2, \ldots, B_t \) satisfying the following conditions:

i) Every block contains \( k \) treatments (the treatments contained in the same block being different from one another).

ii) Among the \( tk(k-1) \) ratios arising out of these \( t \) blocks, the non-unit elements of \( G_1 \) appear \( \lambda_1 \) times, the elements of \( G_1(\overline{X})A \) appear \( \lambda_2 \) times and the elements of \( G_1(\overline{X})B \) appear \( \lambda_3 \) times.

Then on developing these blocks (i.e. multiplying successively by \( 1, c, c^2, \ldots, c^{m_1-1}, a, ac, \ldots, a^{m_1-1}, a^{m_2+m_3} \)) we get the solution of the first type of cyclical design

\[ v = m_1(m_2+m_3+1), \quad b = vt, \quad r = kt, \quad k, \lambda_1, \lambda_2, \lambda_3 \]

\[ n_1 = m_1-1, \quad n_2 = m_1m_2, \quad n_3 = m_1m_3 \]
Example 1. (Subclass A - first type).

Consider the group $G$ formed by the different powers and their products of $a$ and $c$ where $a^3 = c^2 = 1$. Let $G_1 = \langle 1, c \rangle$; $A = \langle a, c, a^5, ac \rangle$, $B = \langle c^2, a^4, a^6 \rangle$. Among the ratios arising out of $G_1$, the element $c$ occurs twice. Among the ratios arising out of $A$, the elements of $B$ occur 4 times. So $\alpha = 0$, $\beta = 4$.

Consider the set of two initial blocks $(1, a, ac)$ and $(1, a^5, a^5c)$. We can easily verify that these two blocks satisfy the conditions of theorem 5.3.1 when $\lambda_1 = 4$, $\lambda_2 = 1$ and $\lambda_3 = 0$.

Hence, on developing these two blocks we get the solution of the design $v = 16$, $b = 32$, $r = 6$, $k = 3$, $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = 0$

$n_1 = 1$, $n_2 = 3$, $n_3 = 6$, $m_1 = 2$, $m_2 = 4$, $m_3 = 3$
Example 2. (Subclass 'A-First type): Consider the abelian group $G$ formed by $a, b$ when $a^2=b^5=l$. Let $G_1: 1, a, b, b^4$;

$B : b^2, b^3$; $\ldots, m_1=m_2=m_3=2$; $a=0, \beta=1, n_1=1, n_2=n_3=4$

$\begin{bmatrix}
0 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{bmatrix}$; $\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 2 \\
0 & 0 & 4
\end{bmatrix}$; $\begin{bmatrix}
0 & 0 & 1 \\
0 & 2 & 2 \\
1 & 2 & 0
\end{bmatrix}$
Consider the initial blocks \((1, b^4, ab^4), (1, a, ab)\) and \((1, a, b^2)\). On developing the blocks we get the solution of the design

\[ v=10, \quad b=30, \quad r=9, \quad k=3, \quad \lambda_1=6, \quad \lambda_2=2, \quad \lambda_3=1. \]

**Example 3.** (Subclass A - First type). Consider the group \(G\) as in example 2. Consider the initial block \((1, a, b, ab)\). On developing this block we get the solution of the design.

\[ v=b=10; \quad r=k=4, \quad \lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = 0. \]

Other parameters same as in example.

**Example 4.** (Subclass A - First type). Consider the same group \(G\) as in example 2. Consider the initial block \((1, a, b^4)\). On developing this block we get the solution of the design

\[ v=b=10; \quad r=k=3, \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0. \]

Other parameters same as in example 2.

**Example 5.** (Subclass B - First type): Consider the group formed by the elements \(a, c\) and their different powers when \(a^{13}=c^2=1\).

Let \(G_1: 1, c; A: a^2, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{12}\)

\[ B: a, a^3, a^4, a^9, a^{10}, a^{12}. \]

We can easily verify that \(a=2, \quad c=3\).

Consider the set of three initial blocks \((1, a^2, a^2c), (1, a^5, a^5c),\) and \((1, a^6, a^6c)\). On developing the initial blocks we get the solution of the design \(v=26, \quad b=78, \quad r=9, \quad k=3, \quad \lambda_1=6, \quad \lambda_2=1, \quad \lambda_3=0, \quad m_1=2, \quad m_2=m_3=6, \quad n_1=1, \quad n_2=n_3=12.\)
Example 6. (Subclass B-First type)

Consider the group \( G \) formed by \( a \) and \( c \) when \( a^6 = c^2 = 1 \).
Let \( G_1 : 1, c \); \( A : a, a^2, a^4, a^5 \); \( B : a^3 \)

So \( \alpha = 2, \beta = 4 \).

Consider the two initial blocks \( (1, a, a^2) \) and \( (1, a^2, a^4) \). On developing the initial blocks we get the solution of the design \( v=12, b=24, r=6, k=3, \lambda_1=4, \lambda_2=1, \lambda_3=0 \),

\( m_1=2, m_2=4, m_3=1, n_1=1, n_2=8, n_3=2 \)

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 4 & 2 \\
0 & 2 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 8 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]
The lay-out is

1. (1, a, ac) 9. (a^2c, a^3c, a^3) 17. (a^4, 1, c)
2. (a, a^2, a^2c) 10. (a^3c, a^4c, a^4) 18. (a^5, a, ac)
3. (a^2, a^3, a^3c) 11. (a^4c, a^5c, a^5) 19. (c, a^2c, a^2)
4. (a^3, a^4, a^4c) 12. (a^5c, c, 1) 20. (ac, a^3c, a^3)
5. (a^4, a^5, a^5c) 13. (1, a^2, a^2c) 21. (a^2c, a^4c, a^4)
6. (a^5, 1, c) 14. (a, a^3, a^3c) 22. (a^3c, a^5c, a^5)
7. (c, ac, a) 15. (a^2, a^4, a^4c) 23. (a^4c, c, 1)
8. (ac, a^2c, a^2) 16. (a^3, a^5, a^5c) 24. (a^5c, ac, a)

Example 7. (Subclass C-First type): Consider the group G formed by a, b and c where \( a^3 = b^3 = c^2 = 1 \). Let \( G_1 = \langle 1, a^2b, ab^2 \rangle \);

\[ A = \langle a^2b^2, ab \rangle, \quad B = \langle c, abc, a^2b^2c \rangle. \]

Among the non-unit ratios arising out of the elements of A, the elements of A appear once. But the elements of B do not appear at all. Hence \( \alpha = 1, \beta = 0 \).

Consider the initial block \( (1, a, b, a^2b^2) \). It may be verified that this initial block gives the following ratios: \( a^2b \) and \( ab^2 \) each repeated thrice and \( a^2b^2, ab, a, b, a^2 \) and \( b^2 \) each repeated once. Hence on developing this initial block we get the solution of the design

\( v = b = 18, \quad r = k = 4, \quad \lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \quad m_1 = 3, \quad m_2 = 2, \quad m_3 = 3, \quad n_1 = 2, \quad n_2 = 6, \quad n_3 = 9 \)
(p_{jk}^1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad ; \quad (p_{jk}^2) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\
(p_{jk}^3) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 6 \\ 2 & 6 & 0 \end{bmatrix}

The lay-out is

1. \((1, a, b, a^2b^2)\) 7. \((a^2b^2, b^2, a^2, ab)\) 13. \((bc, abc, b^2c, a^2c)\)
2. \((a, a^2, ab, b^2)\) 8. \((ab^2, a^2b^2, a, b)\) 14. \((b^2c, ab^2c, c, a^2bc)\)
3. \((a^2, 1, a^2b, ab^2)\) 9. \((a^2b, b, a^2b^2, a)\) 15. \((abc, a^2be, ab^2c, c)\)
4. \((b, ab, b^2, a^2)\) 10. \((c, ac, bc, a^2b^2c)\) 16. \((a^2b^2c, b^2c, a^2c, abc)\)
5. \((b^2, ab^2, 1, a^2b)\) 11. \((ac, a^2c, abc, b^2c)\) 17. \((ab^2c, a^2b^2c, ac, abc)\)
6. \((ab, a^2b, ab^2, 1)\) 12. \((a^2c, c, a^2be, ab^2c)\) 18. \((a^2bc, bc, c, a^2b^2c, ac)\)

Example 8: (Subclass C-First type): Consider \(G, G_1, A\) and \(B\) same as in example 7. Consider the initial block \((1, a^2b, ab^2, c, a^2c, bc)\). This gives the solution of the design

\[v = b = 18, \quad r = k = 6, \quad \lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = 2\]

\[n_1 = 2, \quad n_2 = 6, \quad n_3 = 9, \quad m_1 = 3, \quad m_2 = 2, \quad m_3 = 3\]

\begin{align*}
(p_{jk}^1) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\
(p_{jk}^2) &= \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\
(p_{jk}^3) &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 6 \\ 2 & 6 & 0 \end{bmatrix}
\end{align*}
Example 9. (Subclass 0 - First type): Consider the same group $G$ as in example 7. Let $G = \langle 1, a \rangle$, $A = \langle a, a^2 \rangle$ and $B = \langle b, b^2, ab, ab^2, a^2b, a^2b^2 \rangle$. So, $\alpha = 1$, $\beta = 0$.

Consider the initial block $\langle 1, a, ac \rangle$. On developing this block we get the solution of the design:

$v=b=18$, $r=k=3$, $\lambda_1=2$, $\lambda_2=1$, $\lambda_3=0$, $m_1=m_2=2$, $m_3=6$, $n_1=1$

$n_2 = 4$, $n_3 = 12$.

$$ (p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix} ; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 12 \end{bmatrix} $$

$$ (p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 4 \\ 1 & 4 & 3 \end{bmatrix} $$

The layout is:

1. $\langle 1, a, ac \rangle$
2. $\langle a, a^2, a^2c \rangle$
3. $\langle a^2, 1, c \rangle$
4. $\langle b, ab, abc \rangle$
5. $\langle ab, a^2b, a^2bc \rangle$
6. $\langle a^2b, b, bc \rangle$
7. $\langle b^2, ab^2, ab^2c \rangle$
8. $\langle ab^2, a^2b^2, a^2b^2c \rangle$
9. $\langle a^2b^2, b^2, b^2c \rangle$
10. $\langle ac, a^2c, a^2 \rangle$
11. $\langle ac, a^2c, a^2 \rangle$
12. $\langle a^2c, c, 1 \rangle$
13. $\langle bc, abc, ab \rangle$
14. $\langle abc, a^2bc, a^2b \rangle$
15. $\langle a^2bc, bc, b \rangle$
16. $\langle b^2c, ab^2c, ab^2 \rangle$
17. $\langle ab^2c, a^2b^2c, a^2b^2 \rangle$
18. $\langle a^2b^2c, b^2c, b^2 \rangle$
Example 10. (Subclass 0 - First type): Consider the group $G$ formed by $a^3 = b^5 = c^2 = 1$. Let $G_1 = (l, c); A = \{a, a^2\}; B = \{b, b^2, b^3, b^4, ab, ab^2, ab^3, ab^4, a^2b, a^2b^2, a^2b^3, a^2b^4\}$.

We can easily verify that $a = 1, \beta = 0$. Consider the initial block $(1, a, ac)$. On developing this block we get the solution of the design

$$v=30=b, \quad r=k=5, \quad \lambda_1=2, \quad \lambda_2=1, \quad \lambda_3=0, \quad m_1=m_2=2, \quad m_3=12, \quad n_1=1, \quad n_2=4, \quad n_3=24.$$

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 24 \end{bmatrix} \quad \text{and} \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 24 \end{bmatrix} \quad \text{and} \quad (p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 4 \\ 1 & 4 & 18 \end{bmatrix}
\]

The lay-out is

1. $(1, a, ac)$, 2. $(a, a^2, a^2c)$, 3. $(a, 1, c)$, 4. $(b, ab, abc)$, 5. $(b^2, ab^2, ab^2c)$, 6. $(b^3, ab^3, ab^3c)$, 7. $(b^4, ab^4, ab^4c)$, 8. $(a, a^2b, a^2bc)$, 9. $(ab^2, a^2b^2, a^2b^2c)$, 10. $(ab^3, a^2b^3, a^2b^3c)$, 11. $(ab^4, a^2b^4, a^2b^4c)$, 12. $(a^2b, b, bc)$, 13. $(a^2b^2, b^2, b^2c)$, 14. $(a^2b^3, b^3, b^3c)$, 15. $(a^2b^4, b^4, b^4c)$, 16. $(c, ac, a)$, 17. $(ac, a^2c, a^2)$, 18. $(a^2c, c, 1)$, 19. $(bc, abc, ab)$,
20. \(b^2c, ab^2c, ab^2\), 21. \(b^3c, ab^3c, ab^3\), 22. \(b^4c, ab^4c, ab^4\)
23. \(abc, a^2bc, a^2b\), 24. \(ab^2c, a^2b^2c, a^2b^2\)
25. \(ab^3c, a^2b^3c, a^2b^3\), 26. \(ab^4c, a^2b^4c, a^2b^4\)
27. \(a^2bc, bco, b\), 28. \(a^2b^2c, b^2c, b^2\), 29. \(a^2b^3c, b^3c, b^3\)
30. \(a^2b^4c, b^4c, b^4\).

**Theorem 5.3.1'.** (Second type of generalisation).

Let us change (ii) in theorem 5.3.1 as follows.

(ii') Among the \(tk(k-1)\) ratios arising from these \(t\) blocks, the elements of \(A\) appear \(\lambda_1\) times, the elements of \(B\) appear \(\lambda_2\) times and the elements of \((a_l-1)(\overline{A})a_2\) appear \(\lambda_3\) times.

Then on developing these \(t\) blocks (i.e., multiplying successively by elements of \(A\)) we get the solution of the second type of cyclical design

\[v = m_1(m_2 + m_3 + 1), \quad b = vt, \quad r = kt, \quad k, \quad \lambda_1, \lambda_2, \lambda_3, \quad n_1 = m_2,\]
\[n_2 = m_3, \quad n_3 = (m_1 - 1)(m_2 + m_3 + 1)\]

\[
(p^1_{jk}) = \begin{bmatrix}
\alpha & m_2 - \alpha - 1 & 0 \\
m_2 - \alpha - 1 & m_3 - m_2 + \alpha + 1 & 0 \\
0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}
\]

\[
(p^2_{jk}) = \begin{bmatrix}
\beta & m_2 - \beta & 0 \\
m_2 - \beta & m_3 - m_2 + \beta + 1 & 0 \\
0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}
\]
Example 1 (Subclass A - Second type):

Consider the group $G$ formed by $a, b$ with their different powers and product of powers when $a^4 = b^2 = 1$.

Let $G_1 : l, b; A : a, a^3; B : a^2$. We easily find that $a = 0, \beta = 2$.

Consider the 4 initial blocks $(1, a, a^3), (1, b, a^2b), (1, ab, a^3b)$ and $(1, b, a)$. On developing these initial blocks, we get the solution of the design

$$v = 8, \ b = 32, \ r = 12, \ k = 3, \ \lambda_1 = 3, \ \lambda_2 = 2, \ \lambda_3 = 4$$

$$m_1 = m_2 = 2, \ m_3 = 1, \ n_1 = 2, \ n_2 = 1, \ n_3 = 4.$$

$$(p^3_{jk}) = \begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m_1-2)(m_2+m_3+1) \end{bmatrix}$$

$$(p^1_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \quad (p^2_{jk}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

whose layout is
1. (1, a, a^3), 2. (a, a^2, 1), 3. (a^2, a^3, a), 4. (a^3, 1, a^2)
5. (b, ab, a^3b), 6. (ab, a^2b, b), 7. (a^2b, a^3b, ab),
8. (a^3b, b, a^2b), 9. (1, b, a^2b), 10. (a, ab, a^3b),
11. (a^2, a^2b, b), 12. (a^3, a^3b, ab), 13. (b, 1, a^2),
14. (ab, a, a^3), 15. (a^2b, a^2, 1), 16. (a^3b, a^3, a),
17. (1, ab, a^3b), 18. (a, a^2b, b), 19. (a^2, a^3b, ab),
20. (a^3, b, a^2b), 21. (b, a, a^3), 22. (ab, a^2, 1),
23. (a^2b, a^3, a), 24. (a^3b, 1, a^2), 25. (1, b, a), 26. (a, ab, a^2)
27. (a^2, a^2b, a^3), 28. (a^3, a^3b, 1), 29. (b, 1, ab),
30. (ab, a, a^2b), 31. (a^2b, a^2, a^3b), 32. (a^3b, a^3, b).

Example 2. (Subclass A - second type): Consider G, G, A and B as in example 1. Consider the initial blocks (1, b, a^2b),
(1, ab, a^3b) and (1, a, b). On developing these blocks we get
the solution of the design v = 8, b = 24, r = 9, k = 3,
\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 4. Other parameters same as in example 1.

Example 3: (Subclass B - Second type): Consider the group G formed by a^3 = b^3 = c^2 = 1. Let G, 1, c; A : a, a^2, b, b^2;
B : ab, a^2b, ab^2, a^2b^2.
Here: \alpha = 1, \beta = 2.
Consider the initial block (1, a, a^2, b, b^2, c, c). We can easily verify that among the ratios arising out of this block,
the elements of A appear thrice while the elements of
\((a_1 - 1) (X) (A U B U 1)\) appear twice. Hence on developing the blocks we get the solution of the design \(v = b = 18, \quad r = k = 6, \quad \lambda_1 = 3, \quad \lambda_2 = 0, \quad \lambda_3 = 2, \quad m_2 = m_3 = 4, \quad m_1 = 2, \quad n_1 = n_2 = 4, \quad n_3 = 9\)

\[
(p^1_{jk}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}; \quad (p^2_{jk}) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}
\]

\[
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 4 \\ 6 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}
\]

The lay-out is

1. \((1, a, a^2, b, b^2, c)\)
2. \((a, a^2, 1, ab, ab^2, a)\)
3. \((a^2, 1, a, a^2b, a^2b^2, c)\)
4. \((b, ab, a^2b, b^2c, c, bc)\)
5. \((b^2, ab^2, a^2b^2, c, bc, b^2c)\)
6. \((ab, a^2b^2, b, ab^2, ac, abc)\)
7. \((ab^2, a^2b^2, b^2, ac, abc, ab^2c)\)
8. \((a^2b, b, ab, a^2b^2, a^2c, ab^2c)\)
9. \((a^2b^2, b^2, ab^2, a^2c, a^2bc, a^2b^2c)\)
10. \((c, ac, a^2c, b, b^2, 1)\)
11. \((ac, a^2c, c, ab, ab^2, a)\)
12. \((a^2c, c, ac, a^2b, a^2b^2, a^2)\)
13. \((bc, abc, a^2bc, b^2, l, b)\)
14. \((b^2c, ab^2c, a^2b^2, l, b, b^2)\)
15. \((abc, a^2bc, bc, ab^2, c, ab)\)
16. \((ab^2c, b^2c, abc, a^2bc, b^2, a, ab)\)
Example 4. (Subclass B - Second type).

Consider the group $G$ formed by $a$ and $c$ when $a^6 = c^2 = 1$.

