PART ONE

CHAPTERS TWO - SIX

SOME CLASSES OF PARTIALLY BALANCED INCOMPLETE BLOCK (P.B.I.B.) DESIGNS
CHAPTER TWO

GENERALISATION OF TWO ASSOCIATE CYCLICAL
PARTIALLY BALANCED ASSOCIATION SCHEME

2.1. Introduction

The class of partially balanced incomplete block designs (P.B.I.B.D) of Bose and Nair (1939), later generalised by Nair and Rao (1942), is very wide and includes many designs suggested from practical considerations from time to time. Bose and Shimamoto (1952) have recognized some distinct subclasses of two associate P.B.I.B. designs by the distinguishing nature of the association scheme like group divisible, cyclical, triangular, $L_2$ etc. The group divisible association scheme has been generalised to $m$-associate classes by Roy (1953, 1955, 1962) to hierarchical group divisible schemes, HG$_m$, as also orthogonal group divisible scheme, OGDL$_1$. In this chapter, we shall generalise the cyclical scheme to $m$-associate classes.

The two-associate cyclical scheme has been defined by Bose and Shimamoto (1952) as follows:-- Let the $v-1$ non-zero elements of a module $M$ be divided into two sets containing respectively the elements $(d_1, d_2, \ldots, d_{n_1})$ and $(e_1, e_2, \ldots, e_{n_2})$ where the first set of elements is such that among the $n_1(n_1-1)$ non-zero differences arising out of them, the elements $(d_1, d_2, \ldots, d_{n_1})$ are repeated $\alpha$ times each and the elements $(e_1, e_2, \ldots, e_{n_2})$ $\beta$ times each. The first associates of the treatment $i$ are

$$i + d_1, i + d_2, \ldots, i + d_{n_1} \pmod{v},$$
while the others are second associates.

The parameters $p^i_{jk}$ are given by

$$
(p^1_{jk}) = \begin{bmatrix}
\alpha & n^1_{-1} - \alpha - 1 \\
-1 - \alpha & n^2_{-1} + \alpha + 1
\end{bmatrix}
$$

$$
(p^2_{jk}) = \begin{bmatrix}
\beta & n^1 - \beta \\
-\beta & n^2 - n^1 + \beta - 1
\end{bmatrix}
$$

(2.1.1)

The only restriction on $\alpha$ and $\beta$ is

$$n^1\alpha + n^2\beta = n^1(n^1 - 1)$$

(2.1.2)

Though all the examples given by Bose and Shimamoto (1952) correspond to $n^1 = n^2$ and $\beta = n^1 - \alpha - 1$ this is not any necessary condition. It may also be noted that though it has not been explicit, it is a necessary condition to assume $(d_1, d_2, \ldots, d_{n_1}) = (-d_1, -d_2, \ldots, -d_{n_1})$ if $\alpha = \beta$. We shall prove the necessity of this condition by two examples.

**Example 1:** Consider the module of residue classes mod 7. Let us take the seven elements of this module as 0, 1, 2, 3, 4, 5 and 6. Let us take the set $(d_1, d_2, \ldots, d_{n_1})$ as $(1, 2, 4)$ and the set $(e_1, e_2, \ldots, e_{n_2})$ as $(3, 5, 6)$. We can easily verify that in this case $\alpha = \beta = 1$. So, this set of $d$'s satisfies all the conditions of Bose and Shimamoto's cyclical scheme. Let us next construct the association table:
From the above table we can easily verify that though the condition of constancy of $p_{jk}$ parameters is satisfied, the condition of symmetry of association is being violated. Thus, '1' is a first associate of '0' but '0' is a second associate of '1'.

Example 2: Consider the module of residue classes mod 11. Let $(d_1, d_2, \ldots, d_{n_1}) = (1, 2, 3, 7, 10)$ and $(e_1, e_2, \ldots, e_{n_2}) = (4, 5, 6, 8, 9)$. We can verify that $\alpha = \beta = 2$. But here also symmetry of association is absent. Hence this cannot represent an association scheme.

