CHAPTER FOUR

ANALYSIS OF THREE ASSOCIATE DESIGNS
OBTAINED BY CYCLICAL GENERALISATION
OF VARIOUS ASSOCIATION SCHEME

4.1 Introduction

The analysis of partially balanced incomplete block designs was discussed by Bose and Nair (1939) considering only the intra-block information. Methods of recovering inter-block information have been given by Nair (1944). These methods of analysis have been further improved by Bose (1950), Rao (1947) and others.

Our object, here, is to simplify, as much as possible, the above methods for the particular classes of three associate designs obtained by cyclical generalisation of various association schemes discussed in the earlier chapters. We shall start with the somewhat simplified form given by Nair (1952) for the general three associate designs.

4.2 Combinatorial conditions of cyclic P.B.I.B. designs

We shall assume that \( v \) treatments are being tried in \( b \) blocks of size \( k \) each and each variety is replicated \( r \) times, any two treatments which are \( i \)-th associates occur together in \( \lambda_i \) blocks, \( i = 1, 2, 3 \).
Different three associate designs discussed earlier have the following parameters of the first kind.

First type of generalisation

\[ v = m_1(m_2 + m_3 + 1), \quad n_1 = m_1 - 1, \quad n_2 = m_1 m_2, \quad n_3 = m_1 m_3 \quad (4.2.1) \]

\[ (p_{jk}^1) = \begin{bmatrix}
 m_1 - 2 & 0 & 0 \\
 0 & m_1 m_2 & 0 \\
 0 & 0 & m_1 m_3
\end{bmatrix} \quad (4.2.2) \]

\[ (p_{jk}^2) = \begin{bmatrix}
 0 & m_1 - 1 & 0 \\
 m_1 - 1 & p_{22}^2 & p_{23}^2 \\
 0 & p_{23}^2 & p_{33}^2
\end{bmatrix} \quad (4.2.3) \]

\[ (p_{jk}^3) = \begin{bmatrix}
 0 & 0 & m_1 - 1 \\
 0 & p_{22}^3 & p_{23}^3 \\
 m_1 - 1 & p_{23}^3 & p_{33}^3
\end{bmatrix} \quad (4.2.4) \]

where \[ \sum_{k=2,3} p_{jk}^1 = m_1(m_j + 1) \] or \[ m_j \] according as \( i=j \) or \( i \neq j \) \[ (4.2.5) \]

Second type of generalisation

\[ v = m_1(m_2 + m_3 + 1), \quad n_1 = m_2, \quad n_2 = m_3, \quad n_3 = (m_1 - 1)(m_2 + m_3 + 1) \quad (4.2.6) \]
\( (p_{jk}^1) = \begin{bmatrix}
1 & 1 \\
1 & p_{12} \\
0 & 0 \\
\end{bmatrix} \) \( (m_j-1)(m_2+m_3+1) \) (4.2.7)

\( (p_{jk}^2) = \begin{bmatrix}
2 & 2 \\
2 & p_{12} \\
0 & 0 \\
\end{bmatrix} \) \( (m_j-1)(m_2+m_2+1) \) (4.2.8)

\( (p_{jk}^3) = \begin{bmatrix}
0 & 0 & m_2 \\
0 & 0 & m_3 \\
m_2 & m_3 & (m_j-2)(m_2+m_3+1) \\
\end{bmatrix} \) (4.2.9)

where \( \sum_{k=1,2} p_{jk}^i = (m_j+1-1) \) of \( m_j+1 \) according as

\( i = j \) or \( i \neq j \) (4.2.10)

It may be noted that the unspecified \( p_{jk}^i \) parameters take different values as determined in earlier chapters for the different classes of cyclical designs.

4.3 Intra block analysis of cyclic P.B.I.B. designs.

Let \( y_{ij} \) be the yield of the \( j \)-th treatment in \( i \)-th block.

Let us consider the additive model

\[ y_{ij} = \mu + b_i + t_j + \varepsilon_{ij} \] (4.3.1)
where \( \mu \) = general mean yield,

\[ b_i = i\text{-th block effect (constant or random)}, \]

\[ t_j = j\text{-th treatment effect (constant)} \]

\[ e_{ij} = \text{the error component which is assumed to be} \]

normally distributed with mean zero and variance \( \sigma^2 \); which are considered uncorrelated with other

\[ e_{ij} \] and also with \( b_i \).

Let \( T_j \) stand for the total value of \( y_{ij} \) for the \( j\)-th treatment;

\[ B_i \] stand for the total value of \( y_{ij} \) for the \( i\)-th block and \( G \)

for the grand total of \( y_{ij} \). We define

\[ Q_j = T_j - \frac{1}{r} \text{[Sum of the} r \text{block totals in which} j\text{-th treatment occurs]} \]

\[ = T_j - V_j / r \quad \ldots \quad \ldots \quad \ldots \quad (4.3.2) \]

Let \( \hat{\mu}, \hat{b}_i \) and \( \hat{t}_j \) denote the estimates of \( \mu, b_i \) and \( t_j \)

obtained by minimising

\[ \sum_{ij} (y_{ij} - \mu - b_i - t_j)^2 \quad (4.3.3) \]

subject to the restrictions \( \sum b_i = 0, \sum t_j = 0 \) \( (4.3.4) \)

It is known (Nair, 1952) that for a general three associate

P.B.I.B. design the estimates \( \hat{t}_j \) are given by

\[
t_j = k \quad \begin{vmatrix} Q_j & B_{13} & C_{13} \\ \sum Q_{j1} & B_{23} & C_{23} \\ \sum Q_{j2} & B_{33} & C_{33} \end{vmatrix} \begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{23} & B_{23} & C_{23} \\ A_{33} & B_{33} & C_{33} \end{vmatrix} \] \( (4.3.5) \)
where \( \Sigma Q_{j1} \) stand for the total adjusted treatment totals \( (Q_j) \) of the \( n_1 \) treatments which form the first associates of the \( j \)-th treatment, \( \Sigma Q_{j2} \) corresponding quantities for the second associates, etc.