Let $G_1: 1, c; A: a, a^2, a^4, a^5; B: a^3$

So, $a = 2, \beta = 4$.

Consider the initial block $(1, a^3, c, a^c, a^c)$. We easily verify that this initial block satisfied all the conditions of theorem 5.3.1' having $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 2$. Hence on developing this initial block we get the solution of the design $v = b = 12, r = k = 5, \lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 2; m_1 = 2, m_2 = 4, m_3 = 1; n_1 = 4, n_2 = 1, n_3 = 6$.

\[
(p_{jk}^1) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}
\]

Example 5. (Subclass C - Second type):

Consider the group $G$ formed by $a$, $c$ and their different powers when $a^6 = c^2 = 1$.

Let $G_1: 1, c; A: a^2, a^4; B: a, a^3, a^5$. Hence $a = 1, \beta = 0$.

So the twelve treatments correspond to $1, a, a^2, a^3, a^4, a^5, c, ac, a^2c, a^3c, a^4c, a^5c$. 

\[
(a^6 = c^2 = 1)
\]
Consider the initial blocks \((1, a^2, a^4, c)\) and \((l, a^2, a^4, ac)\). On developing the initial blocks we get the solution of the design

\[ v = 12, \ b = 24, \ r = 8, \ k = 4, \lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 1 \]
\[ m_1 = 2, \ m_2 = 2, \ m_3 = 3, \ n_1 = 2, \ n_2 = 3, \ n_3 = 6 \]

\[
(p^1_{jk}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad (p^2_{jk}) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} 
\]

\[
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}
\]

The lay-out is

1. \((l, a^2, a^4, c)\), 2. \((a, a^3, a^5, ac)\), 3. \((a^2, a^4, l, a^2c)\)
4. \((a^3, a^5, a, a^3c)\), 5. \((a^4, l, a^2, a^4c)\), 6. \((a^5, a, a^3, a^5c)\)
7. \((c, a^2c, a^4c, l)\), 8. \((ac, a^3c, a^5c, a)\), 9. \((a^2c, a^4c, c, a^2)\)
10. \((a^3c, a^5c, ac, a^3)\), 11. \((a^4c, c, a^2c, a^4)\), 12. \((a^5c, ac, a^3c, a^5)\)
13. \((l, a^2, a^4, ac)\), 14. \((a, a^3, a^5, a^2c)\), 15. \((a^2, a^4, l, a^3c)\)
16. \((a^3, a^5, a, a^4c)\), 17. \((a^4, l, a^2, a^5c)\), 18. \((a^5, a, a^3, c)\),
19. \((c, a^2c, a^4c, a)\), 20. \((ac, a^3c, a^5c, a^2)\), 21. \((a^2c, a^4c, c, a^3)\)
22. \((a^3c, a^5c, ac, a^4)\), 23. \((a^4c, c, a^2c, a^5)\),
24. \((a^5c, ac, a^3c, l)\).
Example 6. (Subclass C - second type)

Consider the group $G$ formed by $a$, $c$ and their different powers when $a^9 = c^2 = 1$. Let $G_1 = 1, c; A = a^3, a^6$, $B = G_2 - A - 1$ where $G = G_1(\bar{X})G_2$, Hence $\alpha = 1, \beta = 0$.

Consider the set of three initial blocks $(1, a, a^2, ac)$, $(1, a^2, a^5, a^3c)$ and $(1, a^2, a^3, a^5c)$. On developing these blocks we shall get the solution of the design

$v = 18, b = 54, r = 12, k = 4, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 2$

$m_1 = m_2 = 2, m_3 = 6, n_1 = 2, n_2 = 6, n_3 = 9.$

\[
(p^1_{\bar{jk}}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 9
\end{bmatrix} ; (p^2_{\bar{jk}}) = \begin{bmatrix}
0 & 2 & 0 \\
2 & 3 & 0 \\
0 & 0 & 9
\end{bmatrix} ; (p^3_{\bar{jk}}) = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 6 \\
2 & 6 & 0
\end{bmatrix}
\]

Example 7. (Subclass C - Second type)

Consider the group $G$ formed by $a$, $c$ and their different powers when $a^{15} = c^2 = 1$.

Let $G_1 = 1, c; A = a^5, a^{10}; B = G_2 - A - 1$, when $G = G_1(\bar{X})G_2$

Hence $\alpha = 1, \beta = 0.$
Consider the set of five initial blocks (1, ac, a_6c, a^{11}c),
(l, a^2c, a^7c, a^{12}c), (l, a^3c, a^8c, a^{13}c), (l, a^4c, a^9c, a^{14}c)
and (l, a^5, a^{10}, c), On developing these blocks we get the solution
of the design

\[ v = 30, \ b = 150, \ r = 20, \ k = 4, \ \lambda_1 = 15, \ \lambda_2 = 0, \ \lambda_3 = 2 \]
\[ m_1 = m_2 = 2, \ m_3 = 12, \ n_1 = 2, \ n_2 = 12, \ n_3 = 15 \]

\[
(p^{1}_{jk}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 12 & 0 \\
0 & 0 & 15 \\
\end{bmatrix}; \quad (p^{2}_{jk}) = \begin{bmatrix}
0 & 2 & 0 \\
2 & 9 & 0 \\
0 & 0 & 15 \\
\end{bmatrix}
\]

\[
(p^{3}_{jk}) = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 12 \\
2 & 12 & 0 \\
\end{bmatrix}
\]

**Theorem 5.3.2.** (First type of generalisation)

Let \( G \) be an abelian group formed by different powers of \( a \),
when \( a^{m_2 + m_3 + 1} = 1 \).

Let it be possible to divide the non-unit elements of \( G \) into
two disjoint sets \( A \) and \( B \) such that among the ratios arising
out of the element of \( A \), the elements of \( A \) appear \( \alpha \) times and
the elements of \( B \) appear \( \beta \) times. To any element \( a^i \) let there
correspond \( m_1 \) treatments \( a^i_1, a^i_2, \ldots, a^i_{m_1} \). Treatments denoted
by symbols with the same lower suffix \( j \) may be said to belong
to the \( j \)-th class.
Let it be possible to select a set of $t$ blocks $B_1, B_2, \ldots, B_t$ such that:

i) Every block contains exactly $k$ treatments (the treatments contained in the same block being different from one another).

ii) Among the $kt$ treatments occuring in the $t$ blocks, exactly $r$ treatments should belong to each of the $m_1$ classes. Obviously, $m_1 r = kt$.

iii) Among the $tk (k-1)$ ratios arising from these $t$ blocks, the ratios of the type $l_{i,j}$ will appear $\lambda_1$ times, for $l_{i,j} = 1, 2, \ldots, m_1$; the ratios of the type $a_{i,j}$, where $a \in A$, will appear $\lambda_2$ times, for $i j = 1, 2, \ldots, m_1$ and the ratios of the type $b_{i,j}$, where $b \in B$, will appear $\lambda_3$ times for $i j = 1, 2, \ldots, m_1$.

So, $k(k-1)t = m_1 (m_1-1) \lambda_1 + m_1^2 \lambda_2 + m_1^2 m_3 \lambda_3$. Then on developing these set of $t$ blocks (i.e., multiplying by elements of $G$) we get the solution of the design

$$v = m_1 (m_2 + m_3 + 1), \quad b = (m_2 + m_3 + 1) t, \quad r, \quad k,$$

$$\lambda_1, \quad \lambda_2, \quad \lambda_3, \quad n_1 = m_1 - 1, \quad m_2 = m_1 m_2, \quad n_3 = m_1 m_3$$

$p_{ijk}$ are the same as in theorem 1.

**Example 1.** (Subclass A – First type). Consider the group formed by $1, a, a^2, a^3, a^4, a^5$. Let $A : a, a^3, a^5, B : a^2, a^4$. So $\alpha = 0, \beta = 3$. With each element of $u$ associate two treatments $u_1, u_2$. Consider the set of six initial blocks $(l_1, a_1, l_2)$,
(l_1, a_1^3, l_2), (l_1, a_1^5, l_2), (l_1, l_2, a_2), (l_1, l_2, a_2^3), (l_1, l_2, a_2^5).

On developing these blocks we get the solution of the design

\( v = 12, \ b = 36, \ r = 9, \ k = 3, \ \lambda_1 = 6, \ \lambda_2 = 2, \ \lambda_3 = 0, \ m_1 = 2 \)

\( m_2 = 3, \ m_3 = 2, \ n_1 = 1, \ n_2 = 6, \ n_3 = 4, \ \alpha = 0, \ \beta = 3. \)

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{pmatrix}
\quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 4 \\
0 & 4 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 6 & 0 \\
1 & 0 & 2
\end{pmatrix}
\]

Theorem 5.3.2'. (Second type of generalisation). Let us change (ii) in theorem 5.3.2' as follows:

(ii') Among the ratios of the type \([i, i], \ i = 1, 2, \ldots, m_1\), arising out of \( t \) blocks, the elements of \( A \) appear \( \lambda_1 \) times and the elements of \( B \) appear \( \lambda_2 \) times; among the ratios of the type \([i, j], \ i, j = 1, 2, \ldots, m_1 \), the elements of \( G \) appear \( \lambda_3 \) times.

i \neq j

On developing the blocks, [i.e. multiplying successively by elements of the group \( G \)], we get the solution of the design

\( v = m_1(m_2+m_3+1), \ b = (m_2+m_3+1)t, \ r, k, \lambda_1, \lambda_2, \lambda_3 \)

\( n_1 = m_2, \ n_2 = m_3, \ n_3 = (m_1-1)(m_2+m_3+1), \ \mathbf{p}_{jk} \) same as in theorem 1'.
Example 1. (Subclass A - Second type)

Consider for example the same group \( G \) and the same sets \( A \) and \( B \) as in example 1 of theorem 5.3.2. Consider the following 2 initial blocks

\[(l_1, a^2_1, a^4_1, l_2) \text{ and } (l_2, a^2_2, a^4_2, a_1).\]

It is easy to verify that among the pure differences of the type \((1, 1)\) [or \((2, 2)\)] the elements of \( B \) appear thrice and elements of \( A \) do not appear at all, but among the mixed differences of the type \((1, 2)\) [or \((2, 1)\)] the elements of \( G \) appear once. So, on developing the blocks [i.e. multiplying successively by \( 1, a, a^2, a^3, a^4, a^5 \)] we get the solution of the design \( v = b = 12, r = k = 4, \lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 1, m_1 = 2, m_2 = 3, m_3 = 2, n_1 = 3, n_2 = 2, n_3 = 6.\)

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad (p^2_{jk}) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad (p^3_{jk}) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}
\]

Theorem 5.3.3 (First type of generalisation)

Consider a group \( G \) consisting of \( n(m_2 + m_3 + 1) \) elements, \( a, c \) and their different powers when \( a^{m_2 + m_3 + 1} = c^2 = 1. \) With each
element of $G$, let us associate a set of $m_1/n$ treatments. (It is assumed that $n$ is a divisor of $m_1$). Let it be possible to decompose the group $G$ into direct factors $G = G_1(X)G_2$. Let further $G_2 = AU BU_1$, where $A = A^{-1}$, and among the ratios arising out of $A$, the elements of $A$ appear $\alpha$ times and $B$ appear $\beta$ times.

Let it be possible to select a set of $t$ blocks $B_1, B_2, \ldots, B_t$ such that

1. Each block contains $k$ treatments
2. Each suffix is represented $r$ times
3. Among the pure ratios of the type $[i, i], i = 1, 2, \ldots, m_1/n$ the non-unit elements of $G_1$ appear $\lambda_1$ times, and among the mixed ratios of the type $(i, j)$ the elements of $G_1$ appear $\lambda_1$ times, while among both the pure and the mixed ratios the elements of $G_1(X)A$ appear $\lambda_2$ times and $G_1(X)B$ appear $\lambda_3$ times.

Then on developing the blocks, (i.e. multiplying successively by the elements of $G$) we get the solution of the design

$$v = m_1(m_2+m_3+1), \quad b = (m_2+m_3+1)nt, \quad r = knt/m_1,$$

$$k, \ m_1, \ m_2, \ m_3, \ \lambda_1, \ \lambda_2, \ \lambda_3$$

$$(p_{jk}^1) = \begin{bmatrix}
m_1-2 & 0 & 0 \\
0 & m_1m_2 & 0 \\
0 & 0 & m_1m_3
\end{bmatrix};
(p_{jk}^2) = \begin{bmatrix}
0 & m_1-1 & 0 \\
m_1-1 & m_1 & m_1(m_2-\alpha-1) \\
0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\beta-1)
\end{bmatrix};
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & m_1-1 \\
0 & m_1m_2 & m_1(m_2-\beta) \\
m_1-1 & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1)
\end{bmatrix}$$
Example 1. (Subclass A - First type)

Consider for example, the group $G$ formed by $a$ and $c$ when
\[ a^4 = c^2 = 1. \]


So $\alpha = 0$, $\beta = 2$.

Consider the two initial blocks $(l_1, c_1, l_2, c_2, (a^2)_1)$ and $(l_1, c_1, l_2, c_2, (a^2)_2)$. The ratios arising out of these blocks are found to satisfy the condition of the theorem 5.3.3 and on developing these blocks we get the solution of the design $v = b = 16$,
\[ r = k = 5, \quad m_1 = 4, \quad m_2 = 2, \quad m_3 = 1, \quad \lambda_1 = 4, \quad \lambda_2 = 0, \quad \lambda_3 = 2. \]

Theorem 5.3.3\textsuperscript{1}. (Second type of generalisation). Let us change (3) in theorem 5.3.3 as follows: (3') Among the pure ratios of the type $[i, i], i = 1, 2, \ldots, \frac{m_1}{n}$ arising from the $t$ blocks the elements of $A$ appear $\lambda_1$ times, the elements of $B$ appear $\lambda_2$ times and the elements of $(G_1-l)(\bar{X})(AU BU l)$ appear $\lambda_3$ times; while among the ratios of the type $(i, j)$ the elements of $G$ appear $\lambda_3$ times, $i, j = 1, 2, \ldots, \frac{m_1}{n}$. Then on developing the
set of \( t \) blocks we get the solution of the design

\[ v = m_1(m_2 + m_3 + 1), \quad b = (m_2 + m_3 + 1) t, \quad r, \quad k, \]

\( \lambda_1, \lambda_2, \lambda_3, m_1, m_2, m_3. \)

\[
\begin{bmatrix}
\alpha & m_2 - \alpha - 1 & 0 \\
m_2 - \alpha - 1 & m_3 - m_2 + \alpha + 1 & 0 \\
0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta & m_2 - \beta & 0 \\
m_2 - \beta & m_3 - m_2 + \beta + 1 & 0 \\
0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & m_2 \\
0 & 0 & m_3 \\
m_2 & m_3 & (m_1 - 2)(m_2 + m_3 + 1)
\end{bmatrix}
\]

**Example 1.** (Subclass A– second type). Consider the group \( G \) formed by \( a^4 = c^2 = 1. \)

Let \( A : a, a^3, B : a^2 \) So, \( \alpha = 0, \beta = 2. \)

Consider the set of 12 initial blocks:

\( (l_1, (a^2)_1, c_1), (l_1, (a^3)_1, c_1), (l_2, (a^3)_2, c_2), (a_2, (a^3)_2, c_2) \)
\( (l_1, (a^2)_1, c_2), (l_1, (a^3)_1, c_2), (a_1, (a^3)_1, l_2), (l_1, (a^2)_1, l_2) \)
\( (l_1, l_2, (a^2)_2), (l_2, (a^2)_2, c_1), ((a^3)_2, (a^3)_2 c_1), ((a^2)_2, (a^3)_2 l_2) \)
On developing these blocks we get the solution of the design

\[ v = 16, \ b = 96, \ r = 18, \ k = 3, \ \lambda_1 = 0, \ \lambda_2 = 12, \ \lambda_3 = 2, \]
\[ m_1 = 4, \ m_2 = 2, \ m_3 = 1. \]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 12
\end{bmatrix}
\]

\[
(p_{jk}^1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix} ;
(p_{jk}^2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 8 \end{bmatrix}
\]

Remark:

By obvious extension, theorem 5.3.1 to 5.3.3' can be utilised for getting rectangular type cyclic designs. Thus, let us suppose, the group \( G_2 \) be such that \( G_2 \) can be expressed as the direct product of two groups

\[ G_2 = (1 + A_1)(X)(1 + A_2). \]

In theorem 5.3.1, in selecting the \( t \) blocks, one should ensure that the non-unit elements of \( G_1 \) appear \( \lambda_1 \) times, the elements of \( G_1(X)A_1 \) appear \( \lambda_2 \) times, the elements of \( G_1(X)A_2 \) appear \( \lambda_3 \) times and the elements of \( G_1(X)A_1(X)A_2 \) appear \( \lambda_4 \) times.

Similar modification will be needed for theorems 5.3.2 and 5.3.3.
In theorem 5.3.1', we require that the elements of \( A_1 \) appear \( \lambda_1 \) times, the elements of \( A_2 \) appear \( \lambda_2 \) times, the elements of \( A_1(\overline{X})A_2 \) appear \( \lambda_3 \) times, and the elements of \((G_1 - 1)(\overline{X})G_2 \) appear \( \lambda_4 \) times. Similar modification will be needed for theorem 5.3.2' and 5.3.3'.

**Example 1.** (Rectangular (0.G.D.) - First type).

Consider the group \( G \) formed by \( a^3 = b^4 = c^2 = 1 \). The association scheme is

\[
\begin{array}{ccc}
(1, c) & (a, ac) & (a^2, a^2c) \\
(b, bc) & (ab, abc) & (a^2b, a^2bc) \\
(b^2, b^2c) & (ab^2, ab^2c) & (a^2b^2, a^2b^2c) \\
(b^3, b^3c) & (ab^3, ab^3c) & (a^2b^3, a^2b^3c)
\end{array}
\]

So, \( A_1 = a, a^2 \); \( A_2 = b, b^2, b^3 \)

Consider the two initial blocks \((1, a, c, ac)\) and \((1, b, b^3, b^2c)\).

On developing the blocks we get the solution of the design:

\( v = 24, b = 48, r = 8, k = 4, \lambda_1 = 4, \lambda_2 = \lambda_3 = 2, \lambda_4 = 0 \)

\( m_1 = 2, m_2 = 2, m_3 = 3, m_4 = 6 \).

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 6 & 6
\end{bmatrix}
\]
Example 2. (Rectangular (O.G.D.) - First type).

Consider the group $G$ formed by $a^3 = b^4 = 1$. With each element of $G$ let us associate two treatments, $u_1$ and $u_2$. Let us consider the set of 8 initial blocks:

$$(l_1, a_1, a_2), (l_1, l_2, (b^2)_1), (l_1, l_2, b_1), (l_1, l_2, b_1^3)$$

$$(l_2, a_2, a_1), (l_1, l_2, (b^2)_2), (l_1, l_2, b_2)$$

On developing these blocks we shall get the solution of the design whose second type of parameters are

$$b = 96, \ r = 12, \ k = 3, \ \lambda_1 = 8, \ \lambda_2 = 1, \ \lambda_3 = 2, \ \lambda_4 = 0$$

while the first type of parameters are the same as in example 1 above.