For convenience, the group operation will be taken here to be 'multiplication' instead of 'addition'. The 'differences' will change to 'ratios' and the null element '0' to the unit element '1'. In this multiplicative notation, the above two-associate cyclical scheme may be restated as follows: Consider an abelian group $G$ consisting of $v$ elements. Let it be possible to divide the non-unit elements of $G$ into two disjoint sets $A$ and $B$ with $n_1$ and $n_2$ elements respectively, one containing its inverse elements also. Let the elements of the set $A$ be such that among the $n_1(n_1 - 1)$ non-unit ratios arising out of them, the elements of $A$ are repeated $\alpha$ times and those of $B$ $\beta$ times each. Then the first associates of any treatment $\Theta$ are $\Theta A$ and the second associates of $\Theta$ are $\Theta B$. The parameters of the

<table>
<thead>
<tr>
<th>Tr.</th>
<th>1st associates</th>
<th>2nd associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1, 2, 4</td>
<td>3, 5, 6</td>
</tr>
<tr>
<td>1</td>
<td>2, 3, 5</td>
<td>0, 4, 6</td>
</tr>
<tr>
<td>2</td>
<td>3, 4, 6</td>
<td>0, 1, 5</td>
</tr>
<tr>
<td>3</td>
<td>4, 5, 0</td>
<td>1, 2, 6</td>
</tr>
<tr>
<td>4</td>
<td>5, 6, 1</td>
<td>0, 2, 3</td>
</tr>
<tr>
<td>5</td>
<td>6, 0, 2</td>
<td>1, 3, 4</td>
</tr>
<tr>
<td>6</td>
<td>0, 1, 3</td>
<td>2, 4, 5</td>
</tr>
</tbody>
</table>
association scheme are known to be $v, n_1, n_2$ and the $p_{jk}^i$'s the same as in (2.1.1).

Some known two-associate association schemes become special cases of this cyclical association scheme.

A) Group divisible association scheme is obtained from the above cyclical association scheme by putting (i) $\beta = 0$ and (ii) $\alpha = n_1 - n_2 - 1$.

Proof: Bose and Connor (1952) have proved that if for any two associate partially balanced association scheme $p_{12}^1 = 0$, $i = 1, 2$, then the association scheme reduces to G.D. association scheme. We shall show that due to condition (i) $p_{12}^1 = 0$ while condition (ii) implies $p_{12}^2 = 0$.

1) Obviously, if $\beta = 0$, then from (2.1.2) $n_1 \alpha = n_1(n_1 - 1)$ or $\alpha = n_1 - 1$. So, $p_{11}^1 = n_1 - 1$. This implies $p_{12}^1 = 0$. (Following the arguments of Bose and Connor (1952).

ii) Here from (2.1.2) $n_1(n_1 - n_2 - 1) + n_2 \beta = n_1(n_1 - 1)$,

or, $\beta = n_1$. So, $p_{11}^2 = n_1 \ldots p_{12}^2 = 0$.

B) $L_2$ association scheme is a particular subclass of the cyclical one. To see this, consider the abelian group $G$ of $n^2$ elements formed by the powers of $a$, $b$ and their products where $a^n = b^n = 1$. Take the set $A$ to be $(a, a^2, \ldots a^{n-1}, b, b^2, \ldots b^{n-1})$ and $B = G - A$. Writing the elements in the form of a square as

$$
\begin{array}{cccccc}
1 & a & a^2 & \ldots & \ldots & a^{n-1} \\
\hline
b & ab & a^2b & \ldots & \ldots & a^{n-1}b \\
\hline
b^{n-1} & ab^{n-1} & a^2b^{n-1} & \ldots & \ldots & a^{n-1}b^{n-1}
\end{array}
$$
and remembering that the first associates of \( \Theta \) are \( \Theta A \), second associates of \( \Theta \) are \( \Theta B \), it is easily seen that two elements in a row or a column are first associates and those not in a row or column are second associates. It may be noted that Vartak (1955) attempted to show that \( L_2 \) association scheme did not belong to the cyclical association scheme though some of its necessary conditions are satisfied. But his approach was wrong and hence his proof is also incorrect.