\[
A_{13} = r(k-1)+\lambda_3 \\
A_{23} = (\lambda_3-\lambda_1)(n_1-p_{11}^3)-(\lambda_3-\lambda_2)p_{12}^3 \\
A_{33} = (\lambda_3-\lambda_2)(n_2-p_{22}^3) - (\lambda_3-\lambda_1)p_{12}^3 \\
B_{13} = \lambda_3 - \lambda_1 \\
B_{23} = r(k-1)+\lambda_3+(\lambda_3-\lambda_1)(p_{11}^1-p_{11}^3)+(\lambda_3-\lambda_2)(p_{12}^1-p_{12}^3) \\
B_{33} = (\lambda_3-\lambda_1)(p_{12}^1-p_{12}^3)+(\lambda_3-\lambda_2)(p_{22}^1-p_{22}^3) \\
C_{13} = \lambda_3 - \lambda_2 \\
C_{23} = (\lambda_3-\lambda_1)(p_{11}^2-p_{11}^3)+(\lambda_3-\lambda_2)(p_{12}^2-p_{12}^3) \\
C_{33} = r(k-1)+\lambda_3+(\lambda_3-\lambda_1)(p_{12}^2-p_{12}^3)+(\lambda_3-\lambda_2)(p_{22}^2-p_{22}^3)
\]

As shown by Bose and Nair (1939), the sum of squares due to treatments (eliminating block effects) is given by

\[
\sum_{j=1}^{r} \hat{t}_j Q_j
\]

The sum of squares due to blocks (eliminating treatment effects) can be expressed as
\[
\frac{V}{\sum_{j=1}^{v} \hat{t}_j Q_j + \frac{1}{k} \sum_{i=1}^{b} B_i^2} \quad \frac{1}{b-1} \sum_{i=1}^{b} B_i^2 = \frac{V}{\sum_{j=1}^{V} \hat{t}_j Q_j} \quad (4.3.8)
\]

The residual or intra-block error sum of squares is

\[
\sum_{i,j} y_{ij}^2 = \sum_{i=1}^{v} \hat{t}_j Q_j - \frac{1}{k} \sum_{i=1}^{b} B_i^2 \quad (4.3.9)
\]

The treatment effects can be tested for significance using the variance ratio \( F = (\text{M.S. due to treatments})/(\text{Error M.S.}) \).

**Table 1. Analysis of variance.**

<table>
<thead>
<tr>
<th>Variation due to</th>
<th>D.F.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F-ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (Ignoring treatments)</td>
<td>b-1</td>
<td>( \frac{1}{k} \sum B_i^2 ) ( \frac{G^2}{N} )</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Treatments (Eliminating blocks)</td>
<td>v-1</td>
<td>( \sum \hat{t}_j Q_j )</td>
<td>T</td>
<td>( T/E = F_{v-1, b-v+1} )</td>
</tr>
<tr>
<td>Error</td>
<td>N-b-v+1</td>
<td>*</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>N-1</td>
<td>( \sum y_{ij}^2 ) ( \frac{G^2}{N} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* By subtraction.

The variance \( V_1 \) of difference between two first associates is given by \( 2\sigma_k^2 \) times the value of right hand side of (4.3.5) when the elements of the first column of the determinant in the numerator are replaced by 1, -1, 0 respectively.
Similarly, $V_2$ is obtained when we replace this column by 1, 0, -1 and $V_3$ when we replace this by 1, 0, 0.

**First type of generalisation**

Consider $\Delta_c = \begin{bmatrix} A_{13} & B_{13} & C_{13} \\ A_{23} & B_{23} & C_{23} \\ A_{33} & B_{33} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{13} & -B_{13} & B_{13} & C_{13} \\ A_{23} & -B_{23} & B_{23} & C_{23} \\ A_{33} & -B_{33} & B_{33} & C_{33} \end{bmatrix}$

\[
\begin{bmatrix}
 r(k-1)+\lambda_1 & B_{13} & C_{13} \\
 -[r(k-1)+\lambda_1] & B_{23} & C_{23} \\
 0 & B_{33} & C_{33}
\end{bmatrix} = \begin{bmatrix}
 1 & B_{13} & \lambda_1 \\
 -1 & B_{23} & C_{23} \\
 0 & B_{33} & C_{33}
\end{bmatrix}
\]

\[
= [r(k-1)+\lambda_1], \Delta_1 \quad \cdots \quad \cdots \quad \cdots \quad (4.3.10)
\]

where $\Delta_1 = \begin{bmatrix} 1 & B_{13} & C_{13} \\ -1 & B_{23} & C_{23} \\ 0 & B_{33} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{13}+B_{23} & \lambda_1+C_{23} \\ 0 & B_{33} & C_{33} \end{bmatrix}$

\[
= m_1m_2\lambda_2+m_1(1+m_3)\lambda_3+(\lambda_3-\lambda_2)(p_{22}^2-p_{22}^3)]m_1m_2\lambda_2+m_1(1+m_2\lambda_3)]
\]

\[
-m_1(\lambda_3-\lambda_2)^2 p_{23}^3
\]

\[
= A^2 + A \cdot B - m_1^2 p_{23}^3 \quad \cdots \quad \cdots \quad \cdots \quad (4.3.11)
\]
where $\Lambda = m_1 [m_2 \lambda_2 + (1+m_3)\lambda_3] \cdots \cdots \cdots (4.3.12)$