Example 3. (O.G.D. - First type). Consider the group $G$ formed by $a$, $b$ and $c$ and their different powers when

$$a^3 = b^2 = c^2 = 1.$$ Let $G_1 : 1, c$. $A : a$, $a^2$, $B : b$. So, $v = 12$, $n_1 = 1$, $n_2 = 4$, $n_3 = 2$, $n_4 = 4$. 

$$
\begin{align*}
(p_{jk})^3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 4 & 0 \\ 0 & 4 & 0 & 3 \end{bmatrix}, \\
(p_{jk})^4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 4 \\ 1 & 2 & 4 & 4 \end{bmatrix}
\end{align*}
$$
Consider the set of 24 blocks

1. \((l, a, ac)\)  
2. \((a, a^2, a^2c)\)  
3. \((a^2, 1, c)\)  
4. \((b, ab, abc)\)  
5. \((ab, a^2b, a^2bc)\)  
6. \((a^2b, b, bc)\)  
7. \((c, ac, a)\)  
8. \((ac, a^2c, a^2)\)  
9. \((a^2c, c, l)\)  
10. \((bc, abc, ab)\)  
11. \((abc, a^2bc, a^2b)\)  
12. \((a^2bc, bc, b)\)  
13. \((l, ab, abc)\)  
14. \((a, ab, abc)\)  
15. \((a^2, a^2b, a^2bc)\)  
16. \((b, 1, c)\)  
17. \((ab, a, ac)\)  
18. \((a^2b, a^2, a^2c)\)  
19. \((a, bc, b)\)  
20. \((ac, abc, ab)\)  
21. \((a^2c, a^2bc, a^2b)\)  
22. \((bc, c, l)\)  
23. \((abc, ac, a)\)  
24. \((a^2bc, a^2c, a^2)\)

which gives the solution of the design

\[ b = 24, \; r = 6, \; k = 3, \; \lambda_1 = 4, \; \lambda_2 = 1, \; \lambda_3 = 2, \; \lambda_4 = 0. \]

**Example 3.** (Rectangular (O.G.B.)-Second type).

Consider the group \(G\) formed by \(a^3 = b^4 = c^2 = 1\). As before, let \(A_1 : a, a^2; \; A_2 : b, b^2, b^3\). Consider the three initial blocks

\((1, b, a^2c, a^2b^2c), (1, ab^2, abc, c)\) and \((1, ab^3c, bc, ac)\)
On developing these blocks, we get the solution of the design

\[ v = 24, \quad b = 72, \quad r = 12, \quad k = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 1, \quad \lambda_4 = 2, \]

\[ m_1 = 2, \quad m_2 = 2, \quad m_3 = 3, \quad m_4 = 6 \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 3 & 3 & 0 \\
0 & 0 & 0 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 \\
2 & 0 & 4 & 0 \\
0 & 0 & 0 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 2 & 0 \\
1 & 2 & 2 & 0 \\
0 & 0 & 0 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 6 \\
2 & 3 & 6 & 0
\end{bmatrix}
\]

Some series of cyclic design of the first type having \( k=3, m_1=2 \).

The method of symmetrically repeated ratios discussed above, can be gainfully utilized to derive some series of designs for a given value of \( k \), the block size. Some such series have been obtained for the first type of generalisation having \( k=3 \) and \( m_1=2 \) where, giving different values to \( m_2 \) and \( m_3 \), one may get various designs belonging to the same series. To economise space the \( p_{jk} \) parameters have been omitted but are evident from the context. It may be noted that these series are indicative in nature and does not attempt to exhaust all possible series that can be derived for \( k = 3 \) and \( m_1 = 2 \).
Let \( m_1 = 2 \). Consider the group \( G \) formed by \( 1, a, a^2, \ldots, a^{n-1} \), where \( a^n = 1 \). Let us denote the \( m_2 \) elements of \( A \) by \( d_1, d_2, \ldots, d_{m_2} \).

Consider the set of \( 2m_2 \) initial blocks:

- \( (l_1, (d_1)_1, l_2) \),
- \( (l_1, (d_1)_2, l_2) \),
- \( (l_1, (d_2)_1, l_2) \),
- \( (l_1, (d_2)_2, l_2) \),
- \( (l_1, (d_{m_2})_1, l_2) \),
- \( (l_1, (d_{m_2})_2, l_2) \).

On multiplying these blocks by the elements of the group \( G \), we get the solution of the design:

\[
\begin{align*}
\nu &= 2(m_2 + m_3 + 1), \\
b &= 2m_2(m_2 + m_3 + 1), \\
r &= 3m_2, \\
k &= 3, \\
\lambda_1 &= 2m_2, \\
\lambda_2 &= 3, \\
\lambda_3 &= 0.
\end{align*}
\]

**Series II.** Let \( m_1 = 2 \). Consider the group \( G \) formed by \( 1, a, \ldots, a^{n-1} \) where \( a^n = 1 \). Let us denote the \( m_3 \) elements of \( B \) by \( e_1, e_2, \ldots, e_{m_3} \).

Consider the set of \( 2m_3 \) initial blocks:

- \( (l_1, (e_1)_1, l_2) \),
- \( (l_1, (e_1)_2, l_2) \),
- \( (l_1, (e_2)_1, l_2) \),
- \( (l_1, (e_2)_2, l_2) \),
- \( (l_1, (e_{m_3})_1, l_2) \),
- \( (l_1, (e_{m_3})_2, l_2) \).

On multiplying these blocks by the elements of the group \( G \), we get the solution of the design:

\[
\begin{align*}
\nu &= 2(m_2 + m_3 + 1), \\
b &= 2m_3(m_2 + m_3 + 1), \\
r &= 3m_3, \\
k &= 3, \\
\lambda_1 &= 2m_3, \\
\lambda_2 &= 0, \\
\lambda_3 &= 2.
\end{align*}
\]

**Series III.** Let \( G \) be the group formed by \( a^{3p} = c^2 = 1 \). Let \( G = G_1(G_2)G_3 \), \( G_1: 1, c \); \( A : a^b, a^{2p} \); \( B : G_1 - A - 1 \). So, \( a = 1, \beta = 0, m_1 = m_2 = 2, m_3 = 3(p-1) \).

Consider the initial block \((1, a^p, a^{pc})\). On developing this block (i.e., multiplying by the elements of \( B \)) we get the solution of the design:

\[
\begin{align*}
\nu &= 6p = b, \\
r &= k = 3, \\
\lambda_1 &= 2, \\
\lambda_2 &= 3, \\
\lambda_3 &= 0, \\
m_1 &= m_2 = 2, \\
m_3 &= 3(p-1).
\end{align*}
\]
**Series IV.** Let $G$ be the group formed by $a^{(t+1)p} = c^2 = 1$.
Let $G = G_1 \oplus G_2$ where $G_1 = 1$, $c = a^p, a^{2p}, \ldots a^{tp}$;
$B : G_2 - A = 1$. Consider the set of $t$ blocks $(1, a^p, a^{2p})$,
$(1, a^{2p}, a^{3p}), \ldots, (1, a^{tp}, a^{tp+1})$. On developing the blocks (i.e.
multiplying these set of $t$ blocks successively by the elements
of $G$) we get the solution of the design $v = 2(t+1)p$, $b = 2(t+1)p$, $r = 3t$, $k = 3$,
$\lambda_1 = 2t$, $\lambda_2 = 2$, $\lambda_3 = 0$, $m_1 = 2$, $m_2 = t$,
$m_3 = (t+1)(p-1)$.

**Series V.** Consider the same group $G$ as above but consider the set
of initial blocks

$(1, a, ac), (1, a^2, a^2c), \ldots, (1, a^{p-1}, a^{p-1}c)$
$(1, a^{p+1}, a^{p+1}c), (1, a^{p+2}, a^{p+2}c), \ldots, (1, a^{2p-1}, a^{2p-1}c)$
$\vdots$
$(1, a^{tp+1}, a^{tp+1}c), (1, a^{tp+2}, a^{tp+2}c), \ldots, (1, a^{(t+1)p-1}a^{(t+1)p-c})$

Then on developing these blocks we get the solution of the design

$v = 2(t+1)p$ $b = 2(p-1)(t+1)^2p$, $r = 3(p-1)(t+1)$, $k = 3$
$\lambda_1 = 2(p-1)(t+1)$ $\lambda_2 = 0$ $\lambda_3 = 2$
$m_1 = 2$ $m_2 = t$ $m_3 = (t+1)(p-1)$

**Series VI.** Let us consider the group $G$ formed by $a^{6t} = c^2 = 1$
when $t$ is odd. Let $A$: $a, a^3, \ldots a^{6t-1}$; $B$: $a^2, a^4, \ldots, a^{6t-2}$.
Consider the set of $(3t-1)/2$ initial blocks $(1, a^2, a^{6t-2}c)$,
(1, \ a^4, \ a^{6t-4}, ... \ (1, \ a^{3t-1}, \ a^{3t+1}). \ Then \ on \ developing \ these
blocks, \ we \ get \ the \ solution \ of \ the \ design \ v = 12t, \ b = \frac{3t-1}{2} \cdot 12t
= 6t(3t-1), \ r = 2(3t+1), \ k = 3, \ \lambda_1 = 3t-1, \ \lambda_2 = 0, \ \lambda_3 = 2
\ m_1 = 2, \ m_2 = m_3 = 3t - 1.

**Series VII.** Consider the group G formed by \ a^{3t+1} = c^2 = 1, \ when
t(>1) is of the form \ t = 4m+3, \ m = 0, 1, 2,...

Let \ G_1 : \ 1, \ c.
A : \ a^2, \ a^6, ... \ a^{3t-1}
B : \ a, \ a^3, ... \ a^t

So, \ a = \frac{3(t-1)}{2}, \ \beta = 0.

Consider the set of \ (3t-1)/4 \ initial \ blocks

(1, \ a^2, \ c), \ (1, \ a^4, \ c), ... \ (1, \ a^{(3t-1)/2}, \ c)

On developing these blocks, we get the solution of the design
\ v = 6t+2, \ b = (3t+1)(3t-1)/2, \ r = 3(3t-1)/4, \ k = 3, \ \lambda_1 = (3t-1)/2
\ \lambda_2 = 1, \ \lambda_3 = 0, \ m_1 = 2, \ m_2 = (3t-1)/2 \ m_3 = (3t+1)/2.

**Series VIII.** Consider the group G formed by a and c where
\ a^{3t+1} = c^2 = 1 \ when \ t > 1 \ is an odd prime.

Let \ G_1 : \ 1, \ c.
A : \ a^2, \ a^4, ... \ a^{3t-1}
B : \ a, \ a^3, ... \ a^t

So, \ a = \frac{3(t-1)}{2}, \ \beta = 0.
Consider the set of \((3t-1)/2\) initial blocks \((1, a_2, c), (1, a_4, c), \ldots\) \((1, a^{3t-1}, c)\). On developing these blocks we get the solution of \(v=6t+2\), \(b=(3t-1)(3t+1)\), \(r=3(3t-1)c^2\), \(k=3\), \(\lambda_1=(3t-1)\), \(\lambda_2=2\), \(\lambda_3=0\), \(m_1=2\), \(m_2=(3t-1)/2\), \(m_3=(3t+1)/2\).

5.3. (ii) Method of symmetrically repeated ratios for two-associate triangular designs and the extension of this method to three associate cyclic - triangular designs of both the types of generalisation.

The method of symmetrically repeated differences was introduced by Bose (1939). This method was utilised for constructing P.B.I.B. designs by Bose and Nair in their paper introducing P.B.I.B. designs (1939). The idea behind this method is that if we have two numbers \(x(u)\) and \(x(u')\) in any block of the initial set which differ by a number \(\theta\), then as we go on developing the blocks, all the elements of the module which differ by an element \(\theta\) occur together \(\lambda\) times in our blocks. So, it was necessary to select the initial blocks such that all possible differences appear \(\lambda\) times. It has been shown in chapters three and four that for many of the known P.B.I.B. designs with \(m\)-associate classes one can select \(m\) sets of treatments \(A_1, A_2, \ldots, A_m\) such that \(i\)-th associates of any treatment \(\theta\) are \(\theta A_i\). So, for such designs, it was necessary to select blocks so that among the ratios (differences, while writing these in additive notation) arising out of these blocks, the elements of the set \(A_1\) appear \(\lambda_1\) times (Bose and Nair, 1939, Bhattacharya, 1950,
Bose, Shrikhande and Bhattacharjya 1952; Theorems 5.3.1 to 5.3.3' of this chapter). But for the triangular type designs such a representation is not possible.

This difficulty can be obviated by developing some of the ideas of Masuyama (1965) which however have not been clearly explained by him. The $\frac{n(n-1)}{2}$ treatments in a two associate class triangular design are so arranged that the cyclic property is preserved among the treatments of the following $(m-1)$ sets of treatments for $n = 2m-1$.

$$A_0 = (0, m-1, 2(m-1), \ldots, (n-1)(m-1))$$
$$A_1 = (1, m, 2m-1, \ldots, (n-1)(m-1)+1)$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$A_{m-2} = (m-2, 2m-3, 3m-4, \ldots, (n-1)(m-1)+m-2)$$

and among the following $m$ sets of treatments for $n = 2m$.

$$A_0 = (0, m, 2m, \ldots, (n-2)m)$$
$$A_1 = (1, m+1, 2m+1, \ldots, (n-2)m+1)$$
$$\ldots \quad \ldots \quad \ldots$$
$$A_{m-1} = (m-1, 2m-1, 3m-1, \ldots, m(n-1)-1)$$

The treatments in the association scheme may be arranged as follows:

$n = 2m - 1$
0-th row: (*, 0, 1, 2, ..., m-2, (m+1)(m-1)-1, (m+2)(m-1)-2, ...
... (m+m-2)(m-1)-(m-2), (m+m-1)(m-1)-(m-1)).

1st row: (0, *, m-1, m, ..., 2m-3, (m+2)(m-1)-1, (m+3)(m-1)-2, ...
... (2m-1)(m-1)-(m-2)).

2nd row: (1, m-1, *, 2(m-1), 2m-1, ..., 3m-4, (m+3)(m-1)-1, (m+4)(m-1)-2
... (2m-1)(m-1)-(m-3)).

3rd row: (2, m, 2(m-1), *, 3(m-1), 3m-2, ..., (2m-1)(m-1)-(m-4)).

... ... ...

last row: ((2m-2)(m-1),(2m-3)(m-1)+1 ... ... (n-2)(m-1), *).

For \( n = 2m \),

0-th row: (*, 0, m, 2m, ..., ..., ..., (n-2)m.

1st row: (0, *, 1, 2, ..., m-1, (m+1)m-1, (m+2)m-2, (m+3)m-3, ...
[m+(m-1)]m - (m-i))

2nd row: (m, 1, *, (m+1), (m+2), ..., 2m-1(m+2)m-1, (m+3)m-2, (m+4)m-3,
... (2m-1)m-(m-2).

3rd row: (2m, 2, (m+1), *, (2m+1), (2m+2), ..., (2m-1)m-(m-3))

... ... ...

last row: (n-2)m, (2m-1)m - (m-1), ..., ..., (n-3)m+1, *).

It is clear that the above arrangement can be described as follows:

For \( n = 2m-1 \), the \((n-1)\) elements in the 0-th row are 0, 1, 2, ..., m-2, m(m-1)+(n-m)-1, (m+1)(m-1) + (n-m)-2 ..., (n-1)(m+1) in this order.

The \((n-1)\) elements of the \(k\)-th row are obtained by adding \(k(m-1)\)
to each of these elements respectively. For \( n = 2m \), an element \( (j-1)m \) is put in the position \((0, j)\) for \( j=1,2,\ldots, n-1 \). The \((n-1)\)-elements on the row for which \( i=1 \), are \( 0, 1, 2,\ldots, m-1, m^2+(n-m)-1, (m+1)m+(n-m)-2, \ldots(n-2)m+1 \) in this order. The \((n-1)\) elements of the \( k \)-th row are obtained by adding \((k-1)m\) to each of these elements respectively for \( k \geq 1 \).

After such an arrangement has been achieved, it is possible to derive \( h \) sets of elements \( D_0, D_1, D_2,\ldots, D_{h-1} \) (when \( h=m-1 \) for \( n = 2m-1 \) and \( h = m \) for \( n = 2m \)) where the set \( D_j, j=0, 1, 2,\ldots, h-1 \) consists of the \( n_1 = 2(n-2) \) elements (omitting \( j \)) which appear in the same rows as the element \( j \). From \( D_j \), we can derive another set \( D_j^0 \) by subtracting \( j \) from every element of \( D_j \), which possesses the property that for any element \( \theta \in A_j \), the set of first associates of \( \theta \) is \( \phi(\pi)D_j^0 \).  

Now, we can describe the method of block formation as follows:

Let it be possible to select a set of \( t \) blocks satisfying the following properties:

1. The blocks are of constant size, \( k \).
2. Among the \( kt \) treatments occurring in the \( t \) blocks, the total number of elements appearing from \( A_j \), \( j=0,1,2,\ldots, h-1 \) is constant, equal to \( r \). So, \( kt = hr \).

The all possible differences arising within blocks can be classified in the following:
Consider the elements within blocks which belong to \( A_j \).
Let \( \alpha \) be any such element occurring in say \( j \)-th block. Then from the differences \( \alpha \gamma - \alpha \), where \( \alpha \gamma \) is any element other than \( \alpha \) in the \( j \)-th block. Among the differences obtained from all the elements of \( A_j \) occurring in the \( t \)-blocks, the elements of \( D_j^0 \) will be represented \( \lambda_1 \) times and the other non-zero elements of the module will be represented \( \lambda_2 \) times. This will hold good for \( j = 0, 1, 2, \ldots, h-1 \).

Obviously, \( tk(k-1) = h(n_1^1 \lambda_1 + n_2^2 \lambda_2) \). Then on developing these blocks as explained \( \Theta \) below a \( P.B.I.B \) design with triangular association scheme will follow.

Proof: We now change our method of developing the initial blocks by adding 0, \( h \), 2\( h \), 3\( h \), \ldots, (\( \alpha \)-1)\( h \) in succession (where \( \alpha \)\( h = v \)) instead of adding 0, 1, 2, \ldots, \( v \) in succession. Suppose \( x(u) \) and \( x(u') \) are any two elements of the module, where \( x(u) \in A_j \) and \( x(u') \in A_j \). Let \( x(u) - x(u') = \Theta \),
\[ x(u') = x(u) + \Theta', \] when \( \Theta + \Theta' = 0 \pmod{v} \). So, \( x(u) = x(u') + \Theta \) and \( x(u') = x(u) + \Theta' \). Hence, as we develop the blocks by adding 0, \( h \), 2\( h \), \ldots, (\( \alpha \)-1)\( h \) in succession, the following pairs are generated from this pair \( (x(u), x(u')), (x(u)+h, x(u')+h), \ldots, (x(u)+(\alpha-1)h, x(u')+(\alpha-1)h) \). Hence, if \( \Theta \in D_j^0 \), \( \Theta' \in D_j^0 \), then the elements of \( A_j \) will appear once with that first associate of it which can be obtained from it by adding \( \Theta \).
(\( \text{It may be noted that if } x(u) \in A_j, \text{ then } x(u)+ih \in A_j \).
\[ i = 0, 1, \ldots (a-1) \). Similarly, if \( x^{(u')} \in A_j \), then
\[ x^{(u')} + ih \in A_j, \quad i = 0, 1, \ldots, (a-1) \).