3) The rectangular (O.G.D.) association scheme can be represented in this way. Consider the abelian group \( G \) consisting of \( n_1 \cdot n_2 \) elements formed by \( a, b \) and their products when \( a_1 = b_1 = 1 \). Take the set \( A \) to be \( \{ a, a^2, \ldots, a^{n_1-1} \} \) and \( B \) to be \( \{ b, b^2, \ldots, b^{n_2-1} \} \). Then we can decompose the group \( G \) as \( G = (1+A)(X)(1+B) = 1 \cup A \cup B \cup (A \cap X \cap B) \), where \( 1+A \) and \( 1+B \) are subgroups. Take the first associates of any treatment \( \Theta \) as \( \Theta A \), its second associates as \( \Theta B \) and its third associates as \( \Theta(A \cap X \cap B) \). Then we get the rectangular association scheme of Vartak (1959) which is the same as the Orthogonal Group Divisible (O.G.D.) association scheme of Roy (1954, 1955). Conversely, every O.G.D. (rectangular) design is obtainable this way. It will be seen from subsequent development that this representation of the rectangular association scheme implies that it is a particular case of three associate cyclical association scheme (described later in this chapter.)

2.2 General Association Scheme

We first prove a lemma which will be used later.
Lemma. In a group consisting of $n$ elements, among the set of $n(n-1)$ ratios that arise by taking the ratios of distinct elements of the group, the non-unit elements of the group are each repeated $n$ times.

Proof: Let $c$ be any non-unit element of $G$. This will occur among the ratios as many times as the number of distinct pair of elements $a, b$ satisfying $ac^{-1} = b$. As $a$ takes the $n$ distinct values, $b$ also takes $n$ distinct values. Hence the result.

Consider an abelian group $G$ consisting of $v$ elements. Let it be possible to decompose $G$ into $(m-f)$ direct factors as

$$G = G_1 \times G_2 \times \cdots \times G_{m-f}$$  (2.2.1)

where $G_i$ consists of $m_i$ elements. Let among these $(m-f)$ groups, a set of $f$ groups, say the $i(1)$-th, $i(2)$-th, ... $i(f)$-th groups admit the following further decomposition:

$$G_{i(j)} = a_{i(j)} \cup B_{i(j)}, \quad j = 1, 2, \ldots, f$$  (2.2.2)

where $a_{i(j)} \cap B_{i(j)} = 0$, $a_{i(j)}$ consists of $n_{i(j)}$ distinct elements and $B_{i(j)}$ $m_{i(j)}$ distinct elements and further $a_{i(j)}$ contains also its inverse elements. Obviously,

$$n_{i(j)} + m_{i(j)}^* = m_{i(j)} = 1$$  (2.2.3)

Let further, the elements of $a_{i(j)}$ be such that among the non-unit ratios arising out of them, the elements of $a_{i(j)}$ appear $a_{i(j)}$ times and the elements of $B_{i(j)}$ $b_{i(j)}$ times each. So
\[ m_i(j) a_i(j) + m_i(j) b_i(j) = m_i(j) (m_i(j)^{-1}) \]  

Without any loss of generality, it may be assumed that 

\[ i(1) < i(2) < \ldots < i(\ell). \]

Let us define the different associates of any element \( \theta \) as follows:

1st associates: \( \theta(g_1 - 1) \)

2nd associates: \( \theta g_1(x)(g_2 - 1) \) \hfill (2.2.5)

\[ \ldots \]

\( (i(1)-1) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x)(g_1(1) - 1) \)

\( i(1) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x) g_1(1) - 1(x) A_1(1) \)

\( (i(1)+1) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x) g_1(1) - 1(x) B_1(1) \)

\( (i(1)+2) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x) g_1(1+1) - 1 \)

\[ \ldots \]

\( i(2) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x)(g_1(2) - 1) \)

\( (i(2)+1) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x) g_1(2) - 1(x) A_1(2) \)

\( (i(2)+2) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x) g_1(2) - 1(x) B_1(2) \)

\( (i(2)+3) \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x)(g_1(2+1) - 1) \)

\[ \ldots \]

\( m \)-th associates: \( \theta g_1(x) g_2(x) \ldots (x)(g_{m-\ell} - 1) \).
Theorem 2.1. The association scheme defined by $(2, 2, 1)$ to $(2, 2, 5)$ above is a P.B.I.B. association scheme with the parameters of the first kind as

$$v = m_1 m_2 \cdots m_m.$$ 

$$n_1 = m_1 - 1, \quad n_2 = m_1(m_2 - 1), \ldots \quad n_1(1) - 1 = m_1 m_2 \cdots m_i(1) - 2(m_i(1) - 1)$$