$B = (\lambda_3 - \lambda_2) (p_{22}^2 - p_{22}^3)$

$\Lambda_1 = \lambda_3 - \lambda_2 \cdots \cdots \cdots (4.3.14)$

Now, 

\[ \begin{array}{ccc}
Q \quad B_{13} \quad C_{13} \\
\Sigma Q_{j1} B_{23} C_{23} = Q_j + \Sigma Q_{j1} B_{13} + B_{23} C_{23} + C_{13} \\
\Sigma Q_{j2} B_{33} C_{33} 
\end{array} \]

\[ = q_j \cdot \Delta_1 - (q_j + \Sigma Q_{j1}) [B_{13} C_{33} - C_{13} B_{33}] + \Sigma Q_{j2} [B_{13} C_{23} + C_{13} B_{23}] \]

\[ = \Delta_1 q_j + (q_j + \Sigma Q_{j1}) [(\lambda_3 - \lambda_2)^2 p_{23}^3 + 2(\lambda_3 - \lambda_1)^2 - (\lambda_3 - \lambda_1) m_1 \{m_2 \lambda_2 + (1+m_3)\lambda_3\}] \\
- (\lambda_3 - \lambda_1) [(\lambda_3 - \lambda_2) (p_{23}^3 - p_{23}^2 - m_1)] \\
+ \Sigma Q_{j2} \{m_1 (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_2) - (\lambda_3 \lambda_2) m_1 [m_2 \lambda_2 + (1+m_3)\lambda_3]\} \]

\[ = \Delta_1 q_j + (q_j + \Sigma Q_{j1}) [\lambda_2^2 p_{23}^3 + \lambda_1^2 - \lambda_1 A - \lambda_1 B] + \Sigma Q_{j2} \{m_1 \lambda_1 \lambda_2 \lambda_3\} \]

where $\Lambda_1 = \lambda_3 - \lambda_1 \cdots \cdots \cdots (4.3.16)$

where

\[ p = (q_j + \Sigma Q_{j1}) [\lambda_2^2 p_{23}^3 + \lambda_1^2 - \lambda_1 A - \lambda_1 B] + \Sigma Q_{j2} \{m_1 \lambda_1 \lambda_2 \lambda_3\} \]

\[ (4.3.17) \]
From (4.3.5), (4.3.10) and (4.3.17).

\[ t_j = \frac{k}{r(k-1)+\lambda_1} Q_j + \frac{k p}{[r(k-1)+\lambda_1] \Delta_1} \]  

(4.3.18)

It may be noted that this estimate depends on \( p_{23} \) and \( p_{23} \) and on no other \( p_{ij} \).

The variance of treatment differences for any two treatments which are first associates is \( \frac{2k^2 \sigma_k^2}{r(k-1)+\lambda_1} \). That between any two treatments which are second associates is \( \frac{2k \sigma_k^2 \Delta_2}{r(k-1)+\lambda_1} \) and between any two treatments which are third associates,

\[ \frac{2k \sigma_k^2 \Delta_3}{r(k-1)+\lambda_1} \]

where \( \Delta_2 = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{23} & C_{23} \\ -1 & B_{33} & C_{33} \end{vmatrix} \)

\[ = B_{23} (C_{13} + C_{33}) - C_{23} (B_{13} + B_{33}) \]

where \( B_{13} = \lambda_1 \)

\[ B_{23} = r(k-1)+\lambda_3+(-\lambda_1)(m_1-2)=(\lambda_1-\lambda_3)+m_1 [m_2\lambda_2+(1+m_3)\lambda_0] \]

\[ = A \]

\[ B_{33} = \lambda_2 \sigma_k^2 \]

\[ C_{13} = -1 \]

\[ C_{23} = \lambda_1(m_1-1) \]

\[ C_{33} = (m_1-1)+m_1 m_2 \lambda_2 + \lambda_5 m_3 \lambda_6 + \lambda_3 + (\lambda_5-\lambda_1)(m_1-1) \]

\[ = A + B \]
\[ \Delta_2 = (A + B + \Lambda_1)(A + \Lambda_1) - \Lambda_1^2(m_1-1)(\Lambda_1 + A_2 B_{23}) \]  

(4.3.19)

and \[ \Delta_3 = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{23} & C_{23} \\ 0 & B_{33} & C_{33} \end{vmatrix} \]

\[ = (A + B)(A + \Lambda_1) - \Lambda_1^2(m_1-1)B_{23}^3 \]  

(4.3.20)

The average variance is

\[ \bar{V} = \frac{2k\sigma_k^2}{[r(k-1)+\Lambda_1](v-1)} \left[ (m_1-1)m_2\frac{\Delta_2}{\Lambda_1} + m_1m_3\frac{\Delta_3}{\Lambda_1} \right] \]

(4.3.21)

Hence, efficiency factor

\[ \frac{[r(k-1)+\Lambda_1]^3(v-1)}{r_k\bar{V}} \left[ (m_1-1)\frac{\Delta_1}{\Lambda_1} + m_1m_2\frac{\Delta_2}{\Lambda_1} + m_1m_3\frac{\Delta_3}{\Lambda_1} \right] \cdot \Lambda_1 \]  

(4.3.22)

Second type of generalisation

Here we may simplify as follows:

\[ \Lambda_{13} = r(k-1)+\Lambda_2 = R(\text{say}) \]

\[ \Lambda_{23} = n_1(\Lambda_3-\Lambda_1) = n_1 \Lambda_1 \text{ (say)} \]

\[ \Lambda_{33} = n_2(\Lambda_3-\Lambda_2) = n_2 \Lambda_2 \text{ (say)} \]

\[ B_{13} = \Lambda_1 \]

\[ B_{23} = R + \Lambda_1 p_{11} + \Lambda_2 p_{12} \]

\[ B_{33} = \Lambda_1 p_{12} + \Lambda_2 p_{22} \]
\[ c_{13} = \lambda_2 \]
\[ c_{23} = \lambda_1 p_{11} + \lambda_2 p_{12} = n_1 \lambda_1 + (\lambda_2 - \lambda_1) p_{12} \quad (4.3.23) \]
\[ c_{33} = R + \lambda_1 p_{12} + \lambda_2 p_{22} \]