Thus, the \( \lambda \)-condition is ensured from (3). Other conditions are obviously satisfied. Hence this will give the solution of the design

\[ v = n(n-1)/2, \quad b = ta, \quad r, k, \lambda_1, \lambda_2, n_1 = 2(n-2) \]
\[ n_2 = (n-2)(n-3)/2 \]

\[
\left( p_{jk}^1 \right) = \begin{bmatrix}
    n-2 & n-3 \\
    n-3 & (n-3)(n-4)/2 \\
\end{bmatrix}, \quad \left( p_{jk}^2 \right) = \begin{bmatrix}
    4 & 2(n-4) \\
    2(n-4)(n-4)(n-5)/2 \\
\end{bmatrix}
\]

We shall next derive some \( A_j, \ D^0_j \) and \( h \) for given values of \( n \). \( n = 4, v = 6 \). The design reduces to C.D. design.

\[ n = 5, v = 10 \]
\[ A_0 \equiv (0, 2, 4, 6, 8), \quad D^0_0 = (1, 2, 3, 7, 8, 9) \]

Association Scheme:
\[ A_1 \equiv (1, 3, 5, 7, 9), \quad D^0_1 = (1, 3, 4, 6, 7, 9) \]
(See Masuyama)
\[ h = 2 \]

\[ n = 6, v = 15 \]
\[ A_0 \equiv (0, 3, 6, 9, 12), \quad D^0_0 = (1, 2, 3, 6, 9, 11, 12, 13) \]

Association scheme:
\[ A_1 \equiv (1, 4, 7, 10, 13), \quad D^0_1 = (1, 2, 3, 4, 10, 12, 13, 14) \]
(See Masuyama)
\[ A_2 \equiv (2, 5, 8, 11, 14), \quad D^0_2 = (2, 4, 5, 6, 9, 11, 13, 14) \]
\[ h = 3 \]
\[ n = 7, \; v = 21 \]

**Association Scheme:**

\[
\begin{align*}
* & : 0 & 1 & 2 & 14 & 16 & 18 & \quad h = 3 \\
0 & * & 3 & 4 & 5 & 17 & 19 \\
1 & 3 & * & 6 & 7 & 8 & 20 \\
2 & 4 & 6 & * & 9 & 10 & 11 \\
14 & 5 & 7 & 9 & * & 12 & 13 \\
16 & 17 & 8 & 10 & 12 & \quad * & 15 \\
18 & 19 & 20 & 11 & 13 & 15 & * \\
\end{align*}
\]

\[ n = 8, \; v = 28 \]

**Association Scheme:**

\[
\begin{align*}
* & : 0 & 4 & 8 & 12 & 16 & 20 & 24 & \quad h = 4 \\
0 & * & 1 & 2 & 3 & 19 & 22 & 25 \\
4 & 1 & * & 5 & 6 & 7 & 23 & 26 \\
8 & 2 & 5 & * & 9 & 10 & 11 & 27 \\
12 & 3 & 6 & 9 & * & 13 & 14 & 15 \\
16 & 19 & 7 & 10 & 13 & * & 17 & 18 \\
20 & 22 & 23 & 11 & 14 & 17 & * & 21 \\
24 & 25 & 26 & 27 & 15 & 18 & 21 & * \\
\end{align*}
\]

We next state our theorems for constructing cyclic triangular designs by generalising the above method.

**Theorem 5.3.4:** Let \( G \) be a cyclic group formed by \( a, c \) and their different powers when \( a^{n(n-1)/2} = c^m = 1 \). Let \( G = G_1(X)G_2 \), where
Let \( G_2 \) admit two-associate triangular representation and \( A_j \) and \( D_j \) be subsets of \( G_2 \) and be known for \( j = 0, 1, 2, \ldots, h-1 \).

Let us select a set of \( t \) blocks each of size \( k \) such that

i) Among the \( kt \) treatments occurring in the \( t \)-blocks, the number of elements appearing from \( A_j(G_1) \), \( j = 0, 1, \ldots, h-1 \) is constant, equal to \( r \). Obviously, \( kt = hr \).

ii) The all possible ratios arising within these \( t \) blocks can be classified in the following:

Consider the elements within blocks which belong to \( A_j(G_1) \). Let \( \alpha \) be any such element occurring in say \( k \)-th block. Then form the ratios \( \alpha / \alpha' \), where \( \alpha' \) is any element other than \( \alpha \) in the \( k \)-th block. Among the ratios obtained from all the elements of \( A_j \), occurring in the \( t \)-blocks, the elements of \( D_j(G_1) \) appear \( \lambda_1 \) times, the elements of \( D_j(G_1) \) appear \( \lambda_2 \) times and the other non-unit elements of the group will appear \( \lambda_3 \) times, \( j = 0, 1, 2, \ldots, h-1 \). So,

\[
\frac{(m_1-1)\lambda_1+2m_1(n-2)\lambda_2+\frac{m_1-1}{2}(n-3)(n-4)\lambda_3}{nk(k-1)}
\]

Then on developing the blocks, (i.e. multiplying successively by \( 1, a, a^2, \ldots, a^{(s-1)}h, c, a^hc, \ldots, a^{(s-1)}h^c, \ldots, a^{m_1-1}, \ldots, a^{(s-1)}h^{m_1-1} \)) where \( sh = \frac{n(n-1)}{2} \), we get the solution of the cyclic triangular design \( v = m_1n(n-1)/2, b = st m_1, r, k, \lambda_1, \lambda_2, \lambda_3, n_1 = m_1-1, n_2 = 2m_1(n-2), n_3 = m_1(n-2)(n-3)/2. \)
Theorem 5.3.4'. If we replace (ii) in the theorem as:

(ii') Among the different types of ratios that can be formed from the blocks by taking one element from $A_j(x)G_1$, the elements of $D_j^0$ appear $\lambda_1$ times, the elements of $(A_j^2 - D_j^0 - 1)$ appear $\lambda_2$ times and the elements of $(A_j^1)(x)G_2$ appear $\lambda_3$ times then on developing the $t$ blocks we get the solution of the design

$v = m_1 n(n-1)/2$, $b = m_1 t a$, $r, k, \lambda_1, \lambda_2, \lambda_3$

$n_1 = 2(n-2)$, $n_2 = (n-2)(n-3)/2$, $n_3 = (m_1-1)n(n-1)/2$

$$
\begin{bmatrix}
\begin{array}{ccc}
\frac{n-2}{m_1-1} & 0 & 0 \\
0 & m_1m_2 & 0 \\
0 & 0 & \frac{m_1m_3}{m_1-1}
\end{array}
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & m_1-1 & 0 \\
\frac{m_1(n-2)}{m_1-1} & m_1(n-3) & 0 \\
0 & \frac{m_1(n-3)}{m_1-1} & \frac{m_1(n-3)(n-4)}{2}
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & 0 & \frac{m_1-1}{4m_1} \\
0 & \frac{2m_1}{2m_1(n-4)} & \frac{2m_1(n-4)}{m_1-1} \\
\frac{m_1-1}{2m_1(n-4)} & \frac{2m_1(n-4)}{m_1-1} & \frac{m_1(n-4)(n-5)}{2}
\end{bmatrix}
$$
\[
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & 2(n-2) \\
0 & 0 & (n-2)(n-3)/2 \\
2(n-2) & (n-2)(n-3)/2 & (m-2)n(n-1)/2
\end{bmatrix}
\]

which belongs to second generalisation of triangular association scheme.

Note. It may be observed that sometimes by partial developments of some blocks by the above method and by the complete development of other blocks, cyclic triangular designs can be obtained.

Example 1. (First type). Consider the group \( G \) formed by \( a^{10} = c^2 = 1 \).

Let \( G_1 : l, c ; G_2 : l, a, a^2, \ldots a^9 \).

\( A_0 = (1, a^2, a^4, a^6, a^8) \quad D_0 = (a, a^2, a^3, a^7, a^8, a^9) \)

\( A_1 = (a, a^3, a^5, a^7, a^9) \quad D_1 = (a, a^3, a^4, a^6, a^7, a^9) \)

\( h = 2, \alpha = 5 \).

Consider the set of 6 initial blocks:

\( (a, a^3, ac), (a^2, a^3, a^6c), (a, a^3, a^7c), (a^2, a^8, a^6c) \)

\( (1, a^2, c); (1, a^5, a^5c) \).

We can easily verify that these blocks satisfy all the conditions of theorem 1 when \( r = 9, \lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 2 \). Hence on developing the blocks (i.e. multiplying these blocks successively by the elements \( 1, a^2, a^4, a^6, a^8, c; a^2c, a^4c, a^6c, a^8c \)) we get the solution of the design.
$v = 20$, $b = 60$, $r = 9$, $k = 3$, $\lambda_1 = 6$, $\lambda_2 = 0$, $\lambda_3 = 2$

$n_1 = 1$, $n_2 = 12$, $n_3 = 6$.

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 6 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 8 & 4 \\ 1 & 4 & 0 \end{pmatrix}$

Example 2: (Second type). Consider the group $G$ formed by

Let $G_1: 1, c, c^2$; $G_2: 1, a, a^2, \ldots, a^{14}$. $h = 3$, $\alpha = 5$.

$A_0: (1, a^3, a^6, a^9, a^{12})$  
$B_0: (a, a^2, a^3, a^6, a^9, a^{11}, a^{12}, a^{13})$

$A_1: (a, a^4, a^7, a^{10}, a^{13})$  
$B_1: (a, a^2, a^3, a^4, a^6, a^{10}, a^{12}, a^{13}, a^{14})$

$A_2: (a^2, a^5, a^8, a^{11}, a^{14})$  
$B_2: (a^2, a^4, a^5, a^6, a^{11}, a^{13}, a^{15}, a^{14})$

Let us develop the blocks $(1, a^4, a^{10})$, $(1, a^5, a^8)$, $(1, a^7, a^{14})$ partially by multiplying successively by the elements of $A_0(x)G_1$ and develop the blocks $(1, a^2c, a^3c^2)$, $(1, a^9c, a^7c^2)$, $(1, a^4c, c^2)$, $(1, a^6c, a^5c^2)$ and $(1, a^7c, a^{10}c^2)$ completely by multiplying successively by the elements of the group $G$.

Then we get the solution of the design $v = 45$, $b = 270$, $r = 18$. 
5.4. Miscellaneous methods

Method I. Consider a P.B.I.B. design with parameters

\[ v, b; r, k, \lambda_i, n_i, p_{jk}^i, i, j, k = 1, 2, \ldots, m \]

having a particular type of association scheme. Then, it has been observed by Guerin (1965) that the design complementary to the above P.B.I.B. design will also be a \( m \)-associate P.B.I.B. design having the same value for \( v, b, n_i, p_{jk}^i \); while \( r' = b - r, k' = v - k \) and \( \lambda_i' = b - 2r + \lambda_i \),

\[ i = 1, 2, \ldots, m. \]

This method is applicable to both the types of generalisation and for all the association schemes.

Method II. Omitting the blocks of any P.B.I.B. design \( v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_m \) from the unreduced B.I.B. design obtained by taking all possible combinations of \( k \) at a time from \( v \), we get

\[ k = 3, \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1, n_1 = 3, m_2 = 6, n_3 = 30 \]

\[
\begin{bmatrix}
4 & 3 & 0 \\
3 & 3 & 0 \\
0 & 0 & 30 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & 4 & 0 \\
0 & 0 & 30 \\
0 & 0 & 15 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 8 \\
0 & 0 & 6 \\
8 & 6 & 15 \\
\end{bmatrix}
\]
a P.B.I.B. design of the same type having the same association scheme.

If the parameters of the B.I.B. design be

\[ v, b = \binom{\nu}{k}, \quad r = \binom{\nu-1}{k-1}, \quad \lambda = \binom{\nu-2}{k-2} \]

and that of the P.B.I.B. design be

\[ v' = v, \quad b' = b-b', \quad r' = r-r', \quad k' = k, \quad \lambda'_1 = \lambda - \lambda'_1 \]

\[ p_{jk}^i, \quad i, j, k = 1, 2, \ldots, m. \]

**Example:** Consider, for example, the unreduced B.I.B. design

\[ v = 8, \quad b = 56, \quad r = 21, \quad k = 3, \quad \lambda = 6 \]

and the 3-associate cyclical P.B.I.B. design of the first type

\[ v = b = 8, \quad r = k = 3, \quad m_1 = m_2 = m_3 = 2, \quad n_1 = 1; \quad n_2 = n_3 = 4, \]

\[ \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4
\end{pmatrix} \quad \left( p_{jk}^1 \right) \quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 2 \\
0 & 2 & 2
\end{pmatrix} \quad \left( p_{jk}^2 \right)
\]

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 4 & 0 \\
1 & 0 & 2
\end{pmatrix} \quad \left( p_{jk}^3 \right)
\]
whose solution is obtained by developing the initial block 
(1, a, ac) by the elements of the group G: \( a^4 = c^2 = 1 \) (i.e. 
multiplying successively by 1, a, a^2, a^3, c, ac, a^2c and a^3c) 
as (1, a, ac), (a, a^2, a^2c), (a^2, a^3, a^3c), (a^3, 1, c), (c, ac, a), 
(ac, a^2c, a^2), (a^2c, a^3c, a^3) and (a^3c, c, 1).

Then by the above method we get the solution of the design

\[ v = 8, b = 48, r = 18, k = 3, \lambda_1 = 4, \lambda_2 = 5, \lambda_3 = 6, m_1 = m_2 = m_3 = 2 \]
\[ n_1 = 1, n_2 = n_3 = 4, \alpha = 0, \beta = 2 \]

which belongs also to the first type.

Method III. By dualising partially linked block designs. Partially 
linked block design (P.L.B.) (C.R. Nair 1965) is defined to be an 
arrangement of \( v \) treatments in blocks of \( k \) plots each where 
each treatment is replicated \( r \) times such that its dual is a 
P.B.I.B. design.

Let it be possible to arrange the \( b \) blocks in a partially 
linked block design in \( pq \) cells of \( m_1 \) blocks each so that any 
two blocks in the same cell has \( \mu_1 \) treatments in common. Let 
\( m_1 \) it be further possible to arrange the \( pq \) cells in the form of 
a rectangle of \( p \) rows and \( q \) columns such that any two blocks 
within the same row but not on the same cell have \( \mu_2 \) treatments 
in common and any two blocks within the same column but not in the 
same cell have \( \mu_3 \) treatments in common; while two blocks not 
in the same row and same column have \( \mu_4 \) treatments in common.

Then on dualising this design we get the solution of the 
4-associate cyclical rectangular P.B.I.B. design whose parameters 
are
\( v^* = m_1 p q = b, \ r^* = k, \ r^* = r, \ b^* = v \)

\( \lambda_1^* = \mu_1, \ \lambda_2^* = \mu_2, \ \lambda_3^* = \mu_3, \ \lambda_4^* = \mu_4 \)

\( m_1^* = m_1, \ m_2^* = p - 1, \ m_3^* = q - 1, \)

\( m_1^* = m_1 - 1; \ m_2^* = m_1(p - 1), \ m_3^* = m_1(q - 1), \ m_4^* = m_1(p - 1)(q - 1) \)

\[
(p_{jkk}^1) = \begin{bmatrix}
m_1 - 2 & 0 & 0 & 0 \\
0 & m_1(p - 1) & 0 & 0 \\
0 & 0 & m_1(q - 1) & 0 \\
0 & 0 & 0 & m_1(p - 1)(q - 1)
\end{bmatrix}
\]

\[
(p_{jkk}^2) = \begin{bmatrix}
0 & m_1 - 1 & 0 & 0 \\
m_1 - 1 & m_1(p - 2) & 0 & 0 \\
0 & 0 & 0 & m_1(q - 1) \\
0 & 0 & m_1(q - 1) & m_1(q - 1)(p - 2)
\end{bmatrix}
\]

\[
(p_{jkk}^3) = \begin{bmatrix}
0 & 0 & m_1 - 1 & 0 \\
0 & 0 & 0 & m_1(p - 1) \\
m_1 - 1 & 0 & m_1(q - 1) & 0 \\
0 & m_1(p - 1) & 0 & m_1(p - 1)(q - 2)
\end{bmatrix}
\]

\[
(p_{jkk}^4) = \begin{bmatrix}
0 & 0 & 0 & m_1 - 1 \\
0 & 0 & m_1 & m_1(p - 2) \\
0 & m_1 & 0 & m_1(q - 2) \\
m_1 - 1 & m_1(p - 2) & m_1(q - 2) & m_1(p - 2)(q - 2)
\end{bmatrix}
\]
This belongs to the first type of cyclical generalisation of the three associate O.G.D. (rectangular) association scheme to four associate classes.

**Corollary 5.3.1.** If \( m_2^* = m_3^* \) and \( \mu_2 = \mu_3 \), then the design reduces to 3-associate cyclical L_2-type design with \( \lambda_1^* = \mu_1, \lambda_2^* = \mu_2, \lambda_3^* = \mu_4, m_1^* = 2(p-1), m_2^* = (p-1)^2, n_2^* = 2m_1(p-1), n_3^* = m_1(p-1)^2 \) all other parameters remain unaltered.

**Corollary 5.3.2.** If \( \mu_3 = \mu_4 \), then we get a 3-associate cyclical design whose parameters will be \( m_3^* = p(q-1), m_2^* = p-1, \alpha = m_2^*-1, \beta = 0. \)

**Example:** Consider the following set of 12 blocks

\[
\begin{align*}
B_1 & : (1, 2, 3, 4) & B_1^* & : (5, 6, 7, 8) \\
B_2 & : (1, 5, 11, 12) & B_2^* & : (3, 6, 9, 10) \\
B_3 & : (2, 7, 9, 11) & B_3^* & : (4, 8, 10, 12) \\
B_4 & : (1, 3, 7, 8) & B_4^* & : (2, 4, 5, 6) \\
B_5 & : (1, 6, 10, 11) & B_5^* & : (3, 5, 9, 12) \\
B_6 & : (2, 7, 10, 12) & B_6^* & : (4, 8, 9, 11)
\end{align*}
\]

We can arrange the blocks in the following way

\[
\begin{array}{c|c}
(B_1, B_1^*) & (B_4, B_4^*) \\
(B_2, B_2^*) & (B_5, B_5^*) \\
(B_3, B_3^*) & (B_6, B_6^*)
\end{array}
\]
Here any two blocks on the same cell has no treatments in common, two blocks from different cells in the same row has 2 treatments in common and any two blocks not on the same row has one treatment in common. So, $m_1^* = 2, m_2^* = 1, m_3^* = 4, r^* = k^* = 4, b^* = v^* = 12, \mu_1 = 0, \mu_2 = 2, \mu_3 = 1$.