$$n_i(1) = m_1 m_2 \cdots m_i(1) - 1 m'_i(1)$$

$$n_i(1) + 1 = m_1 m_2 \cdots m_i(1) - 1 m''_i(1)$$

$$n_i(1) + 2 = m_1 m_2 \cdots m_i(1) (m_i(1) + 1)$$

$$\quad \ldots \quad \ldots \quad \ldots$$

$$n_i(2) = m_1 m_2 \cdots m_i(2) - 2 (m_i(2) - 1)$$

$$n_i(2) + 1 = m_1 m_2 \cdots m_i(2) - 1 m'_i(2)$$

$$n_i(2) + 2 = m_1 m_2 \cdots m_i(2) - 1 m''_i(2)$$

$$n_i(2) + 3 = m_1 m_2 \cdots m_i(2) (m_i(2) + 1 - 1)$$

$$\quad \ldots \quad \ldots \quad \ldots$$

$$n_m = m_1 m_2 \cdots m_{m-1} (m_m - 1).$$

$$(p_{jk}^w) = \begin{bmatrix}
0(w-1 \times w-1) & x(w-1) & 0(w-1 \times m-w) \\
x'(w-1) & D(m-w+1 \times m-w+1) \\
0(m-w \times w-1) & D(m-w+1 \times m-w+1)
\end{bmatrix}$$
for \( w = 1, 2, \ldots, m \) except \( i(1), i(1) + 1, \ldots, i(k') + (k' - 1), i(k') + k' \); where \( 0(i \times j) \) is an \( i \times j \) null matrix, \( x'(i) = (n_1, n_2, \ldots, n_i) \) and \( D(m-w+1 \times m-w+1) \) is a diagonal matrix whose diagonal elements are \( m_1 m_2 \cdots m_{w-1} (m_w - 2), n_{w+1} \cdots n_m \) respectively.

\[
\begin{pmatrix}
A(t+1 \times t+1) & 0(t+1 \times m-t-1) \\
0(m-t-1 \times t+1) & D(m-t-1 \times m-t-1)
\end{pmatrix}
\]

\[
\begin{pmatrix}
B(t+1 \times t+1) & 0(t+1 \times m-t-1) \\
0(m-t-1 \times t+1) & D(m-t-1 \times m-t-1)
\end{pmatrix}
\]

where

\[
A(t+1 \times t+1) = \begin{bmatrix} 0(t-1 \times t-1) & x(t-1) & 0(t-1 \times 1) \\ x'(t-1) & 0(1 \times t-1) & x \\ 0(1 \times 1) & 0(1 \times t-1) & 0(t-1 \times 1) \end{bmatrix}
\]

\[
B(t+1 \times t+1) = \begin{bmatrix} 0(t-1 \times t-1) & 0(t-1 \times 1) & x(t-1) \\ 0(1 \times t-1) & 0(1 \times 1) & x'(t-1) \end{bmatrix}
\]

\[
x = \begin{bmatrix} m_1 m_2 \cdots m_{t-1} \alpha_t \\ m_1 m_2 \cdots m_{t-1} (m_t' - \alpha_t - 1) \\ m_1 m_2 \cdots m_{t-1} (m_t' - \alpha_t - 1) m_1 m_2 \cdots m_{t-1} (m_t'' - m_t' + \alpha_t + 1) \end{bmatrix}
\]

\[
y = \begin{bmatrix} m_1 m_2 \cdots m_{t-1} \beta_t \\ m_1 m_2 \cdots m_{t-1} (m_t' - \beta_t) \\ m_1 m_2 \cdots m_{t-1} (m_t' - \beta_t) m_1 m_2 \cdots m_{t-1} (m_t'' - m_t' + \beta_t - 1) \end{bmatrix}
\]
$D(m-t-1 \times m-t-1)$ is a diagonal matrix with diagonal elements as $n_{t+2}, n_{t+3}, \ldots, n_m$; other matrices have been defined already and

$$(t, \alpha_t) = (i(1), \alpha_i(1)), (i(2)+1, \alpha_i(2)), \ldots, (i(k)+k-1, \alpha_i(k))$$

$$(t, \beta_t) = (i(1), \beta_i(1)), (i(2)+1, \beta_i(2)), \ldots, (i(k)+k-1, \beta_i(k))$$