So, \[ \Delta_0 = \begin{vmatrix} \lambda_{13} & B_{13} & C_{13} \\ \lambda_{23} & B_{23} & C_{23} \\ \lambda_{33} & B_{33} & C_{33} \end{vmatrix} \]

\[ = \begin{vmatrix} R & \lambda_1 & \lambda_2 \\ n_1 \lambda_1 & R + \lambda_1 p_{11} + \lambda_2 p_{12} & \lambda_1 p_{11} + \lambda_2 p_{12} \\ n_2 \lambda_2 & \lambda_1 p_{12} + \lambda_2 p_{22} & R + \lambda_1 p_{12} + \lambda_2 p_{22} \end{vmatrix} \]

\[ =(R + n_1 \lambda_1 + n_2 \lambda_2) \begin{vmatrix} R & \lambda_1 & \lambda_2 \\ n_1 \lambda_1 & R + \lambda_1 p_{11} + \lambda_2 p_{12} & \lambda_1 p_{11} + \lambda_2 p_{12} \\ 1 & 1 & 1 \end{vmatrix} \]

Adding first two rows to the third row and taking \((R + n_1 \lambda_1 + n_2 \lambda_2) \)
common from the third row, \( \Delta_0 \) can be expressed as

\[ v\lambda_3 [(R - \lambda_1)(R - \lambda_2) - R(\lambda_1 - \lambda_2)(p_{12} - \lambda_1 p_{12}) + (\lambda_1 - \lambda_2)(\lambda_2 p_{12} - \lambda_1 p_{12})] \quad (4.3.24) \]
Since \( R + n_1 \Lambda_1 + n_2 \Lambda_2 = \nu \Lambda_3 \) \hfill (4.3.25)

\[
\begin{bmatrix}
Q_j & B_{13} & C_{13} \\
\Sigma Q_{j1} & B_{23} & C_{23} \\
\Sigma Q_{j2} & B_{33} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\nu \Lambda_3 \\
\nu \Lambda_3 \\
\nu \Lambda_3
\end{bmatrix}
= \begin{bmatrix}
Q_j & B_{13} & C_{13} \\
\Sigma Q_{j1} & B_{23} & C_{23} \\
\Sigma Q_{j2} & B_{33} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\nu \Lambda_3 \\
\nu \Lambda_3 \\
\nu \Lambda_3
\end{bmatrix}
= \nu \Lambda_3 \begin{bmatrix}
Q_j & B_{13} & C_{13} \\
\Sigma Q_{j1} & B_{23} & C_{23} \\
\Sigma Q_{j2} & B_{33} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\nu \Lambda_3 \\
\nu \Lambda_3 \\
\nu \Lambda_3
\end{bmatrix}
\]

\[
= \nu \Lambda_3 \left[ (k-1) + \lambda_3 + (\lambda_3 - \lambda_1) \left\{ n_1 - \frac{1}{p_{12}} - n_1 + \frac{2}{p_{12}} \right\} + (\lambda_3 - \lambda_2) \left( \frac{1}{p_{12} - p_{12}} \right) \right] Q_j
\]

\[
+ \nu \Lambda_3 (\lambda_2 - \lambda_1) \Sigma Q_{j1} + \nu \Lambda_3 (\lambda_3 - \lambda_1) \Sigma Q_{j2}
\]

\[
= \nu \Lambda_3 \left[ (R - \Lambda_1) - (\lambda_1 - \lambda_2) \left( \frac{1}{p_{12}} - \frac{2}{p_{12}} \right) \right] Q_j + \nu \Lambda_3 (\lambda_1 - \lambda_2) \Sigma Q_{j1}
\]

\[
+ [\lambda_2 (R - \Lambda_1) - n_1 \Lambda_1 (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_2) \left( \frac{1}{p_{12} - p_{12}} \right) ] \Sigma Q_{j2}
\]

\[
\text{Let } S = (\lambda_1 - \lambda_2) \left( \frac{1}{p_{12}} - \frac{2}{p_{12}} \right) \hfill (4.3.26)
\]

\[
H = (\lambda_1 - \lambda_2) \left( \frac{1}{p_{12} - p_{12}} \right) \hfill (4.3.27)
\]

\[
\hat{\varphi}_j = \frac{(R - \Lambda_1) \Sigma Q_{j1} + (\lambda_1 - \lambda_2) \Sigma Q_{j2} + [\lambda_2 (R - \Lambda_1) - n_1 \Lambda_1 (\lambda_1 - \lambda_2) - H] \Sigma Q_{j3}}{(R - \Lambda_1) (R - \Lambda_2) - RS + H} \hfill (4.3.28)
\]
Here also, the estimates depend only on $p_{12}$ and $p_{12}^2$ and on no other $p_{jk}$.

Let $A_1 = \begin{vmatrix} 1 & B_{13} & C_{12} \\ -1 & B & C \\ 0 & B_{33} & C_{33} \end{vmatrix} = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{13} + B_{23} + B_{33} & C_{13} + C_{23} + C_{33} \\ 0 & B_{33} & C_{33} \end{vmatrix}$

$$= \left( R + n_1 \wedge_1 + n_2 \wedge_2 \right) (C_{33} - B_{33}) = \nu \Lambda_3 (R - S - \Lambda_2) \quad (4.3.29)$$

Let $A_2 = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{23} & C_{23} \\ -1 & B_{33} & C_{33} \end{vmatrix} = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{23} & C_{23} \\ 0 & B_{13} + B_{23} + B_{33} & C_{13} + C_{23} + C_{33} \end{vmatrix}$

$$= \left( R + n_1 \wedge_1 + n_2 \wedge_2 \right) (B_{23} - C_{23}) = \nu \Lambda_3 (R - S - \Lambda_1) \quad (4.3.30)$$

Let $A_3 = \begin{vmatrix} 1 & B_{13} & C_{13} \\ 0 & B_{23} & C_{23} \\ 0 & B_{33} & C_{33} \end{vmatrix} = B_{23} C_{33} - C_{23} B_{33}$