**Method IV.**

Let us consider the following three associate cyclic association scheme (having $\rho_{33}^2 > 0$ for cyclic designs) of the first type of generalisation involving $v = m_1(m_2 + m_3 + 1)$ treatments.

<table>
<thead>
<tr>
<th>First associate group</th>
<th>Second associate group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$G_{11}, G_{12}, \ldots, G_{1m_2}$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$G_{21}, G_{22}, \ldots, G_{2m_2}$</td>
</tr>
<tr>
<td>\vdotsof$G_{m_2+m_3+1}$</td>
<td>\vdotsof$G_{m_2+m_3+1,m_2}$</td>
</tr>
</tbody>
</table>

when $G_{ij}$ and $G_{kl}$ are rearrangements of $G_2, \ldots, G_{m_2+m_3+1}$ and are not necessary distinct, i.e., $k, l = 1, 2, \ldots m_2 + m_3 + 1$; $j, \ell = 1, 2, \ldots m_2$. (This is possible because of the lemma 5.1)

Consider any treatment $\Theta$ from $G_1$. Form a block by taking $m_1 - 1$ elements of $G_1$ (excluding $\Theta$) all of which are first associates to $\Theta$ together with any element from $G_{11}$, i.e., $(1, 2, \ldots m_2)$. The element $\Theta$ can be chosen from $G_1$ in $m_1$ distinct ways.
Also any element from $G_{i_1}$ can be chosen in $m_1$ distinct ways. So, this will give us $m_2^2$ blocks. The group $G_{i_1}$ can be chosen in $m_2$ distinct ways for $i = 1, 2, \ldots, m_2$. So, considering the $m_2$ distinct groups of second associates, we get $m_1^2 m_2$ blocks.

Next, consider any treatment from $G_{i_1}, i = 2, 3, \ldots, m_2 + m_3 + 1$. Form a block by taking $m_1 - 1$ element from $G_{i_1}$ ($i_2, \ldots, m_2 + m_3 + 1$) and any treatment from its group of second associate treatments. This will again give us $m_1^2 m_2$ blocks for a given value of $i$. Varying $i$ from 2 to $m_2 + m_3 + 1$ we get $m_1^2 m_2 (m_2 + m_3)$ blocks.

Considering these with $m_1^2 m_2$ blocks obtained earlier we get the solution of a first type of cyclic design of the three associate cyclic, triangular, and association scheme whose parameters are

\[ v = m_1 (m_2 + m_3 + 1), \quad b = m_1^2 m_2 (m_2 + m_3 + 1), \quad r = m_1^2 m_2 \]

\[ k = m_1, \quad \lambda_1 = m_1 m_2 (m_1 - 2), \quad \lambda_2 = 2 (m_1 - 1), \quad \lambda_3 = 0 \]

\[ \rho_{jk} \] will depend on the association scheme.

The proof follows easily from the method of construction.

**Corollary 5.4.1.** Instead of second associate treatments, third associate treatments can also be used.

**Corollary 5.4.2.** For four associate cyclic rectangular (O.G.D.) design, this method will lead to disconnected design. So, we replace second associate treatments by fourth associate treatments.

**Example.** Consider the following three associate cyclic $L_2$ association scheme.
where two treatments are first associates if they occur in the same cell, they are second associates if they occur in cells within the same row or column and they are third associates if otherwise.

Parameters of the association scheme are

\[ v = 12, \; m_1 = 3, \; m_2 = 2, \; m_3 = 1, \]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 3 \\
0 & 3 & 0
\end{pmatrix}
\]

By the above procedure we get the solution of the design whose parameters of the second kind will be

\[ b = 72, \; r = 18, \; k = 3, \; \lambda_1 = 6, \; \lambda_2 = 4, \; \lambda_3 = 0 \]

The blocks are

(1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 4), (1, 3, 5), (1, 3, 6)
(2, 3, 4), (2, 3, 5), (2, 3, 6), (1, 2, 7), (1, 2, 8), (1, 2, 9)
(1, 3, 7), (1, 3, 8), (1, 3, 9), (2, 3, 7), (2, 3, 8), (2, 3, 9)
(4, 5, 1), (4, 5, 2), (4, 5, 3), (4, 5, 10), (4, 5, 11), (4, 5, 12)
(4, 6, 1), (4, 6, 2), (4, 6, 3), (4, 6, 10), (4, 6, 11), (4, 6, 12),
(5, 6, 1), (5, 6, 2), (5, 6, 3), (5, 6, 10), (5, 6, 11), (5, 6, 12),
(7, 8, 1), (7, 8, 2), (7, 8, 3), (7, 8, 10), (7, 8, 11), (7, 8, 12)
(7, 9, 1), (7, 9, 2), (7, 9, 3), (7, 9, 10), (7, 9, 11), (7, 9, 12)
Method V. Consider the first type of cyclic association scheme exhibited in method IV. Consider the following set of $m_2+m_3+1$ blocks:

<table>
<thead>
<tr>
<th>Block No.</th>
<th>treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_1, G_{11}, G_{12}, \ldots, G_{1m_2}$</td>
</tr>
<tr>
<td>2</td>
<td>$G_2, G_{21}, G_{22}, \ldots, G_{2m_2}$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$(m_2+m_3+1)$</td>
<td>$G_{m_2+m_3+1}, G_{m_2+m_3+1}, \ldots, G_{m_2+m_3+1}, m_2$</td>
</tr>
</tbody>
</table>

These blocks give us a solution of the first type of three associate cyclic design (having any association scheme) with parameters, 

$$v = (m_2 + m_3 + 1)m_1, \quad b = m_2 + m_3 + 1, \quad r = 1 + m_2,$$

$$k = m_1(l + m_2), \quad \lambda_1 = 1 + m_2, \quad \lambda_2 = p_{22}/m_1 + 2, \quad \lambda_3 = p_{22}/m_1 + 2,$$

$$n_1 = m_1 - 1, \quad n_2 = m_1 m_2, \quad n_3 = m_1 m_3, \quad p_{jk}^i \text{ parameters depend on the association scheme.}$$

**Proof:** The parameters $v, b, r, k$ are obvious, while $n_1, n_2, n_3$ and $p_{jk}^i$ being association parameters remain unchanged after forming the blocks. We shall explain $\lambda_2$ and $\lambda_3$. Consider any two
second associate treatments, say \( \Theta \in G_1 \) and \( \Phi \in G_{11} \). So, \( \Theta \) and \( \Phi \) occur together in block 1. Again, \( G_{11} \in (G_2, G_3, \ldots, G_{m_2+m_3+1}) \).

We may assume \( G_{11} = G_1 \). So, in the \( i \)-th block \( \Theta \) and \( \Phi \) again occur together. Also, the two treatments \( \Theta \) and \( \Phi \) have a group of \( p_{22}/m_1 \) first associate groups as common second associate groups. These groups may be taken to be \( G_i(1), G_i(2), \ldots, G_i(p_{22}/m_1) \). So, \( \Theta \) and \( \Phi \) occur together in the blocks having numbers \( i(1), i(2), \ldots, i(p_{22}/m_1) \). Thus, \( \lambda_2 = p_{22}/m_1 + 2 \). Similarly, we can prove that \( \lambda_3 = p_{22}/m_1 \).

**Example:**

Consider the following three associate cyclical association scheme.

<table>
<thead>
<tr>
<th>Groups of first associate treatments</th>
<th>Second associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, c))</td>
<td>((a, ac), (a^3, a^3c), (a^5, a^5c))</td>
</tr>
<tr>
<td>((a, ac))</td>
<td>((a^2, a^2c), (a^4, a^4c), (1, c))</td>
</tr>
<tr>
<td>((a^2, a^2c))</td>
<td>((a^3, a^3c), (a^5, a^5c), (a, ac))</td>
</tr>
<tr>
<td>((a^3, a^3c))</td>
<td>((a^4, a^4c), (1, c), (a^2, a^2c))</td>
</tr>
<tr>
<td>((a^4, a^4c))</td>
<td>((a^5, a^5c), (a, ac), (a^3, a^3c))</td>
</tr>
<tr>
<td>((a^5, a^5c))</td>
<td>((a, c), (a^2, a^2c), (a^4, a^4c))</td>
</tr>
</tbody>
</table>

Consider the following set of six blocks:

1. \((1, c, a, ac, a^3, a^3c, a^5, a^5c)\)
2. \((a, ac, a^2, a^2c, a^4, a^4c, l, c)\)
3. \((a^2, a^2c, a^3, a^3c, a^5, a^5c, a, ac)\)
4. \((a^3, a^3c, a^4, a^4c, 1, c, a^2, a^2c)\)
5. \((a^4, a^4c, a^5, a^5c, a, ac, a^3, a^3c)\)
6. \((a^5, a^5c, 1, c, a^2, a^2c, a^4, a^4c)\)

This gives the solution of the first type of cyclic design

\(v=12, b=6, r=4, k=8, n_1=1, n_2=6, n_3=4, \lambda_1=4, \lambda_2=2, \lambda_3=3\)

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 4 \\
0 & 4 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 1 \\
0 & 6 & 0 \\
1 & 0 & 2
\end{bmatrix}
\]

**Corollary 5.5.1** Instead of second associates, third associate treatments can also be used.

**Method VI** Consider the following three associate cyclic association scheme of the first type of generalisation involving

\(v = m_1(m_2 + m_3 + 1)\) treatments.

<table>
<thead>
<tr>
<th>First associate group</th>
<th>Third associate group</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_1)</td>
<td>(G_{11}, G_{12}, \ldots, G_{1m_3})</td>
</tr>
<tr>
<td>(G_2)</td>
<td>(G_{21}, G_{22}, \ldots, G_{2m_3})</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(G_{m_2+m_3+1})</td>
<td>(G_{m_2+m_3+1,1}, G_{m_2+m_3+1,2}, \ldots, G_{m_2+m_3+1,m_3})</td>
</tr>
</tbody>
</table>
where \((G_{11}, G_{12}, \ldots, G_{1m_3}) \subseteq (G_1, G_2, \ldots, G_{m_2+m_3+1})\).

Consider the following set of \((m_2+m_3+1)\) - blocks:

$$
\begin{align*}
\text{Block No.} & \quad \text{Treatments} \\
1 & \quad G_{11}, G_{12}, \ldots, G_{1m_3} \\
2 & \quad G_{21}, G_{22}, \ldots, G_{2m_3} \\
& \quad \ldots \\
(m_2+m_3+1), & \quad G_{m_2+m_3+1, 1}, G_{m_2+m_3+1, 2}, \ldots, G_{m_2+m_3+1, m_3}
\end{align*}
$$

This gives us the solution of a three associate cyclic design (having cyclical, triangular or rectangular association scheme) whose parameters will be

\(v = m_1(m_2+m_3+1), \quad b = m_2+m_3+1, \quad r = m_2, \quad k = m_1 m_2\)

\(\lambda_1 = m_3, \quad \lambda_2 = p^2_{33}/m_1, \quad \lambda_3 = p^3_{33}/m_1, \quad p^i_{jk}\) parameters will depend on the association scheme.

Example: Consider the three associate cyclic triangular association scheme.

<table>
<thead>
<tr>
<th></th>
<th>((1, o))</th>
<th>((a, ac))</th>
<th>((a^2, a^2c))</th>
<th>((a^3, a^3c))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, o))</td>
<td>*</td>
<td>((a^4, a^4c))</td>
<td>((a^5, a^5c))</td>
<td>((a^6, a^6c))</td>
</tr>
<tr>
<td>((a, ac))</td>
<td>((a^4, a^4c))</td>
<td>*</td>
<td>((a^7, a^7c))</td>
<td>((a^8, a^8c))</td>
</tr>
<tr>
<td>((a^2, a^2c))</td>
<td>((a^5, a^5c))</td>
<td>((a^7, a^7c))</td>
<td>*</td>
<td>((a^9, a^9c))</td>
</tr>
<tr>
<td>((a^3, a^3c))</td>
<td>((a^6, a^6c))</td>
<td>((a^8, a^8c))</td>
<td>((a^9, a^9c))</td>
<td>*</td>
</tr>
</tbody>
</table>

We can represent this association scheme in the above form as follows:
First associate group | Third associate group
---|---
(1, c) | (a^7, a^7c), (a^8, a^8c), (a^9, a^9c)
(a, ac) | (a^5, a^5c), (a^6, a^6c), (a^9, a^9c)
(a^2, a^2c) | (a^4, a^4c), (a^6, a^6c), (a^8, a^8c)
(a^3, a^3c) | (a^4, a^4c), (a^5, a^5c), (a^7, a^7c)
(a^4, a^4c) | (a^2, a^2c), (a^3, a^3c), (a^9, a^9c)
(a^5, a^5c) | (a, ac), (a^3, a^3c), (a^8, a^8c)
(a^6, a^6c) | (a, ac), (a^2, a^2c), (a^7, a^7c)
(a^7, a^7c) | (l, c), (a^3, a^3c), (a^6, a^6c)
(a^8, a^8c) | (l, c), (a^2, a^2c), (a^5, a^5c)
(a^9, a^9c) | (l, c), (a, ac), (a^4, a^4c)

By the above procedure we get the solution of the three associate cyclic triangular design with the parameters

\[ v=20, h=10, r=3, k=6, \lambda_1=3, \lambda_2=1, \lambda_3=0 \]
\[ n_1=1, n_2=12, n_3=6, m_1=2, m_2=6, m_3=3 \]

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 6 & 4 \\ 0 & 4 & 2 \end{bmatrix}
\]

\[
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 8 & 4 \\ 1 & 4 & 0 \end{bmatrix}
\]
whose lay-out is

<table>
<thead>
<tr>
<th>Block No.</th>
<th>Treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(1, c, a, ac, a^c, a^c)</td>
</tr>
<tr>
<td>2.</td>
<td>(1, c, a^2, a^2c, a^5, a^5c)</td>
</tr>
<tr>
<td>3.</td>
<td>(1, c, a^3, a^3c, a^6, a^6c)</td>
</tr>
<tr>
<td>4.</td>
<td>(a, ac, a^2, a^2c, a^7, a^7c)</td>
</tr>
<tr>
<td>5.</td>
<td>(a, ac, a^3, a^3c, a^8, a^8c)</td>
</tr>
<tr>
<td>6.</td>
<td>(a^2, a^2c, a^3, a^3c, a^9, a^9c)</td>
</tr>
<tr>
<td>7.</td>
<td>(a^4, a^4c, a^6, a^6c, a^8, a^8c)</td>
</tr>
<tr>
<td>8.</td>
<td>(a^4, a^4c, a^5, a^5c, a^7, a^7c)</td>
</tr>
<tr>
<td>9.</td>
<td>(a^5, a^5c, a^6, a^6c, a^9, a^9c)</td>
</tr>
<tr>
<td>10.</td>
<td>(a^7, a^7c, a^8, a^8c, a^9, a^9c)</td>
</tr>
</tbody>
</table>

Method VII. Suppose the following two three associate cyclical designs belonging to the first type of generalisation exist:

\[ n_1 : v = m_1 (m_1 + m_3 + 1), \quad b, r, k, \lambda_1, \lambda_2, \lambda_3, \]

\[ n_1 = m_1 - 1, \quad n_2 = m_1 m_2, \quad n_3 = m_1 m_3 (1 + m_2), \]

where \( m_3^l = m_3 (1 + m_2) \)
\[
(p_{jk}^1) = \begin{bmatrix}
m_1 - 2 & 0 & 0 \\
0 & m_1 m_2 & 0 \\
0 & 0 & m_1 m_3 (1 + m_2)
\end{bmatrix}
\]

\[
(p_{jk}^2) = \begin{bmatrix}
0 & m_1 - 1 & 0 \\
m_1 - 1 & m_1 (m_2 - 1) & 0 \\
0 & 0 & m_1 (1 + m_2) m_3
\end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & m_1 - 1 \\
0 & m_1 m_2 & m_1 (1 + m_2) (m_3 - 1)
\end{bmatrix}
\]

and \( D_2 \): \( v' = m_1 (m_2 + m_3 + 1), b', r', k' = k, \lambda_1', \lambda_2', \lambda_3' \)

\[
\begin{align*}
& n_1' = m_1 - 1, \\
& n_2' = m_1 m_3, \\
& n_3' = m_1 m_2 (1 + m_3), \\
& m_2' = m_2 (1 + m_3)
\end{align*}
\]

\[
(p_{jk}^1) = \begin{bmatrix}
m_1 - 2 & 0 & 0 \\
0 & m_1 m_3 & 0 \\
0 & 0 & m_1 m_2 (1 + m_3)
\end{bmatrix}
\]

\[
(p_{jk}^2) = \begin{bmatrix}
0 & m_1 - 1 & 0 \\
m_1 - 1 & m_1 (m_3 - 1) & 0 \\
0 & 0 & m_1 (1 + m_3) m_2
\end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & m_1 - 1 \\
0 & 0 & m_1 m_3 \\
m_1 - 1 & m_1 m_3 & m_1 (1 + m_3) (m_2 - 1)
\end{bmatrix}
\]
Then we can utilize these two designs to get a solution of the cyclic rectangular design having four associate classes in the following manner:

Let us arrange the treatments in an \((1+m_2)(1+m_3)\) rectangle each cell whereof contains \(m_1\) treatments. Form a design \(D_1\) by considering the treatments which are in the same cells as first associates, the treatments which are in the cells within the same row as second associates and the treatments which are in cells not in the same row as third associates. In \(D_2\), let the treatments which are in the same cell be first associates, those in cells in the same column as second associates, and those in cells not in the same column as third associates. It may be observed that both the association schemes are three associate cyclical association scheme of the first type of generalisation. Then by combining \(s\) times the blocks of \(D_1\) with \(t\) times the blocks of \(D_2\) we get the solution of the design

\[
D^*: \quad r^* = m_1(m_2 + m_3 + l), \quad b^* = sb + tb'
\]
\[
r^* = sr + tr', \quad k^* = k
\]
\[
\lambda_1 = s\lambda_1 + t\lambda_1', \quad \lambda_2 = s\lambda_2 + t\lambda_2'
\]
\[
\lambda_3 = s\lambda_3 + t\lambda_3', \quad \lambda_4 = s\lambda_4 + t\lambda_4'
\]
\[
n_1^* = m_1 - l, \quad n_2^* = m_1m_2, \quad n_3^* = m_1m_3, \quad n_4^* = m_1m_2m_3
\]
Example. Consider, for example, the following two disconnected designs:

\[ D_1: v=b=12, r=\lambda=3, n_1=3, n_2=2, n_3=8, \lambda_1=\lambda_2=2, \lambda_3=0 \]

whose association rule is
where treatments in the same cell are first associates, those in cells in the same column are second associates otherwise they are third associates whose solution is obtained by developing the block $(1, a, ac)$ where $a^2 = b^3 = c^2 = 1$, and the design $B_2$ given by

$$
B_2: v' = b' = 12, \ r' = k' = 3, \ n_1' = 1, \ n_2' = 4, \ n_3' = 6
$$

$$\lambda_1' = 2, \ \lambda_2' = 1, \ \lambda_3' = 0$$

whose association scheme is the same as in $D_1$, but the association rule is changed as follows:

Two treatments in the same cell are first associates, those in cells in same column are second associates, otherwise they are third associates. A solution of $D_2$ is obtained by developing the block $(1, b, bc)$.