**Proof:** We prove the different parts of the theorem separately.

(a) We first prove the symmetry of associateship. Let $\phi$ be any treatment which is $i$-th associate of $\Theta$. We are to prove that $\Theta$ is an $i$-th associate of $\phi$. As $\phi$ is an $i$-th associate of $\Theta$, so $\phi \in \Theta G_1(x)G_2(x) \cdots (X)G_{i-1}$, say. Hence $\phi$ can be expressed as

$$\phi = \Theta g_1 g_2 \cdots g_{i-1} g_i'$$

where $g_j \in G_j$, $j = 1, 2, \ldots, i-1$ and $g_i'$ is a non-unit element of $G_i$. Since $G_1, G_2, \ldots, G_i$ are all groups, $g_1, g_2, \ldots, g_{i-1}, g_i'$ have unique inverses in the corresponding groups. So, $\Theta = \phi (g_1 g_2 \cdots g_{i-1} g_i')^{-1} = \phi (g_i'^{-1} g_{i-1}^{-1} \cdots g_2^{-1} g_1^{-1}) \in \phi (g_i)_{G_i} G_{i-1} (X) G_2 (X) G_1$. Hence $\Theta \in \Theta G_1(x)G_2(x) \cdots (X)G_{i-1} (X)(G_i - 1)$ by definition of direct product, and is $i$-th associate of $\phi$. For other types of $i$-th associates the proof is similar, remembering the sets $A_i(j), B_i(j)$ are such by definition that they contain their respective inverses.

(b) As regards the number of associates of different types the result is evident by counting. In fact we can count the number of $i$-th associates as the number of elements contained in $\Theta G_1(x)G_2(x) \cdots (X)(G_i - 1)$ which is $n_i$ given above independently of the element $\Theta$. 
(c) Here we shall actually calculate some \( p_{jk}^i \) values in course of which it will be automatically proved that \( p_{jk}^i \) are constants independently of the elements.

For values of \( w \) and \( t \) as stated in the theorem \( p_{ww}^w = \text{Number of elements common between the } w\text{-th associates of } \phi \text{ and } w\text{-th associates of } \psi \text{ when } \phi \text{ and } \psi \text{ are } w\text{-th associates of each other.}

\[
= \text{Number of elements common between } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \\
\text{and } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \text{ when } \phi \in \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}).
\]

As \( \phi \in \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \), \( \psi \) can be expressed as

\[
\psi = g_1g_2\ldots g_{w-1}g'_w \quad \text{where } g_j \in G_j, \ j = 1,2,\ldots, w-1 \text{ and } g'_w \text{ is a non-unit element of } G_w.
\]

So, \( p_{ww}^w = \text{No. of elements common between } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \)

\[
\quad \text{and } \phi g_1g_2\ldots g_{w-1}g'_w g_1(x)g_2(x)\ldots(x)(g_{w-1})
\]

\[
= \text{Number of elements common between } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \\
\text{and } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1})(g_{w-2}).
\]

\[
= m_1m_2\ldots m_{w-1} (m_w = 2).
\]

For \( i \neq w \),

\[
p_{iw}^w = \text{Number of elements common between } \phi g_1(x)g_2(x)\ldots(x)(g_{i-1}) \\
\text{and } \phi g_1(x)g_2(x)\ldots(x)(g_{w-1}) \text{ when } \phi = g_1g_2\ldots g_{w-1} g'_w.
\]
Number of elements common between $G_1(x)G_2(x)\ldots G_{w-1}(x)(a_x - g)\quad m_1m_2\ldots m_{i-1} = n_i$ for $w > i$

$= 0$ for $w < i$.

$P_{tt} = \text{Number of elements common between } G_1(x)G_2(x)\ldots (x)G_{t-1}(x)A_t$

and $G_1(x)G_2(x)\ldots (x)G_{t-1}(x)A_t$ when

$\phi = \theta G_1G_2\ldots G_{t-1}a_t, \quad \theta \in G_j, \quad a_t \in A_t$

$m_1m_2\ldots m_{t-1}$ [Number of elements common between $A_t$ and $a_tA_t$]

$= m_1m_2\ldots m_{t-1} \alpha_t$

Other $p_{jk}^i$ parameters of the theorem can be verified in the same way, making use of the lemma proved earlier.

**Corollary 1:** If $\lambda = 0$, then the $m$-associate cyclical scheme reduces to an $m$-associate hierarchical group divisible association scheme ($H \in D_m$) and conversely.