Consider $\begin{vmatrix} B_{23} - n_1 \wedge_1 & C_{23} - n_1 \wedge_1 \\ B_{33} - n_2 \wedge_2 & C_{33} - n_2 \wedge_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & B_{23} - n_1 \wedge_1 & C_{23} - n_1 \wedge_1 \\ 0 & B_{33} - n_2 \wedge_2 & C_{33} - n_2 \wedge_2 \end{vmatrix}$

$$= \frac{1}{\nu \Lambda_3} \begin{vmatrix} R + n_1 \wedge_1 + n_2 \wedge_2 & B_{13} + B_{23} + B_{33} & C_{13} + C_{23} + C_{33} \\ n_1 \wedge_1 & B_{23} & C_{23} \\ n_2 \wedge_2 & B_{33} & C_{33} \end{vmatrix}$$
\[ \frac{\Delta_3}{\sqrt{\lambda_3}} = \Delta_3 - n_1 \Lambda_1 (c_{33} - b_{33}) - n_2 \Lambda_2 (b_{23} - c_{23}) \]

\[ = \Delta_3 - n_1 \Lambda_1 (R - S - \Lambda_2) - n_2 \Lambda_2 (R - S - \Lambda_1) \]

\[ = \frac{\Delta_3}{\sqrt{\lambda_3}} + n_1 \Lambda_1 (R - S - \Lambda_2) + n_2 \Lambda_2 (R - S - \Lambda_1) \]

\[ \therefore \Delta_3 = \frac{\Delta_3}{\sqrt{\lambda_3}} + n_1 \Lambda_1 (R - S - \Lambda_2) + n_2 \Lambda_2 (R - S - \Lambda_1) \]  \hspace{1cm} (4.3.31)

The variance of treatment differences for any two treatments which are first associates is \(2k \sigma_k^2 \cdot \frac{\Delta_1}{\Delta_o}\), for any two treatments which are second associates, the variance is \(2k \sigma_k^2 \cdot \frac{\Delta_2}{\Delta_o}\) and those for any two treatments which on third associates, the variance is \(2k \sigma_k^2 \cdot \frac{\Delta_3}{\Delta_o}\).

Hence we calculate the average variance as

\[ \bar{v} = \frac{1}{v - 1} \left[ m_2 \Lambda_1 + m_3 \Lambda_2 + (m_1 - 1)(m_2 + m_3 + 1) \Lambda_3 \right] \cdot \frac{2k \sigma_k^2}{\Delta_o} \] \hspace{1cm} (4.3.32)

and efficiency factor = \(\frac{(v - 1) \Lambda_o}{r \cdot k} \cdot \left[ m_2 \Lambda_1 + m_3 \Lambda_2 + (m_1 - 1)(m_2 + m_3 + 1) \Lambda_3 \right]^{-1}\)

\hspace{1cm} (4.3.33)

4.4 Recovery of inter-block information

We have so far considered the block effects to be constant. We now consider the blocks to be random variables with mean zero and standard deviation \(\sigma_1\). Let us define \(P_1 = w Q_1 + w' Q_t\), where \(w = \frac{1}{k}\) and \(w' = \frac{v(r - 1)}{k(b - 1)B - (v - k)E}\) where \(E\) is the error variance and \(B\) is the variance of the block effects.
\( Q_i = \text{sum of means of blocks in which the } i\text{-th treatment occurs minus } r \times \text{the grand mean.} \)

\[
\frac{V_i}{k} = \frac{rG_i}{N} \quad \ldots \quad \ldots \tag{4.4.1}
\]

**First type of generalisation**

Then the combined inter and intra-block estimate for the first type of generalisation is

\[
t_j = k \times \frac{P_i \Delta'_1 + P' \Delta'_2}{\Delta'_s} \quad \ldots \quad \ldots \tag{4.4.2}
\]

where

\[
\Delta'_1 = (w-w')^2 \Delta_1 + (w-w')rw' (2A+B) + (rw')^2 \tag{4.4.3}
\]

\[
\Delta'_2 = \left\{ r(k+1)+\lambda_1 \right\} (w-w') + rw' \right\} \Delta'_1 \quad \ldots \tag{4.4.4}
\]

\[
P' = [P_1 + \Sigma P_{ji} ] (w-w') \left\{ \lambda_1^2 2 + 2 \lambda_1 - \lambda_1 A - \lambda_1 B \right\} (w-w')
\]

\[
+ rw' \lambda_1 \right\}
\]

\[
+ \Sigma P_{ji} (w-w') \left\{ \lambda_1^2 \lambda_2 = \lambda_2 A (w-w') - rw' \lambda_2 \right\} \tag{4.4.5}
\]

The variance between any two treatments which are first associates is \( \frac{2k \Delta'_1}{\Delta'_s} \), that between any two treatments which are second associates is \( \frac{2k \Delta'_2}{\Delta'_s} \) and between any two treatments which are third associates is \( \frac{2k \Delta'_3}{\Delta'_s} \);

where

\[
\Delta'_2 = \Delta_2 (w-w')^2 + (w-w')rw' [2A+B - \lambda_1 + \lambda_2] + (rw')^2 \tag{4.4.6}
\]
and
\[ \Delta_3 = (w-w')^2 \left[ (A+B)(A-\Lambda_1) - \Lambda_2^2 (m_1-1)p_2^3 \right] + rkw'(w-w')(2A+B-\Lambda_1) + (rkw')^2 \]  
(4.4.7)