Then by taking the blocks of $D_1$ and $D_2$ together we get the solution of the design

$$
D*: v^* = 12, \ b^* = 24, \ r^* = 6, \ k^* = 3,
$$

$$n_1^* = 1, \ n_2^* = 2, \ n_3^* = 4, \ n_4^* = 4
$$

$$\lambda_1^* = 4, \ \lambda_2^* = 2, \ \lambda_3^* = 1, \ \lambda_4^* = 0$$
Corollary 5.7.1. This method is applicable to second type of cyclic rectangular designs also. Thus, suppose the following two designs belonging to the three associate cyclical design of the second type exist:

\[ \begin{array}{ccc}
  n_1 : v = m_1(m_2 + m_3 + 1), & b, r, k, \lambda_1, \lambda_2, \lambda_3, \\
  m_3 = m_3(1 + m_2) & n_1 = m_2, & n_2 = m_3', n_3 = (m_1 - 1)(m_2 + m_3 + 1), \\
\end{array} \]

\[
(p_{jk})^1 = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 2 & 0 & 0 \\
  0 & 0 & 4 & 0 \\
  0 & 0 & 0 & 4 \\
\end{bmatrix}
\]

\[
(p_{jk})^2 = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 4 & 0 \\
\end{bmatrix}
\]

\[
(p_{jk})^3 = \begin{bmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 2 \\
  1 & 0 & 2 & 0 \\
  0 & 2 & 0 & 2 \\
\end{bmatrix}
\]

\[
(p_{jk})^4 = \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 2 & 0 \\
  0 & 2 & 0 & 2 \\
  1 & 0 & 2 & 0 \\
\end{bmatrix}
\]
We can arrange the treatments in both $D_1$ and $D_2$ in $m_1$ rectangles: $(1 + m_2) \times (1 + m_3)$ side by side horizontally, where two treatments occurring in the same row of a rectangle are first associates in $D_1$ while two treatments occurring in the same column of a rectangle are first associates in $D_2$; two treatments of a rectangle not occurring in a row are second associates in $D_1$, while two treatments not occurring in a column are second associates in $D_2$; two treatments are third associates both in

\[
\begin{align*}
(p_{jk}^3) &= \begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m_1-1)(m_2+m_3+1) \end{bmatrix} \\
\end{align*}
\]

and $D_2: v'=m_1(m_2'+m_3'+1), b', r', k, \lambda'_1, \lambda'_2, \lambda'_3$

$n'_1=m_3, n'_2=m_2', n'_3=(m_1-1)(m_2'+m_3+1), m'_2=m_2 (1+m_3)$

\[
\begin{align*}
(p_{jk}^1) &= \begin{bmatrix} m_3-1 & 0 & 0 \\ 0 & m_2(l+m_3) & 0 \\ 0 & 0 & (m_1-1)(m_2+m_3+1) \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
(p_{jk}^2) &= \begin{bmatrix} 0 & m_3 & 0 \\ m_3 & (m_3+1)(m_2-1) & 0 \\ 0 & 0 & (m_1-1)(m_2'+m_3+1) \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
(p_{jk}^3) &= \begin{bmatrix} 0 & 0 & m_3 \\ 0 & 0 & m_2 \\ m_3 & m_2 & (m_1-1)(m_2+m_3+1) \end{bmatrix} \\
\end{align*}
\]
$D_1$ and $D_2$, if they do not occur in a rectangle. Then by considering $s$ times the blocks of $D_1$ with $t$ times the blocks of $D_2$, we get the solution of the second type of cyclical rectangular design whose parameters are

$$D^*: \quad v^* = m_1(m_2 + m_3 + 1) = m_1(m_2^r + m_3 + 1) \quad b^* = sb + tb^t$$

$$r^* = sr + tr^t \quad k^* = k,$n_1 = s\lambda_1 + t\lambda_1^t \quad \lambda_2^* = s\lambda_2 + t\lambda_2^t$$

$$\lambda_3^* = s\lambda_3 + t\lambda_3^t \quad \lambda_4^* = s\lambda_4 + t\lambda_4^t$$

$$m_1 = m_2(1 + m_2) \quad m_2^r = m_2(m_2^r + m_3 + 1)$$

$$m_1^* = m_2, \quad m_2^* = m_3, \quad m_3 = m_2m_3, \quad m_4^* = (m_1 - 1)(m_2 + m_3 + 1)$$

$$(p_{jk}^1) = \begin{bmatrix}
    m_2 - 1 & 0 & 0 & 0 \\
    0 & 0 & m_3 & 0 \\
    0 & m_3 & m_3(m_2 - 1) & (m_1 - 1)(m_2 + m_3 + 1) \\
    0 & 0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}$$

$$(p_{jk}^2) = \begin{bmatrix}
    0 & 0 & m_2 & 0 \\
    0 & m_3 - 1 & 0 & 0 \\
    m_2 & 0 & m_2(m_3 - 1) & 0 \\
    0 & 0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}$$

$$(p_{jk}^3) = \begin{bmatrix}
    0 & 1 & m_2 - 1 & 0 \\
    1 & 0 & m_3 - 1 & 0 \\
    m_2 - 1 & m_3 - 1 & (m_2 - 1)(m_3 - 1) & 0 \\
    0 & 0 & 0 & (m_1 - 1)(m_2 + m_3 + 1)
\end{bmatrix}$$
Method VIII. Let a two associate P.B.I.B. design exist with the following parameters:

\[
D: v, b, r, k, \lambda_1, \lambda_2, \gamma_1, \gamma_2, (p_{jk}^1) = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix};
\]

\[
(p_{jk}^2) = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\]

With each block of this design, add a set of \(v\) treatments \(v+1, v+2, \ldots, 2v\). With the treatments \(v+1, \ldots, 2v\) let us construct another set of \(b\) blocks of the design \(D\). With each block thus formed, let us add the treatments 1, 2, \ldots, \(v\). The set of \(2b\) blocks thus formed will be a solution of the design \(D^*\) belonging to the second type of generalisation of cyclical design. If \(D\) is triangular, then \(D^*\) will be cyclic triangular; if \(D\) is of the \(L_1\) association class, then \(D^*\) will follow cyclic \(L_1\) association class and if \(D\) is two-associate cyclical design, \(D^*\) will be three-associate cyclical design. The parameters of \(D^*\) will be
Example 1. Consider the cyclical design

\( v = 6, b = 4, r = 2, k = 3, \lambda_1 = 1, \lambda_2 = 0 \)
\( n_1 = 4, n_2 = 1 \)

\[
\begin{pmatrix}
1 & 1 \\
2 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 \\
2 & 0
\end{pmatrix}
\]

whose solution is

\((1, 2, 3), (3, 4, 5), (5, 1, 6), (6, 4, 2)\).

By the above method we get the solution of the design

\( v = 12, b = 8, r = 6, k = 9, \lambda_1 = 5, \lambda_2 = 4, \lambda_3 = 4 \)

\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 6
\end{pmatrix} \quad \begin{pmatrix}
4 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 6
\end{pmatrix}
\]
Example 2. Consider the triangular design

\[ v=10, \ b=5, \ r=2, \ k=4, \ \lambda_1=1, \ \lambda_2=0, \ \eta_1=6, \ \eta_2=3, \]

\[
(p_{jk}^1) = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} ; \quad (p_{jk}^2) = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}
\]

whose solution is

1. \(1, a, a^2, a^3\), 2. \((a^4, a^5, a^6)\), 3. \((a, a^4, a^7, a^8)\), 4. \((a^2, a^5, a^7, a^9)\) and \((a^3, a^6, a^2, a^9)\).

By the above method we get the solution of the cyclic triangular design belonging to the second type of generalisation whose parameters are

\[ v=20, \ b=10, \ r=7, \ k=14, \ \lambda_1=6, \ \lambda_2=5, \ \lambda_3=4, \ \eta_1=6, \ \eta_2=3, \ \eta_3=10, \]

\[ m_1=2, \ m_2=6, \ m_3=3 \]
Corollary 5.8.1 This method can be fruitfully employed for generating four associate cyclic rectangular (O.G.D.) design by taking the design $D$ as a three associate rectangular design.

Method IX. Let a solution exist for the P.B.I.B. design

$$D_1: v, b, r, k, \lambda_1, \lambda_2, n_1, n_2$$

$$\begin{pmatrix}
(p_{jk}^1) = \begin{bmatrix}
3 & 2 & 0 \\
2 & 1 & 0 \\
0 & 0 & 10
\end{bmatrix};
(p_{jk}^2) = \begin{bmatrix}
4 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 10
\end{bmatrix};
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & 6 \\
0 & 0 & 3 \\
6 & 3 & 0
\end{bmatrix}
\end{pmatrix}$$

Let $D_2$ be another solution of $D_1$ with new treatment numbers. Each block of $D_1$ is adjoined to the $b$ blocks of the complete solution $D_2$. Then we get the solution of the design $D^*$ belonging to the second type of generalisation whose parameters are:

$$D^*: v^* = 2v, b^* = b^2, r^* = br, k^* = 2k, \lambda_1^* = b\lambda_1, \lambda_2^* = b\lambda_2,$$

$$\lambda_3^* = r^2, n_1^* = n_1, n_2^* = n_2, n_3^* = v.$$
The nature of $D^*$ will be the same as that of $D_1$. Thus, if
$D_1$ is two-associate cyclical design, then $D^*$ will be three
associate cyclical design, if $D_1$ is two associate $L_1$, then
$D^*$ is three associate cyclic $L_1$ type design, if $D_1$ follow
triangular association scheme, then $D^*$ will follow 3-associate
cyclic triangular association scheme.

Example. Consider for example, the $L_2$-design

$$D_1: v=9, b=6, r=2, k=3, \lambda_1=1, \lambda_2=0$$

$$n_1 = n_2 = 4$$

$$\begin{pmatrix}
 1 & 2 \\
 2 & 2
\end{pmatrix} ;
\begin{pmatrix}
 2 & 2 \\
 2 & 1
\end{pmatrix}$$

whose solution is

1. (1, 2, 3), 2. (4, 5, 6), 3. (7, 8, 9), 4. (1, 4, 7),
5. (2, 5, 8), 6. (3, 6, 9).

Let $D_2$ be another solution of $D_1$ with the treatments 10, 11,
..., 18. Then by the above method we get the solution of the three
associate $L_2$ design belonging to the second type of generalisation
\[ D* : v=18, b=36, r=12, k=6, \lambda_1=6, \lambda_2=0, \lambda_3=4, n_1=n_2=4, n_3=9 \]

\[
(p^1_{jk}) = \begin{bmatrix}
1 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 9
\end{bmatrix};
(p^2_{jk}) = \begin{bmatrix}
2 & 2 & 0 \\
0 & 0 & 9
\end{bmatrix};
(p^3_{jk}) = \begin{bmatrix}
0 & 0 & 4 \\
4 & 4 & 0
\end{bmatrix}
\]

whose layout is

1. 1 2 3 10 11 12 15 7 8 9 10 11 12
2. 1 2 3 13 14 15 14 7 8 9 13 14 15
3. 1 2 3 16 17 18 15 7 8 9 16 17 18
4. 1 2 3 10 13 16 16 7 8 9 10 13 16
5. 1 2 3 11 14 17 17 7 8 9 11 14 17
6. 1 2 3 12 15 18 18 7 8 9 12 15 18
7. 4 5 6 10 11 12 19 1 4 7 10 11 12
8. 4 5 6 13 14 15 20 1 4 7 13 14 15
9. 4 5 6 16 17 18 21 1 4 7 16 17 18
10. 4 5 6 10 13 16 22 1 4 7 10 13 16
11. 4 5 6 11 14 17 23 1 4 7 11 14 17
12. 4 5 6 12 15 18 24 1 4 7 12 15 18
Corollary 5.9.1. If the design $D_1$ is taken as a three associate rectangular design, then by the above procedure we get the solution of a four associate cyclic rectangular design belonging to the second type of generalisation.

Method X. Consider the following association scheme of the first type of generalisation (of any three associate association scheme).

Let $G = G_1(\bar{X})G_2$, $G_1 : 1, c, c^2, \ldots, c^{m_1-1}$;
$$G_2 : 1, a, a^2, \ldots, a^{m_2+m_3}$$

Let $G_2 = A \cup B \cup 1$, when $A$ consists of $m_2$ elements and $B$ consists of $m_3$ elements and further the sets $A$ and $B$ satisfy the conditions of the particular two associate class association scheme. Set the first associates of any treatment $\Theta$ as $\Theta(G_1-1)$, second associates of $\Theta$ as $\Theta G_1(\bar{X})A$ and third associates as $\Theta G_1(\bar{X})B$.

Suppose a group divisible design $D$ exists with the parameters:

$D : v=m_1(l+m_2), m=m_1, l=lm_2, b, r, k, \lambda_1, \lambda_2$

Construct the group divisible design $D$ with elements of $G_1(\bar{X})(1+A)$
as treatments when \((1\oplus A)\) is a sub-group of \(G_2\). Let us take the first associates of any treatment \(\phi \in G_1(\overline{\chi})(1\oplus A)\) as \(\phi (G_1-1)\) and second associates of \(\phi\) as \(\phi G_1(\overline{\chi})A\). This gives a set of \(b\) blocks. Consider \((m_2 + m_3)\) such sets of \(b\) blocks by considering the the elements of \(a_1 \in G_1(\overline{\chi})(1\oplus A)\) as the set of treatments in the \(i\)-th set, \(i = 1, 2, \ldots, m_2 + m_3\). By adjoining these \((m_2+m_3)\) \(b\) blocks with the \(b\) blocks obtained earlier, we get the solution of the design belonging to the first type of generalisation of various three associate designs.

**Example:**

Consider the following three associate cyclical association scheme of the first type of generalisation:

\[
G : 1 = a^4 = b^3; \quad G_1 : 1, b, b^2; \quad A : a, a^3; \quad B : a^2, a = 0, \beta = 2.
\]

Consider the following G.B. design:

\[
D : v=b=9, \quad r=k=4, \quad m=3, \quad n=3, \quad \lambda_1=5, \quad \lambda_2=1,
\]

whose solution is

1. \((0, 1, 3, 6)\), 2. \((1, 2, 4, 7)\), 3. \((2, 3, 5, 8)\), 4. \((3, 4, 6, 0)\)
5. \((4, 5, 7, 1)\), 6. \((5, 6, 8, 2)\), 7. \((6, 7, 0, 3)\), 8. \((7, 8, 1, 4)\)
and 9. \((8, 0, 2, 5)\),

whose groups of first associate treatments are \((0, 3, 6)\), \((1, 4, 7)\) and \((2, 5, 8)\).

Let us form four sets of 9 blocks by forming the 9 blocks of the \(i\)-th set from \(D\) by corresponding the treatments of the design \(D\) as follows:
This gives us the solution of the following design:

\[ v=12, \ n_1=2, \ n_2=6, \ n_3=3 \]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 3 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 3 \\
0 & 3 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 2 \\
0 & 6 & 0 \\
2 & 0 & 0 \\
\end{pmatrix}
\]

whose lay-out is

\[
\begin{align*}
(1, b, a, b^2) & \quad (b, ab, b^2, 1) & \quad (b^2, ab^2, 1, b) \\
(a, a^3, ab, ab^2) & \quad (ab, a^3b, ab^2, a) & \quad (ab^2, a^3b^2, a, ab) \\
(a^3, b, a^3b, a^3b^2) & \quad (a^3b, b^2, a^3b^2, a^3) & \quad (a^3b^2, 1, a^3, a^3b) \\
(a, a^2, ab, ab^2) & \quad (a^3, a^3b, a^3b^2, 1) & \quad (a^2, a^2b, a^2b^2, a^3) \\
(a^2, 1, a^2b, a^2b^2) & \quad (1, b, b^2, a^2) & \quad (a^3, a^3b, a^3b^2, a) \\
(1, ab, b, b^2) & \quad (a^3b, a^2, a^2b, a^2b^2) & \quad (a^2b, a, ab, ab^2)
\end{align*}
\]
Corollary 5.10.1  
If a G.D. design exists with parameters

\[ v = m_1(l + m_3), \ b, \ r, \ k, \ m = m_1, \ m = l + m_3, \ \lambda_1, \ \lambda_2 \]

then by constructing \((m_2 + m_3 + 1)\) designs as explained above we get the solution of the design

\[ v^* = m_1(m_2 + m_3 + 1), \ b^* = (m_2 + m_3 + 1)b, \ r^* = (l + m_3)r, \]

\[ k^* = k, \ \lambda_1^* = (l + m_3)\lambda_1, \ \lambda_2^* = 0, \ \lambda_3^* = (l + m_3)\lambda_2 \]

The \(p_{ijk}\) parameters will depend on the association scheme.