**Proof:** Here the $i$-th associates of any treatment $\theta$ are

$\theta G_1(x)G_2(x)\ldots (x)G_{i-1}(x) (a_i - 1) \quad i = 1, 2, \ldots, m$. Hence,

$n_i = m_1m_2\ldots m_{i-1} (m_i - 1), \quad i = 1, 2, \ldots, m$

and

$$
p_{jk}^i = \begin{bmatrix}
0(i-1 \times i-1) & x(i-1) & 0(i-1 \times m-i) \\
x'(i-1) & 0(m-i \times i-1) & D(m-i+1 \times m-i+1)
\end{bmatrix}
$$
Let \( m_i = N_{m-i+1} \).

Then \( n_i = N_m N_{m-1} \cdots N_{m-i+2} (N_{m-i+1} - 1) \)

\[ i = 1, 2, \ldots, m. \]

We can identify these parameters as the parameters of the HGD\(_m\) in the notation of Raghavarao (1960). Both Roy (1955, 1962) and Raghavarao (1960) have proved the uniqueness of the HGD\(_m\) (G.D. \( m \)-associate) association scheme. Hence the results follow immediately.

**Corollary 2.** If \( \alpha_t = m_t - 1, \beta_t = 0, m_t'' = m_t (m_t + 1) \) for \( t, \alpha_t, \beta_t, m_t \) in theorem 2.1 then again the association scheme reduces to that of HGD\(_m\) and conversely.

**Proof:** Here \( n_t = m_1 m_2 \cdots m_{t-1} m_t \)

\[ n_{t+1} = m_1 m_2 \cdots m_{t-1} m_t'' \]

\[ = m_1 m_2 \cdots m_{t-1} m_t (m_t + 1). \]

Put \( m_1 = N_m, m_2 = N_{m-1}, \ldots, m_i(1)-1 = N_{m-i(1)}. \)

\[ m_i(1) = N_{m-i(1)-1} = N_{m-i(1)-2}, \]

\[ m_i(1)+1 = N_{m-i(1)-3}. \]

\[ \vdots \]

\[ m_m = N_1. \]
Here $X$ and $Y$ of theorem 2.1 becomes

$$X = \begin{bmatrix}
    m_1 m_2 \cdots m_{t-1} (m_t^i - 1) & 0 \\
    0 & m_1 m_2 \cdots m_{t-1} m_t^i (m_t^i + 1)
\end{bmatrix}$$

$$Y = \begin{bmatrix}
    0 & m_1 m_2 \cdots m_{t-1} m_t^i \\
    m_1 m_2 \cdots m_{t-1} m_t^i & m_1 m_2 \cdots m_{t-1} (m_t^i + 1)(m_t^i - 1)
\end{bmatrix}$$

So, $A(t+1 \times t+1) = \begin{bmatrix}
    0(t-1 \times t-1) & x(t-1) & 0(t-1 \times 1) \\
    x'(t-1) & m_1 m_2 \cdots m_{t-1} (m_t^i - 1) & 0 \\
    0(1 \times t-1) & 0 & n_{t+1}
\end{bmatrix}$

$$B(t+1 \times t+1) = \begin{bmatrix}
    0(t-1 \times t-1) & 0(t-1 \times 1) & x(t-1) \\
    0(t-1 \times 1) & 0 & m_1 m_2 \cdots m_{t-1}, m_t^i \\
    x'(t-1) & m_1 m_2 \cdots m_{t-1} m_t^i & m_1 m_2 \cdots m_{t-1} (m_t^i + 1)(m_t^i - 1)
\end{bmatrix}$$

Hence

$$(p_{jk}^t) = \begin{bmatrix}
    0(t-1 \times t-1) & x(t-1) & 0(t-1 \times m-t) \\
    x'(t-1) & D(m-t+1 \times m-t+1) \\
    0(m-t \times t-1)
\end{bmatrix}$$

$$(p_{jk}^{t+1}) = \begin{bmatrix}
    0(t \times t) & x(t) & 0(t \times m-t) \\
    x'(t) & D(m-t \times m-t) \\
    0(m-t \times t)
\end{bmatrix}$$
which are of the same form as $p_{jkl}^1$ of Corollary 1. Hence the proof follows from Corollary 1.