**Second type of generalisation**

Here we have
\[
t_j = \frac{k}{\Delta_0^2} \left\{ A \left\{ \frac{R}{\Lambda_1} - S \right\} (w-w') + rkw' \right\} + A(\Lambda_1 - \Lambda_2)(w-w') + \sum \left\{ \Lambda_2 (R-\Lambda_1) - n_2 \Lambda_1 (\Lambda_2 - \Lambda_1) - S \right\} (w-w')^2 + rkw'(w-w')(\Lambda_2 - n_2 \Lambda_1) \right\} \]  
... (4.4.8)

where \( A = n_3 (w-w') + rkw'. \)

\[
\Delta'_0 = \left[ \frac{n_3 (w-w') + rkw'}{\Lambda_3} \right] \left( \frac{w-w'}{\Lambda_3} \right)^2 \]  
(4.4.9)

\[
\Delta'_1 = A \left\{ (R-\Lambda_1^2) (w-w') + rkw' \right\} \]  
... (4.4.10)

\[
\Delta'_2 = A \left\{ (R-\Lambda_2^2) (w-w') + rkw' \right\} \]  
... (4.4.11)

\[
\Delta'_3 = \Delta'_3 (w-w')^2 + rkw'(w-w')(n_3 - \Lambda_1 - \Lambda_2 - S) + (rkw')^2 \]  
(4.4.12)

As in the first type of generalisation, here also, the variance between any two treatments which are first associates is \( \frac{2k \Delta'_1}{\Delta'_0} \), that between any two treatments which are second associates is \( \frac{2k \Delta'_2}{\Delta'_0} \) and between any two treatments which are third associates is \( \frac{2k \Delta'_3}{\Delta'_0} \).
4.5 Numerical examples

Example 1. (First type):

Consider the following design of the cyclic type

\[ v=8, \quad r=k=3 \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0 \]
\[ n_1 = 1, \quad n_2 = 4, \quad n_3 = 2, \quad \alpha = 0, \quad \beta = 2 \]
\[ m_1 = 2, \quad m_2 = 2, \quad m_3 = 1 \]

\[
(p_{jk}^1) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 2
\end{bmatrix}; \quad (p_{jk}^2) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 2 \\
0 & 2 & 0
\end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & 1 \\
0 & 4 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

whose treatments are the elements of the group

\[ G : a^4 = c^2 = 1, \quad \text{where} \ G_1 : 1, \ c ; \ A : a, \ a^3 ; \ B : a^2. \]

The lay-out and yield (fictitious) for 8 treatments are given in table 4.5.1.
Table 4.5.1

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Treatments and yields</th>
<th>Block Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41.1 a 40.2 ac 34.3</td>
<td>115.6</td>
</tr>
<tr>
<td>2</td>
<td>46.3 a 2 39.4 a 2 c 43.2</td>
<td>128.9</td>
</tr>
<tr>
<td>3</td>
<td>38.0 a 3 42.1 a 3 c 35.1</td>
<td>113.2</td>
</tr>
<tr>
<td>4</td>
<td>40.9 a 5 37.0 a 37.5</td>
<td>115.4</td>
</tr>
<tr>
<td>5</td>
<td>27.1 a 33.1 ac 29.1</td>
<td>89.3</td>
</tr>
<tr>
<td>6</td>
<td>36.4 a ac 35.5 a 2 c 38.3</td>
<td>110.2</td>
</tr>
<tr>
<td>7</td>
<td>38.8 a 2 c 39.8 a 3 c 30.5</td>
<td>109.1</td>
</tr>
<tr>
<td>8</td>
<td>43.8 a 38.0 a 3 c 31.3</td>
<td>113.1</td>
</tr>
</tbody>
</table>

To start with, the constants defined in (4.3.11) to (4.3.20) are calculated.

\[
A = 4, \quad B = 4, \quad \lambda_1 = -2, \quad \lambda_2 = -1, \\
r(k-1) + \lambda_1 = 3, \quad \Lambda_1 = 32, \quad \Lambda_2 = 40, \quad \Lambda_3 = 48 \\
P = 3 [ (Q_j + \Sigma Q_{j1}) + \Sigma Q_{j2} ] 
\]

Hence, variance between any two treatments which are first associates is 0.75 \( \sigma_k^2 \), between any two treatments which are second associates is 0.94 \( \sigma_k^2 \) and between any two treatments which are third associates is 1.25 \( \sigma_k^2 \).

From (4.3.22), efficiency factor = 0.6914.
We next calculate different sum of squares.

We have correction factor (C.F.) = \((\sum y_{ij})^2/N = 33561.126\)

Total sum of squares = \(\sum y_{ij}^2 - (\sum y_{ij})^2/N = 533.774\)

Sum of squares due to blocks (ignoring treatments) = \(\sum T_j^2/k - C.F. = 279.840\)

Sum of squares due to treatments (ignoring blocks) = \(\sum T_j^2/(\sum y_{ij})^2/N = 264.310\)

Sum of squares due to treatments (eliminating blocks) = \(\sum \hat{T}_j Q_j = 215.144\)

The analysis of variance leading to test of significance based on intra-block estimates is shown in the following table.
### Table 4.5.3.

**Analysis of variance**

<table>
<thead>
<tr>
<th>Variation due to</th>
<th>D.F.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F-ratio Tabulated $F_{7,21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (Ignoring treatments)</td>
<td>7</td>
<td>279.840</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments (Eliminating blocks)</td>
<td>7</td>
<td>215.144</td>
<td>30.7349</td>
<td>$F_{01} = 3.65$ $F_{05} = 2.49$</td>
</tr>
<tr>
<td>Error</td>
<td>9</td>
<td>38.790*</td>
<td>4.3300</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>533.774</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Obtained by subtraction

Thus, we find that the observed $F$-ratio is highly significant. The variance between any two treatments which are first associates is estimated by $3.2475$, between any two treatments which are second associates by $4.0702$, between any two treatments which are third associates by $5.4125$.

We next perform the following analysis for estimating the weights of the inter-block estimates.