Method XI.  Suppose a resolvable two-associate P.B.I.B. design exists with \(v = r\). With every block of the \(i\)-th replicate let us add a new treatment \(v+1, i = 1, 2, ..., r\) and thus obtain \(b\) blocks. Next, we obtain another set of \(b\) blocks from the original resolvable P.B.I.B. design in this way:
Let us remain the $j$-th treatment of the design as $(v+j)$, and then with the blocks of the $i$-th replicate, thus formed, let us add a new treatment $i$, $i = 1, 2, \ldots, r$. This set of 24 blocks give us a cyclical P.B.I.B design of the second type. The association class of the cyclical three associate design will be the same as the association class of the two associate design. If the parameters of the two associate design are:

$$v, b, r = v, k, \lambda_1, \lambda_2, n_1, n_2,$$

then the parameters of the cyclic design will be

$$v^* = 2v, b^* = 2b, r^* = r + b, k^* = k + 1, \lambda_1^* = \lambda_1, \lambda_2^* = \lambda_2$$

$$\lambda_3^* = 1, n_1^* = n_1, n_2^* = n_2, n_3^* = v$$

\[
\begin{pmatrix}
\begin{array}{ccc}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]
Example. Consider for example the G.D. design \( v=6, b=18, k=2, \lambda_1=0, \lambda_2=2, m=2, n=3 \) whose solution is

\[
\begin{align*}
(1) & \quad 1 & 4 & (4) & 1 & 4 & (7) & 1 & 5 & (10) & 1 & 6 & (13) & 1 & 5 & (16) & 1 & 6 \\
(2) & \quad 2 & 5 & (5) & 2 & 5 & (8) & 2 & 6 & (11) & 2 & 4 & (14) & 2 & 6 & (17) & 2 & 4 \\
(3) & \quad 3 & 6 & (6) & 3 & 6 & (9) & 3 & 4 & (12) & 3 & 5 & (15) & 3 & 4 & (18) & 3 & 5
\end{align*}
\]

By applying the above method we get the design

\[ v*=12, \ b*=36, \ r*=9, \ k*=3, \ \lambda_1^* = 0, \ \lambda_2^* = 2, \ \lambda_3^* = 1 \]

\[
(p_{jk}^1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad (p_{jk}^3) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}
\]

whose solution is

\[
\begin{align*}
(1) & \quad 1 & 4 & 7; \quad (2) & \quad 2 & 5 & 7; \quad (3) & \quad 3 & 6 & 7; \quad (4) & \quad 1 & 4 & 8; \quad (5) & \quad 2 & 5 & 8; \\
(7) & \quad 1 & 5 & 9; \quad (8) & \quad 2 & 6 & 9; \quad (9) & \quad 3 & 4 & 9; \quad (10) & \quad 1 & 6 & 10; \quad (11) & \quad 2 & 4 & 10; \\
(12) & \quad 3 & 5 & 10; \quad (13) & \quad 1 & 5 & 11; \quad (14) & \quad 2 & 6 & 11; \quad (15) & \quad 3 & 4 & 11; \quad (16) & \quad 1 & 6 & 12; \\
(17) & \quad 2 & 4 & 12; \quad (18) & \quad 3 & 5 & 12; \quad (19) & \quad 7 & 10 & 1; \quad (20) & \quad 8 & 11 & 1; \quad (21) & \quad 9 & 12 & 1; \\
(22) & \quad 7 & 10 & 2; \quad (23) & \quad 8 & 11 & 2; \quad (24) & \quad 9 & 12 & 2; \quad (25) & \quad 7 & 11 & 2; \quad (26) & \quad 8 & 12 & 2; \\
(27) & \quad 9 & 10 & 2; \quad (28) & \quad 7 & 12 & 3; \quad (29) & \quad 8 & 10 & 3; \quad (30) & \quad 9 & 11 & 3; \quad (31) & \quad 7 & 11 & 4; \\
(32) & \quad 8 & 12 & 4; \quad (33) & \quad 9 & 10 & 4; \quad (34) & \quad 7 & 12 & 5; \quad (35) & \quad 8 & 10 & 5; \quad (36) & \quad 9 & 11 & 5
\end{align*}
\]
Method XII:
Suppose a P.B.I.B design exists with the parameters

\[ D_1: v = m_2 + m_3 + l, \quad b, \quad r, \quad k, \quad \lambda_1, \quad \lambda_2, \quad n_1 = m_2, \quad n_2 = m_3 \]

\[
(p_{jk}^1) = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & v
\end{bmatrix}
\]

and a B.I.B. design exists with the parameters

\[ D_2: v' = m_1 (m_2 + m_3 + l), \quad b', \quad r', \quad k' = k, \quad \lambda' \]

Let us form \( m_1 \) sets of blocks of \( D_1 \) (call it \( D_1' \)) by replacing the treatments 1, 2, ..., \( v \) in \( D_1 \) by \((i-1)v+1, (i-1)v+2, ..., iv\) in the \( i \)-th set, \( i = 1, 2, ..., m_1 \). With these \( m_1 \) \( b \) blocks add the \( b' \) blocks of \( D \) to get the solution of the design

\[ v^* = m_1 (m_2 + m_3 + l), \quad b^* = m_1 b + b', \quad r^* = r + r', \quad k^* = k, \]

\[ \lambda'_1 = \lambda_1 + \lambda, \quad \lambda'_2 = \lambda_2 + \lambda, \quad \lambda'_3 = \lambda \]

\[ n'_1 = n_1, \quad n'_2 = n_2, \quad n'_3 = (m_1-1)(m_2 + m_3 + l) \]

\[
(p_{jk}^2) = \begin{bmatrix}
2 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & v
\end{bmatrix}
\]
The association class of the three-associate cyclical design of the second type will be the same as that of the associate class.

**Example 1.** Consider for example the following G.D. design with the parameters:

\[ D_1: v=8, b=4, r=3, k=3, n_1=1, n_2=6, m=4, \lambda_1=3, \lambda_2=1 \]

\[
\begin{bmatrix}
0 & 0 & n_1 \\
0 & 0 & n_2 \\
n_1 & n_2 & (m_1-2)(m_2+m_3+1)
\end{bmatrix}
\]

whose solution as given by Bose, Glatworthy and Shrikhande is

\[
\begin{array}{cccccc}
(1) & 1 & 5 & 2 & 7 & 3 & 6 \\
(2) & 2 & 6 & 1 & 5 & 4 & 3 \\
(3) & 3 & 7 & 4 & 8 & 1 & 5 \\
(4) & 4 & 8 & 2 & 6 & 3 & 7 \\
\end{array}
\]

Let us form 4 more blocks by housing the treatment 8+i in block j if the i-th treatment occurs in j-th block of \( D_1 \). This gives the following 4 blocks:

\[
\begin{array}{cccccccc}
(5) & 9 & 13 & 10 & 15 & 11 & 14 \\
(6) & 10 & 14 & 9 & 13 & 12 & 16 \\
(7) & 11 & 15 & 12 & 16 & 9 & 13 \\
(8) & 12 & 16 & 10 & 14 & 11 & 15 \\
\end{array}
\]
With these 8 blocks let us join the 16 blocks of the B.I.B. design (Cochran and Cox P 525 Plan 13, 9) \( v=b=16, r=k=6, \lambda=2 \) whose solution is

\[
\begin{align*}
(1) & \quad (1 2 3 4 5 6) \\
(2) & \quad (1 2 7 8 9 10) \\
(3) & \quad (1 3 7 11 12 13) \\
(4) & \quad (1 4 8 11 14 15) \\
(5) & \quad (1 5 9 12 14 16) \\
(6) & \quad (1 6 10 13 15 16) \\
(7) & \quad (2 3 7 14 15 16) \\
(8) & \quad (2 4 8 12 13 16)
\end{align*}
\]

and

\[
\begin{align*}
(9) & \quad (2 5 9 11 13 15) \\
(10) & \quad (2 6 10 11 12 14) \\
(11) & \quad (3 4 9 10 11 16) \\
(12) & \quad (3 5 8 10 12 15) \\
(13) & \quad (3 6 8 9 13 14) \\
(14) & \quad (4 5 7 10 13 14) \\
(15) & \quad (4 6 7 9 12 15) \\
(16) & \quad (5 6 7 8 11 16)
\end{align*}
\]

to get the solution of the design \( v=18, b=24, k=6, r=9, \lambda_1=5, \lambda_2=3, \lambda_3=2, n_1=1, n_2=6, n_3=8 \).

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} ; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 6 \\ 1 & 6 & 0 \end{bmatrix}
\]

**Example 2.**

Consider the triangular design

\( D_1: v=10, b=6, r=5, k=5, \lambda_1=1, \lambda_2=2 \)
whose solution is

1. (2, 3, 5, 7, 8); 2. (0, 4, 5, 7, 9); 3. (1, 2, 6, 7, 9); 4. (1, 3, 4, 8, 9); 5. (0, 1, 3, 5, 6); 6. (0, 2, 4, 6, 8)

and the B.I.B. design $D$: $v=20$, $b=26$, $k=5$, $r=19$, $\lambda=4$

whose solution (as given in Takeuchi (1962) is obtained by developing the initial blocks $(\infty, 0, 2, 3, 7)$, $(0, 1, 4, 9, 11)$, $(0, 2, 3, 7, 13)$, $(0, 4, 5, 7, 13) \pmod{19}$.

We obtain a second set of 6 blocks from the design $D'$ (by adding 10 as

7. (12, 13, 15, 17, 18); 8. (16, 14, 15, 17, \infty); 9. (11, 12, 16, 17, \infty); 10. (11, 13, 14, 16, \infty); 11. (10, 11, 13, 15, 16); and 12. (10, 12, 14, 16, 18).

Joining these 12 blocks with the solution of the B.I.B. design we get the solution of the design of the 2nd generalisation of triangular type whose parameters are

$v=20$, $b=35$, $k=5$, $r=22$, $\lambda_1=5$, $\lambda_2=6$, $\lambda_3=4$, $n_1=6$, $n_2=3$, $n_3=10$

$$(p_{jk}^1) = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$
Method XIII

If the design D in (XII) is a P.B.I.B. design of the cyclical type (2nd) with parameters
\[ v', b', r', k' = k, \lambda_1', \lambda_2', \lambda_3', n_1' = n_1, n_2' = n_2, n_3' = n_1 + n_2 + 1 \]

whose association rule for the \( n_1 \) sets (of D) are the same as that in \( D_1 \), then also we shall get a P.B.I.B. design of the second type with the association scheme as in \( D \) and parameters as
\[ v^* = v' = m_1(m_2 + m_3 + 1), \ b^* = m_1 b + b', \ r^* = r + r', \ k^* = k = k' \]
\[ \lambda_1^* = \lambda_1 + \lambda_1', \lambda_2^* = \lambda_2 + \lambda_2', \lambda_3^* = \lambda_3' \]

Example. Consider for example the \( L_2 \) design \( D_1: v = 9, b = 6, r = 4, k = 2, \lambda_1 = 2, \lambda_2 = 3, n_1 = 4, n_2 = 4 \)

\[ (p_{jk}^1) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \ \ (p_{jk}^2) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \] whose solution is

\begin{align*}
1. & \quad (1, a^2, ab, a^2b, b^2, ab^2) \\
2. & \quad (1, a^2, b, ab, ab^2, a^2b^2) \\
3. & \quad (a, a, ab, a^2b, b^2, a^2b^2) \\
4. & \quad (1, a, b, a^2b, ab^2, a^2b^2) \\
5. & \quad (a, b, a^2b^2, a^2b, ab^2) \\
6. & \quad (a, b, a^2b^2, ab, a^2b^2)
\end{align*}

Association Scheme

\[
\begin{array}{ccc}
1 & a & a^2 \\
b & ab & a^2b \\
b^2 & ab^2 & a^2b^2 \\
\end{array}
\]
and the 3-associate design

\[ D; v = b = 18, \ n_1 = n_2 = 4, \ n_3 = 9, \ r = k = 6, \ \lambda_1 = 3, \ \lambda_2 = 0, \ \lambda_3 = 2 \]

\[
\begin{bmatrix}
1 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 9
\end{bmatrix} \quad ; \quad \begin{bmatrix}
2 & 2 & 0 \\
2 & 1 & 0 \\
0 & 0 & 9
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 4 \\
0 & 0 & 4 \\
4 & 4 & 0
\end{bmatrix}
\]

Association Scheme:

\[
\begin{array}{cccc}
1 & a & a^2 & c \\
b & ab & a^2b & bc \\
b^2 & ab^2 & a^2b^2 & b^2c \\
\end{array}
\begin{array}{cccc}
c & ac & a^2c & ab \\
bc & abc & a^2bc & ab^2 \\
b^2c & ab^2c & a^2b^2c & \end{array}
\]

whose solution is obtained by developing the initial block \((1, a, a^2, bc, b^2c, c)\).

Consider another set of 6 blocks derived from the solution of \(D_1\):

(7) \((c, a^2c, abc, a^2bc, b^2c, ab^2c)\)
(8) \((c, a^2c, bc, abc, a^2bc, b^2c)\)
(9) \((c, ac, abc, a^2bc, b^2c, a^2b^2c)\)
(10) \((c, ac, bc, a^2bc, ab^2c, a^2b^2c)\)
(11) \((ac, bc, a^2c, b^2c, a^2bc, ab^2c)\)
(12) \((ac, bc, a^2c, b^2c, abc, a^2b^2c)\)
Considering these sets of 30 blocks together, we get the solution of the design

\[ v = 30, b = 30, r = 10, k = 6, \lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 2 \]

\[ n_1 = n_2 = 4, \quad n_3 = 9 \]

\[
\begin{pmatrix}
1 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 9
\end{pmatrix}
\quad \begin{pmatrix}
2 & 2 & 0 \\
2 & 1 & 0 \\
0 & 0 & 9
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 4 \\
0 & 0 & 4 \\
4 & 4 & 0
\end{pmatrix}
\]

**Method XIV**

Let the following two P.B.I.B. designs exist.

1. **D:** \( v, b, r, k, n_1, n_2, \lambda_1, \lambda_2 \)

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\quad \begin{pmatrix}
2 & 2 \\
2 & 2 \\
2 & 2
\end{pmatrix}
\]

and (2) \( D_2: v' = m_1(n_1 + n_2 + 1), \quad b', r', k' = k, \quad \lambda_1', \lambda_2' \)

when \( n_1' \) and \( n_2' \) can have the following possible alternatives.

1. \( n_1' = n_1, \quad n_2' = (m_1 - 1)(n_1 + n_2 + 1) + n_2 \)
2. \( n_1' = n_2, \quad n_2' = (m_1 - 1)(n_1 + n_2 + 1) + n_1 \)
3. \( n_1' = (m_1 - 1)(n_1 + n_2 + 1) + n_2, \quad n_2' = n_1 \)
4. \( n_1' = (m_1 - 1)(n_1 + n_2 + 1) + n_1, \quad n_2' = n_2 \)
where in (i) the $p_{jk}^i$ parameters should be of the form

$$
(p_{jk}^1) = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
P_{11} & 1 \\
P_{12} & 1 \\
P_{12} & 1
\end{bmatrix} + (m_l - 1)(n_1 + n_m + 1)
$$

$$
(p_{jk}^2) = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
P_{11} & 2 \\
P_{12} & 2 \\
P_{12} & 2
\end{bmatrix} + (m_l - 1)(n_1 + n_m + 1)
$$

Let us form $m_1$ sets of blocks of $D_1$ by replacing the treatments $1, 2, \ldots, v$ in $D_1$ by $(i-1)v+1, (i-1)v+2, \ldots$, $iv$ in the $i$-th set, $i=1, 2, \ldots, m_1$. With these $m_1b$ blocks add the $b'$ blocks of $D_2$ to get the solution of the design $B_s$.

$$
D*: \quad v^* = m_1(m_2 + m_3 + 1), \quad b^* = m_1b+b', \quad r^* = r = r^3,
$$

$$
k^* = k, \quad \lambda_1^* = \lambda_1^* + \lambda_1', \quad \lambda_2^* = \lambda_2 + \lambda_2', \quad \lambda_3^* = \lambda_2',
$$

$$
n_1^* = n_1, \quad n_2^* = n_2, \quad n_3^* = (m_1 - 1)(m_2 + m_3 + 1)
$$

$$
(p_{jk}^1) = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & n_3^* \\
\end{bmatrix}; \quad (p_{jk}^2) = \begin{bmatrix}
2 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & n_3^* \\
\end{bmatrix}
$$

$$
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & n_1 \\
0 & 0 & n_2 \\
0 & n_1 & n_2 (m_1 - 2) (n_1 + n_2 + 1) \\
\end{bmatrix}
$$
The association class of the second type of cyclic design $D^*$ will be the same as two associate class design $D_1$.

**Example.** Consider the design $D_1$ and $D_2$ as

$D_1$: $v=9$, $b=6$, $r=2$, $k=3$, $\lambda_1=0$, $\lambda_2=1$, $n_1=n_2=4$

\[
(p^1_{jk}) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad (p^2_{jk}) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}
\]

and $D_2$: $v=18$, $b=36$, $r=6$, $k=3$, $n_1=4$, $n_2=13$, $\lambda_1=3$, $\lambda_2=0$

\[
(p^1_{jk}) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad (p^2_{jk}) = \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}
\]

The solution of $D_2$ is obtained by developing the initial blocks $(1, a, a^2)$ and $(1, b, b^2)$ when $a^3 = b^3 = c^2 = 1$; while the blocks of $D_1$ can be written as

$(1, ab, a^2b^2)$, $(1, ab^2, a^2b)$, $(a, b, a^2b^2)$, $(a, b^2, a^2b)$

$(a^2, b, ab^2)$, $(a^2, ab, b^2)$. With these 42 blocks we add the 6 blocks $(c, abc, a^2bc)$, $(c, ab^2c, a^2bc)$, $(ac, bc, a^2bc)$

$(ac, b^2c, a^2bc)$, $(a^2c, bc, ab^2c)$ to get the solution of the design $D^*$: $v=18$, $b=48$, $r=3$, $n_1=n_2=4$, $n_3=9$, $\lambda_1=3$, $\lambda_2=1$, $\lambda_3=0$

\[
(p^1_{jk}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad (p^2_{jk}) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}
\]
Here, the nature of the design $D_1$ is $L_2$ type, so the resultant design $D^*$ will belong to the three associate class cyclic $L_2$ design of the second type of generalisation.

Corollary 5.14.1. This method can also be utilised for developing four associate class cyclic rectangular (C.G.D.) design belonging to the second type of generalisation.

Method XV. Consider a three associate cyclic triangular association scheme of the first type of generalisation between $m_1(n-1)/2$ treatments arranged in $n(n-1)$ cells of $m_1$ treatments each in the form of a square array whose main diagonal is vacant and where the scheme is symmetrical about the main diagonal.

Let a G.D. design $D^*$ exist with the parameters

$$D^* : v^* = m_1(n-1), b^*, r^*, k^*, \lambda_1^*, \lambda_2^*, m^* = n-1, n^* = m_1.$$ 

Let us correspond these $v^*$ treatments with the $v^*$ treatments appearing in any row (or column) of the cyclical triangular association scheme discussed above. These give us $b^*$ blocks whose treatments are now the treatments appearing in a particular row (or particular column) of the cyclical triangular association scheme. Corresponding the treatments of $D^*$ in turn (after renaming) with the treatments in $n$ distinct rows (or $n$ distinct columns) we shall get $n$ sets of $b^*$ blocks. Considering these $nb^*$ blocks we get the solution of the cyclical triangular design belonging to the
second type of generalisation whose parameters are

\[ v = m_1 n(n-1)/2, \quad b = nb^*, \quad r = 2r^*, \quad k = k^* \]

\[ \lambda_1 = 2 \lambda_1^*, \quad \lambda_2 = \lambda_2^*, \quad \lambda_3 = 0, \quad m_1 \]

\[ m_2 = 2(n-2), \quad m_3 = \frac{1}{2} (n-2)(n-3) \]

\[ n_1 = m_1 - 1, \quad n_2 = 2(n-2)m_1, \quad n_3 = \frac{1}{2} (n-2)(n-3)m_1 \]

\[
(p^1_{jk}) = \begin{bmatrix}
m_1 - 2 & 0 & 0 \\
0 & m_1 m_2 & 0 \\
0 & 0 & m_1 m_3 
\end{bmatrix}
\]

\[
(p^2_{jk}) = \begin{bmatrix}
0 & m_1 - 1 & 0 \\
m_1 - 1 & m_1(n-2) & m_1(n-3) \\
0 & m_1(n-3) & m_1(n-3)(n-4)/2 
\end{bmatrix}
\]

\[
(p^3_{jk}) = \begin{bmatrix}
0 & 0 & m_1 - 1 \\
0 & 4m_1 & 2m_1(n-4) \\
m_1 - 1 & 2m_1(n-4) & m_1(n-4)(n-5)/2 
\end{bmatrix}
\]

**Example 1.** Consider the association scheme.

<table>
<thead>
<tr>
<th></th>
<th>(1, 2)</th>
<th>(3, 4)</th>
<th>(5, 6)</th>
<th>(7, 8)</th>
<th>(9, 10)</th>
<th>(11, 12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td>*</td>
<td>(13, 14)</td>
<td>(15, 16)</td>
<td>(17, 18)</td>
<td>(19, 20)</td>
<td>(21, 22)</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>(13, 14)</td>
<td>*</td>
<td>(23, 24)</td>
<td>(25, 26)</td>
<td>(27, 28)</td>
<td>(29, 30)</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>(15, 16)</td>
<td>(23, 24)</td>
<td>*</td>
<td>(31, 32)</td>
<td>(33, 34)</td>
<td>(35, 36)</td>
</tr>
<tr>
<td>(7, 8)</td>
<td>(17, 18)</td>
<td>(25, 26)</td>
<td>(31, 32)</td>
<td>*</td>
<td>(37, 38)</td>
<td>(39, 40)</td>
</tr>
<tr>
<td>(9, 10)</td>
<td>(19, 20)</td>
<td>(27, 28)</td>
<td>(33, 34)</td>
<td>(37, 38)</td>
<td>*</td>
<td>(41, 42)</td>
</tr>
<tr>
<td>(11, 12)</td>
<td>(21, 22)</td>
<td>(29, 30)</td>
<td>(35, 36)</td>
<td>(39, 40)</td>
<td>(41, 42)</td>
<td>*</td>
</tr>
</tbody>
</table>
Again, consider the G.D. design $v^* = b^* = 12$, $r^* = k^* = 4$, $m^* = 6$, $n^* = 2$, $\lambda_1^* = 2$, $\lambda_2^* = 1$ whose solution is obtained by developing the initial block $(0, 1, 4, 6) \mod 12$ (Bose, Shrikhande and Bhattacharjya, 1953) and Whose first associate groups are $(0, 6)$, $(1, 7)$, $(2, 8)$, $(3, 9)$, $(4, 10)$, $(5, 11)$.