### Table 4.5.4.

<table>
<thead>
<tr>
<th>Variation due to</th>
<th>D.F.</th>
<th>S.S.</th>
<th>M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (Eliminating treatments)</td>
<td>7</td>
<td>230.674</td>
<td>32.9534</td>
</tr>
<tr>
<td>Treatments (Ignoring blocks)</td>
<td>7</td>
<td>264.310</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>9</td>
<td>38.790*</td>
<td>4.3300</td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>533.774</td>
<td></td>
</tr>
</tbody>
</table>
1. By subtraction, 2. By transference from table 4.5.3.

Hence we estimate \( w = \frac{1}{E} = .2309 \)

and \( w' = \frac{v(v-1)}{k(b-1)B-(v-k)E} \)

\[ = \frac{16}{21 \cdot 55} = .0234 \]

Hence \( \frac{w'}{w} = \frac{.0234}{.2309} = .1013. \)

So, the approximate gain in efficiency is 10.13 per cent and we proceed to recover the inter-block information.

Table 4.5.5.

Estimation of combined intra- and inter block estimate.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>( Q_j' )</th>
<th>( P_j )</th>
<th>( \Sigma P_{j1} )</th>
<th>( \Sigma P_{j2} )</th>
<th>( \bar{t}_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.85</td>
<td>2.63</td>
<td>.48</td>
<td>-3.72</td>
<td>3.0300</td>
</tr>
<tr>
<td>c</td>
<td>-5.92</td>
<td>.43</td>
<td>2.63</td>
<td>-3.72</td>
<td>-.4257</td>
</tr>
<tr>
<td>a</td>
<td>-.58</td>
<td>.52</td>
<td>-1.58</td>
<td>3.71</td>
<td>2.0283</td>
</tr>
<tr>
<td>ac</td>
<td>-6.32</td>
<td>-1.58</td>
<td>.52</td>
<td>3.71</td>
<td>-1.3522</td>
</tr>
<tr>
<td>a^2</td>
<td>5.58</td>
<td>-.71</td>
<td>1.31</td>
<td>-3.72</td>
<td>-2.3455</td>
</tr>
<tr>
<td>a^2c</td>
<td>4.22</td>
<td>1.31</td>
<td>-.71</td>
<td>-3.72</td>
<td>.9098</td>
</tr>
<tr>
<td>a^3</td>
<td>.72</td>
<td>1.24</td>
<td>-3.90</td>
<td>3.71</td>
<td>3.1886</td>
</tr>
<tr>
<td>a^3c</td>
<td>-.05</td>
<td>-3.90</td>
<td>1.24</td>
<td>3.71</td>
<td>-5.0833</td>
</tr>
</tbody>
</table>
There is no exact test for testing the treatment effects from this combined inter and intra-block estimates, $\beta_j$. A rough test is however supplied by $\chi^2_{v-1} = \sum \bar{t}_j^2 p_j$. Here $\sum \bar{t}_j^2 p_j = 38.0003$.

But tabulated value of $\chi^2$ with 7 d.f. at 1 per cent is 18.475 and at 5 per cent is 14.067. Hence the observed value of $\chi^2$ is highly significant.

We next calculate $\Delta_1 = 3.5941$, $\Delta'_1 = 1.9296$, $\Delta_2 = 2.3139$ and $\Delta'_3 = 2.6962$.

Hence the combined variance after recovery of inter-block information between two treatments which are first associates is estimated by 3.2213, between two treatments which are second associates is estimated by 3.8628, and between two treatments which are third associates by 4.5044.

**Example 2. (Second type)**

Consider the following design having cyclical association scheme:

$v = b = 12$, $r = k = 5$, $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 2$,

$n_1 = 4$, $n_2 = 1$, $n_3 = 6$, $m_2 = 4$, $m_3 = 1$, $m_1 = 2$, $\alpha = 2$, $\beta = 4$

$\begin{bmatrix}
2 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 6
\end{bmatrix}$

$\begin{bmatrix}
4 & 0 & 0 \\
0 & 0 & \infty \\
0 & 0 & \infty
\end{bmatrix}$

$\begin{bmatrix}
0 & 0 & 4 \\
0 & 0 & 1 \\
4 & 1 & 0
\end{bmatrix}$
whose treatments are elements of the group 0 determined by a, o and their products and the product of their powers when \( a^6 = c^2 = 1 \).

The association scheme is of the second type of three associate cyclic design determined by \( A : a^1, a^2, a^4, a^5 \); \( B : a^1 \); \( G_1 : l, a, a^2, a^3, a^4, a^5 \).

The layout and yield (fictitious) for 12 treatments are given in table 4.5.5.

**Table 4.5.5.**

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Treatments and yields</th>
<th>( B_1 )-Block totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 39 43 47 51 54 56 58</td>
<td>212</td>
</tr>
<tr>
<td>2</td>
<td>42 44 45 50 51 52 53 54</td>
<td>230</td>
</tr>
<tr>
<td>3</td>
<td>48 50 51 52 53 54 55 56</td>
<td>234</td>
</tr>
<tr>
<td>4</td>
<td>47 48 49 50 51 52 53 54</td>
<td>240</td>
</tr>
<tr>
<td>5</td>
<td>47 48 49 50 51 52 53 54</td>
<td>232</td>
</tr>
<tr>
<td>6</td>
<td>54 55 56 57 58 59 60 61</td>
<td>266</td>
</tr>
<tr>
<td>7</td>
<td>54 55 56 57 58 59 60 61</td>
<td>268</td>
</tr>
<tr>
<td>8</td>
<td>54 55 56 57 58 59 60 61</td>
<td>273</td>
</tr>
<tr>
<td>9</td>
<td>54 55 56 57 58 59 60 61</td>
<td>237</td>
</tr>
<tr>
<td>10</td>
<td>53 54 55 56 57 58 59 60</td>
<td>270</td>
</tr>
<tr>
<td>11</td>
<td>53 54 55 56 57 58 59 60</td>
<td>273</td>
</tr>
<tr>
<td>12</td>
<td>53 54 55 56 57 58 59 60</td>
<td>276</td>
</tr>
</tbody>
</table>
We next calculate the following constants

\[ R = 22, \ A_1 = 1, \ A_2 = -2, \ S = 3, \ H = -6, \ \Delta_0 = 10368 \]

\[ \Delta_1 = 504, \ \Delta_2 = 432, \ \Delta_3 = 480 \]

variance between any two treatments which are first associates is
\( 35/72 \sigma_k^2 \), between any two treatments which are second associates
is \( 5/12 \sigma_k^2 \) and between any two treatments which are third
associates is \( 25/72 \sigma_k^2 \). The efficiency factor is calculated
from (4.3.33) as .8562

### Table 4.5.6

Showing calculation for \( \hat{t}_j \), the intra-block estimate.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>( T_j )</th>
<th>( V_j )</th>
<th>( Q_j )</th>
<th>( \Sigma Q_{j1} )</th>
<th>( \Sigma Q_{j3} )</th>
<th>( t_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>233</td>
<td>1233</td>
<td>-3.6</td>
<td>-2.2</td>
<td>19.8</td>
<td>-2.3055</td>
</tr>
<tr>
<td>a</td>
<td>255</td>
<td>1273</td>
<td>-1.6</td>
<td>-15.6</td>
<td>19.8</td>
<td>-1.1042</td>
</tr>
<tr>
<td>a^2</td>
<td>255</td>
<td>1283</td>
<td>-1.6</td>
<td>-15.8</td>
<td>19.8</td>
<td>-1.1111</td>
</tr>
<tr>
<td>a^3</td>
<td>244</td>
<td>1235</td>
<td>-3.0</td>
<td>-8.2</td>
<td>19.8</td>
<td>-1.1389</td>
</tr>
<tr>
<td>a^4</td>
<td>252</td>
<td>1273</td>
<td>-2.6</td>
<td>-15.6</td>
<td>19.8</td>
<td>-1.3125</td>
</tr>
<tr>
<td>a^5</td>
<td>262</td>
<td>1322</td>
<td>-2.4</td>
<td>-15.8</td>
<td>19.8</td>
<td>-1.2778</td>
</tr>
<tr>
<td>c</td>
<td>250</td>
<td>1250</td>
<td>0</td>
<td>17.4</td>
<td>-19.8</td>
<td>.8333</td>
</tr>
<tr>
<td>ac</td>
<td>246</td>
<td>1228</td>
<td>.4</td>
<td>12.4</td>
<td>-19.8</td>
<td>.7431</td>
</tr>
<tr>
<td>a^2c</td>
<td>250</td>
<td>1209</td>
<td>8.2</td>
<td>9.8</td>
<td>-19.8</td>
<td>2.2778</td>
</tr>
<tr>
<td>a^3c</td>
<td>258</td>
<td>1278</td>
<td>2.4</td>
<td>17.4</td>
<td>-19.8</td>
<td>1.3333</td>
</tr>
<tr>
<td>a^4c</td>
<td>253</td>
<td>1230</td>
<td>7.0</td>
<td>12.4</td>
<td>-19.8</td>
<td>2.1180</td>
</tr>
<tr>
<td>a^5c</td>
<td>250</td>
<td>1241</td>
<td>1.8</td>
<td>9.8</td>
<td>-19.8</td>
<td>.9444</td>
</tr>
</tbody>
</table>
Total sum of squares \( = \sum y_{ij}^2 - (\sum y_{ij})^2 \frac{1}{N} = 1240.99 \)

Block sum of squares (ignoring treatments) \( = \sum b_i^2 / k - (\sum y_{ij})^2 / N \)
\( = 1079.39 \)

Treatment sum of squares (ignoring block) \( = \sum t_j^2 / r - (\sum y_{ij})^2 / N \)
\( = 88.19 \)

Treatment sum of squares (eliminating blocks) \( = \sum t_j^2 = 87.2687 \).

The analysis of variance for testing intra-block estimates is performed as follows:

<table>
<thead>
<tr>
<th>Variation due to</th>
<th>D.F.</th>
<th>S.S.</th>
<th>M.S.</th>
<th>F-ratio</th>
<th>Tabulated F&lt;sub&gt;1,37&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (Ignoring treatments)</td>
<td>11</td>
<td>1079.3900</td>
<td>m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments (Eliminating blocks)</td>
<td>11</td>
<td>87.2687</td>
<td>7.9335</td>
<td>3.9491</td>
<td>F&lt;sub&gt;.01&lt;/sub&gt; = 2.7641, F&lt;sub&gt;.05&lt;/sub&gt; = 2.0559</td>
</tr>
<tr>
<td>Error</td>
<td>37</td>
<td>74.3313*</td>
<td>2.0089</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>59</td>
<td>1240.9900</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* By subtraction

Hence the observed F-ratio is highly significant.

The variance between any two treatments which are first associates is estimated by \( 0.9765 \), between any two treatments which are second associates by \( 0.8370 \), and between any two treatments which are
third associates by .6975.

We next perform the following analysis of variance for estimating the weights of inter-block estimates.

<table>
<thead>
<tr>
<th>Variation due to</th>
<th>D.F.</th>
<th>S.S.</th>
<th>M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (Eliminating treatments)</td>
<td>11</td>
<td>1078.4687</td>
<td>98.0426</td>
</tr>
<tr>
<td>Treatments (Ignoring blocks)</td>
<td>11</td>
<td>88.1900</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>37</td>
<td>74.3313²</td>
<td>2.0089</td>
</tr>
</tbody>
</table>

Total: 59

1. By subtraction. 2. By transference from table 4.5.7.

Hence, \( w = \frac{1}{B} = .4978 \), \( w' = \frac{48}{958 - 7E} = .00898 \).

\[ \frac{w'}{w} = .0180. \]

So, we find that approximate gain in efficiency is only 1.8 per cent and recovery of inter block information is not worthwhile.
References


Nair, K. R. (1944): The recovery of inter-block information in incomplete block designs, Sankhya, 6, 383-390.