Form 7 distinct G.D. designs by considering the set of 12 treatments which lie on the cells in the same row of the association scheme as the set of 12 treatments of the G.D. design and by forming first associate groups with treatments which appear in the same cell. This gives the solution of the design:

$$v = 42, \quad b = 48, \quad r = 8, \quad k = 4, \quad m_1 = 2, \quad m_2 = m_3 = 10$$

$$\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = 0$$

$$n = 7, \quad n_1 = 1, \quad n_2 = n_3 = 20$$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 20 & 0 \\
0 & 0 & 20
\end{bmatrix} = (p^1_{jk}), \quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 10 & 8 \\
0 & 8 & 12
\end{bmatrix} = (p^2_{jk})$$

$$\begin{bmatrix}
0 & 0 & 1 \\
0 & 8 & 12 \\
1 & 12 & 6
\end{bmatrix} = (p^3_{jk})$$

Example 2. Consider the cyclical triangular association scheme between 56 treatments arranged in a $8 \times 8$ square array whose main diagonal is vacant and each cell contains two treatments.
Consider the G.D. design

\[ v=14, \ b=7, \ r=3, \ k=6, \ \lambda_1=3, \ \lambda_2=1, \ m=7, \ n=2. \]

By the above procedure we get the solution of the three associate design \( v = b = 56, \ r = k = 6, \ \lambda_1 = 6, \ \lambda_2 = 1, \ \lambda_3 = 0, \ m_1 = 2, \ m_2 = 12, \ m_3 = 15, \ n_1 = 1, \ n_2 = 24, \ n_3 = 30, \ n = 8 \)

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 24 & 0 \\
0 & 0 & 30
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 12 & 10 \\
0 & 10 & 20
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 8 & 16 \\
1 & 16 & 12
\end{pmatrix}
\]

Example 3. Considering the above association scheme between 30 treatments having \( m_1=2 \) and \( n=6 \) and the G.D. design \( v = 10, \ b=6, \ r=4, \ k=5, \ \lambda_1=0, \ \lambda_2=2 \) whose solution is

\( (1, 3, 5, 7, 9), \ (2, 4, 6, 8, 9), \ (1, 3, 6, 8, 10), \ (2, 4, 5, 7, 10) \)

we get the solution of the design \( v=30, \ b=48, \ r=8, \ k=5, \ \lambda_1=0, \ \lambda_2=2, \ \lambda_3=0, \ m_1=2, \ m_2=3, \ m_3=6, \ n_1=1, \ n_2=16, \ n_3=12, \)

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 16 & 0 \\
0 & 0 & 12
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 3 & 6 \\
0 & 6 & 6
\end{pmatrix}
\]
Consider a group divisible design $D: v = m_1 (m_2 + m_3 + 1)$,

$$n = m_2 + m_3 + 1, \quad m = m_1, \quad b, r, k, \lambda_1, \lambda_2(\cdot)$$

and a P.B.I.B. design $D': \ v' = m_2 + m_3 + 1, \ n_1 = m_2, \ n_2 = m_3, \ b', r', \ k' = k, \ \lambda_1', \lambda_2', \ p'_{jk}$ of a specified type of association scheme.

Form $m_1$ different sets of $b'$ blocks by considering the treatments in $D'$ as a set of $n$ treatments of $D$ (This is possible because of lemma 5.2.4). To the $m_1 b'$ blocks thus formed add the blocks of $D$. The resultant design will be a three associate design of the second type with parameters:

$$D*: v^* = m_1 (m_2 + m_3 + 1), \ b^* = m_1 b' + b, \ r^* = r + r', \ k^* = k,$$

$$\lambda_1^* = \lambda_1 + \lambda_1', \ \lambda_2^* = \lambda_2 + \lambda_1', \ \lambda_3^* = \lambda_2$$

$$(p^1_{jk}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & (m_1 - 1)(m_2 + m_3 + 1) \end{bmatrix}$$

$$(p^2_{jk}) = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & (m_1 - 1)(m_2 + m_3 + 1) \end{bmatrix}$$
(p_{j,k}^3) = \begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m_1-2)(m_2+m_3+1) \end{bmatrix}

The three associate cyclic association class of $D^*$ will be the same as that of two associate $D'$.

**Example:** Consider the G.D. design $D$: $v=24$, $b=72$, $r=9$, $k=3$, $m=4$, $n=6$, $\lambda_1=0$, $\lambda_2=1$ whose solution is obtained by developing the blocks

$$(0, 1, 11), (0, 2, 7), (0, 3, 9) \pmod{24}$$

(Bose, Shrikhande and Bhattacharjya 1953) and the cyclical P.B.I.B. design

$D'$: $v=6$, $b=4$, $r=2$, $k=3$, $\lambda_1=1$, $\lambda_2=0$, $n_1=4$, $n_2 = 1$,

$$(p_{j,k}^1) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}; \quad (p_{j,k}^2) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

whose solution is

$$(0, 1, 5), (2, 1, 3), (3, 4, 5) \text{ and } (0, 2, 4)$$

Now, the 4 groups of $D$ are $(0, 4, 8, 12, 16, 20), (1, 5, 9, 13, 17, 21), (2, 6, 10, 14, 18, 22)$ and $(3, 7, 11, 15, 19, 23)$.

By considering the treatments in a group as the treatments of $D'$, we can form the following set of 16 blocks:
Adding these 16 blocks to the blocks of \( D \) we get the solution of the three associate cyclical design belonging to the second type of generalisation having parameters:

\[
\begin{align*}
\lambda_1 &= 1, \\
\lambda_2 &= 0, \\
\lambda_3 &= 1, \\
n_1 &= 4, \\
n_2 &= 1, \\
n_3 &= 18
\end{align*}
\]

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 18
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 18
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 4 \\
0 & 0 & 1 \\
4 & 1 & 12
\end{bmatrix}
\]

**Example 2.** Consider the group divisible design

\( D: \nu=24, \ b=60, \ r=10, \ k=4, \ m=3, \ n=8, \lambda_1=2, \lambda_2=1 \)

and the cyclical P.B.I.B. design

\( D': \nu=6, \ r=4, \ n_1=6, \ n_2=1, \lambda_1=2, \lambda_2=0 \)

\[
\begin{bmatrix}
4 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & 0 \\
0 & 0
\end{bmatrix}
\]
whose solution is

1. (1,2,3,4)  2. (1,3,5,7)  3. (3,4,5,6)  4. (1,4,5,8)
5. (1,2,7,8)  6. (2,4,6,8)  7. (5,6,7,8)  8. (2,3,6,7)

The solution of the design D as given by Bose, Shrikhande and Bhattacharjiya (1953) may be conveniently obtained by developing the initial blocks \((1, a, a^4, a^9)\) and \((1, a^2, a^5, a^9)\) over the elements of the group \(G\) (i.e., multiplying these two blocks successively by the elements of \(G\)) and the initial block \((1, a, a^4, a^9)\) over the elements of \(G_2\) when \(G\) is formed by \(a\) and \(c\) when 
\[a^{12} = c^2 = 1.\]

By applying method XVI we get the solution of the three associate cyclical design belonging to the second type of generalization whose parameters are \(v=24, b=84, r=14, k=4, \lambda_1=4, \lambda_2=2, \lambda_3=1, n_1=6, n_2=1, n_3=16\).

\[
\begin{bmatrix}
4 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 16
\end{bmatrix}
\begin{bmatrix}
6 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 16
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 6 \\
0 & 0 & 1 \\
6 & 1 & 8
\end{bmatrix}
\]

The three groups of treatments \(G_1, G_2, G_3\) whose treatments are either first or second associate within the group but third associates between the groups may be conveniently written as
Method XVII

Suppose the following two three associate cyclic designs \( D \) and \( D' \) (both belonging to the same type of generalisation and having the same association scheme) exist.

\[
\begin{align*}
D: & \quad v = m_1(m_2 + m_3 + 1), \quad b, r, k, \lambda_1, \lambda_2, \lambda_3, n_1, n_2, n_3 \\
D': & \quad v' = m_1(m_2 + m_3 + 1), \quad b', r', k' = k, \lambda_1', \lambda_2', \lambda_3', n_1' = n_1, \\
& \quad n_2' = n_2, \quad n_3' = n_3
\end{align*}
\]

Thereby adding a replicate of \( D \) with \( t \) replicate of \( D' \) we get the solution of the design \( D^* \) whose parameters are

\[
\begin{align*}
D^*: & \quad v^* = v, \quad b^* = sb + tb', \quad r^* = sr + t r', \\
& \quad k^* = k, \quad \lambda_i^* = s \lambda_i + t \lambda_i', \quad i = 1, 2, 3.
\end{align*}
\]

Example: Consider the first type of cyclical association scheme between the elements of the group \( G \), where \( G: a^4 = c^2 = 1 \). Association scheme: \( A: a, a^3, B: a^2, G_1: 1, c \). First associate of any element \( \theta \) of \( G \) is \( \theta c \), second associate of \( \theta \) is \( \theta A(\overline{x})G_1 \) and third associate of \( \theta \) is \( \theta B(\overline{x})G_1 \). Consider the solution of the design \( D \) whose parameters are

\[
\begin{align*}
D: & \quad v = 3 = b, \quad r = k = 3, \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0 \\
& \quad n_1 = 1, \quad n_2 = 4, \quad n_3 = 2, \quad m = 2 = m_2, \quad m_3 = 1
\end{align*}
\]
whose lay-out is obtained by the development of the initial block 
\((1, a, c)\) by successively multiplying this block by the elements 
of the group \(G\).

Consider \(D'\) as follows:

\(D':\) \(\nu = b = 8, \ r = k = 5, \lambda_1 = \lambda_2 = 2, \lambda_2 = 0\)

\(n_1 = 1, \ n_2 = 4, \ n_3 = 2, \ m_1 = m_2 = 2, \ m_3 = 1\)

\[
(p_{jk})^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} ; \quad (p_{jk})^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}
\]

\[
(p_{jk})^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

whose solution is obtained by developing the initial block 
\((1, a^2, a^2c)\) by successively multiplying this initial block by 
the elements of \(G\).
By considering the blocks of $D$ and $D'$ together, we get the solution of the three associate cyclical design having the same association scheme as $D$ or $D'$, but now the parameters of the design become

$v=8, b=16, r=6, k=3, \lambda_1=4, \lambda_2=1, \lambda_3=2$

$n_1=1, n_2=4, n_3=2, m_1=m_2=2, m_3=1$

$$
(p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} ; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

**Corollary 5.17.1.** Suppose a three associate cyclic P.B.I.B. design $D$ (of either type with any association scheme) exists with parameters

$D: v=m_1(m_2+m_3+1), b, r, k, \lambda_1, \lambda_2, \lambda_3, n_1, n_2, n_3,$

$\lambda^i_{jk}, \ i, j, k = 1, 2, 3.$

Let a B.I.B. design $D'$ exist with parameters

$D': v'=m_1(m_2+m_3+1), b', r', k' = k, \lambda'.$

Then by joining $s$ times the blocks of $D$ with $t$ times the blocks of $D'$ we get the solution of the cyclic design $D^*$ of the
same type and having the same association scheme as \( D \). The parameters of \( D^* \) will be

\[
\begin{align*}
D^*: & \quad v^* = m_1(m_2 + m_3 + 1), \quad b^* = sb + tb', \quad r^* = sr + tr', \\
& \quad k^* = k, \quad \lambda_1^* = s\lambda_1 + t\lambda_1', \quad p_{jk}^i \quad \text{parameters}
\end{align*}
\]

The same as in \( D \).

**Proof:** The proof follows easily by considering the fact that a B.I.B. design may be considered to be a P.B.I.B. design with any association scheme having \( \lambda_1 = \lambda_2 = \lambda_3 \).

**Example:** Consider the three associate cyclical design of the first type of generalisation

\( D: v = 10, b = 5, r = 2, k = 4, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0, \)

\( m_1 = 1, n_2 = 4, n_3 = 4, m_1 = m_2 = m_3 = 2 \).

\[
(p_{jk}^1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \quad (p_{jk}^2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}; \quad (p_{jk}^3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}
\]
whose solution is 1. (1, 2, 3, 4), 2. (3, 4, 5, 6), 3. (5, 6, 7, 8)
4. (7, 8, 9, 10), 5. (9, 10, 1, 2) and the B.I.B. design

\[ D': \quad v=10, \quad b=15, \quad r=6, \quad k=4, \quad \lambda=2 \]
whose solution (Cochran and Cox (1957) Plan 11.16) is

1. (1, 2, 3, 4), 2. (1, 2, 5, 6), 3. (1, 3, 7, 8), 4. (1, 4, 9, 10),
5. (1, 5, 7, 9), 6. (1, 5, 7, 9), 7. (2, 3, 6, 9), 8. (2, 4, 7, 10),
9. (2, 5, 8, 10), 10. (3, 5, 9, 10), 11. (3, 6, 7, 10),
12. (2, 7, 8, 9), 13. (3, 4, 5, 8), 14. (4, 5, 6, 7), 15. (4, 6, 8, 9)
together the solution of the three associate cyclical design of the
first type of generalisation having the parameters:

\[ D*: \quad v^*=10, \quad b^*=20, \quad r^*=8, \quad k^*=4, \quad \lambda_1^*=4, \quad \lambda_2^*=3, \quad \lambda_3^*=2 \]

\[ n_1^*-l, \quad n_2^*-n_3^*-4, \quad m_1^*-m_2^*-m_3^*-2. \]

**Corollary 5.17.2.** Suppose a three associate cyclical P.B.I.B.
[...]

\[ D: \quad v=m_1(m_2+m_3+1), \quad b, \quad r, \quad k, \quad \lambda_1, \quad \lambda_2, \quad \lambda_3, \quad n_1=m_1-1, \quad n_2=m_1 m_2 \]

\[ n_3=m_1 m_3 \]

\[ (p^1_{jk}) = \begin{bmatrix}
    m_1-2 & 0 & 0 \\
    0 & m_1 m_2 & 0 \\
    0 & 0 & m_1 m_3
\end{bmatrix}, \quad (p^2_{jk}) = \begin{bmatrix}
    0 & m_1-1 & 0 \\
    0 & p^{2}_{22} & p^{2}_{23} \\
    0 & p^{2}_{23} & p^{2}_{33}
\end{bmatrix} \]
and a two associate P.B.I.B. design $D'$ exists with parameters:

$$D': v'= m_1 (m_2 + m_3 + 1), \ b', r', k'= k, \lambda_1', \lambda_2', \ n_1', \ n_2'= n_2 + n_3$$

$$\begin{bmatrix}
0 & 0 & m_1 - 1 \\
0 & p_{22}^3 & p_{23}^3 \\
m_1 - 1 & p_{23}^3 & p_{33}^3 
\end{bmatrix}$$

Then by considering $s$ times the blocks of $D$ with $t$ times the blocks of $D'$ we get the solution of the design $D^*$ of the same type and having the same association scheme as $D$.

$$D^*: \ v^* = m_1 (m_2 + m_3 + 1), \ b^* = s b + t b', \ r^* = s r + t r'$$

$$k^* = k, \ \lambda_1^* = s \lambda_1 + t \lambda_1', \ \lambda_2^* = s \lambda_2 + t \lambda_2'$$

$$\lambda_3^* = s \lambda_3 + t \lambda_2', \ n_1^* = n_1, \ n_2^* = n_2, \ n_3^* = n_3$$
Proof: Proof follows easily by considering the facts that (5.17.2.1) and (5.17.2.2) together with \( \lambda_2 = \lambda_3 \) are exactly the conditions laid down by Rao (1947) and Vartak (1955) that a three associate P.B.I.B. design having \( \lambda_2 = \lambda_3 \) will reduce to a 2-associate P.B.I.B. design.

Example. Consider the abelian group \( G \) formed by \( a \) and \( b \) when \( a^2 = b^5 = 1 \). Let \( G_1 : 1, a; A : b, b^4, B : b^2, b^3 \)

\( n_1=1, n_2 = 4, \lambda_3 = 4, m_1 = m_2 = m_3 = 2 \)

\[
(p^{1}_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \quad (p^{2}_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}
\]

Consider the initial block \((1, b, ba)\). On developing this block we get the solution of the design
D: b=10, r=3, k=3, \lambda_1=2, \lambda_2=1, \lambda_3=0

Again consider the initial blocks (1, b^4, ab^4) and (1, b, b^2).
On multiplying these two initial blocks successively by the
elements of the group \( G \), we get the solution of the design

\[ D': \ v'=10, \ b'=20, \ r'=3, \ k'=3, \ \lambda_1'=4, \ \lambda_2'=1 \]
\[ n_1^4 = 1, \ n_2^4 = 8 \]

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 0 \\ 0 & 8 \\ 0 & 0 \end{bmatrix}; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix};
\]

One can easily verify that the conditions (5.17.2.1) and (5.17.2.2)
are being satisfied. Hence considering the blocks of \( D \) and \( D' \)
together we get the solution of the design

\[ D^*: \ v^*=10, \ b^*=30, \ r^*=9, \ k^*=3, \ \lambda_1^*=6, \ \lambda_2^*=2, \ \lambda_3^*=1 \]

\[
(p^1_{jk}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \quad (p^2_{jk}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix};
\]

\[
(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}
\]
REFERENCES