2.3 Three Associate Cyclical Schemes

The above generalisation of two associate cyclical association scheme gives rise to a large number of possibilities. Thus, for $m = \xi$, we may take $k = 0, 1, 2, \ldots \xi/2$ [or $\xi-1$] according as $\xi$ is even or odd. Again, there are $(\xi-1)$ distinct types depending on the positions of the $k$ groups to be decomposed further. So, there are $\sum_k (\xi/k)$ distinct types of cyclical schemes for the $\xi$-associate scheme. Thus, for the two associate scheme there are only two types of cyclical association schemes, viz. when $k = 0$ and 1. For the three associate case also, there are two values of $k$, viz. $k = 0, 1$; but there are altogether three types of generalisations. For $k = 0$, we get the well-known HGD designs of Roy (1953). For $k = 1$, there are two distinct types according as the first or second direct factor group is decomposed. Conventionally, we may suppose that the group $G_2$ is decomposed in both types of generalization.

Consider an abelian group $G$ consisting of $v = m_1n$ elements. Let it be possible to decompose $G$ into two direct factors:

$G = G_1 (\overline{X}) G_2$ where $G_1$ consists of $m_1$ elements $1, d_1, d_2, \ldots d_{m_1-1}$ while the non-unit elements of $G_2$ can be divided into two disjoint
sets A and B of $m_2$ and $m_3$ elements respectively, i.e.,
$A = \{e_1, e_2, \ldots, e_{m_2}\}$ and $B = \{f_1, f_2, \ldots, f_{m_2}\}$. Further let the
elements of $A$ be such that all the inverse elements are also in
$A$ and that among the $m_2(m_2-1)$ non-unit ratios arising out of
them, the elements of $A$ are repeated $\alpha$ times and those of $B$
$\beta$ times each. Obviously, $n = m_2 + m_3 + 1$, $m_2\alpha + m_3\beta = m_2(m_2-1)$.

First type of generalization. Let the first associates of any
treatment $\Theta$ be $\Theta(G_1-1)$, its second associates $\Theta G_1(X) A$ and its
third associates $\Theta G_1(X) B$.

The parameters of the association scheme are:

$$v = m_1(m_2+m_3+1) = m_1^n, n_1=m_1-1, n_2=m_1m_2, n_3=m_1m_3 \quad (2.3.1)$$

$$(p^{1}_{jk}) = \begin{bmatrix}
m_1^{-2} & 0 & 0 \\
0 & m_1m_2 & 0 \\
0 & 0 & m_1m_3
\end{bmatrix} \quad (2.3.2)$$

$$(p^{2}_{jk}) = \begin{bmatrix}
0 & m_1^{-1} & 0 \\
m_1^{-1} & m_1\alpha & m_1(m_2 - \alpha - 1) \\
0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\alpha+1)
\end{bmatrix} \quad (2.3.3)$$

$$(p^{3}_{jk}) = \begin{bmatrix}
0 & 0 & m_1^{-1} \\
0 & m_1\beta & m_1(m_2 - \beta) \\
m_1^{-1} & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1)
\end{bmatrix} \quad (2.3.4)$$
It follows from the above that if \( \Theta \) and \( \phi \) are \( i \)-th associates of each other, then all the first associates of \( \Theta \) (with the exception of \( \phi \) in case \( i = 1 \)) are \( i \)-th associates of \( \phi \) and Conversely; \( i = 1, 2, 3 \).

Second type of generalization. Let the first associates of any treatment \( \Theta \) be \( \Theta A \), second associates \( \Theta B \) and third associates \( \Theta G_2 (X)(G_1-1) \). The parameters of the association scheme are

\[
v = m_1 (m_2 + m_3 + 1) = m_1 n, \quad n_1 = m_2, \quad n_2 = m_3, \quad n_3 = (m_1-1)(m_2+m_3+1)
\]

\[
(p_{jk}^1) = \begin{bmatrix}
\alpha & m_2 - \alpha - 1 & 0 \\
m_2 - \alpha - 1 & m_3 - m_2 + \alpha + 1 & 0 \\
0 & 0 & (m_1-1)(m_2+m_3+1)
\end{bmatrix}
\]

\[
(p_{jk}^2) = \begin{bmatrix}
\beta & m_2 - \beta & 0 \\
m_2 - \beta & m_3 - m_2 + \beta + 1 & 0 \\
0 & 0 & (m_1-1)(m_2+m_3+1)
\end{bmatrix}
\]

\[
(p_{jk}^3) = \begin{bmatrix}
0 & 0 & m_2 \\
0 & 0 & m_3 \\
m_2 & m_3 & (m_1-2)(m_2+m_3+1)
\end{bmatrix}
\]

It immediately follows that if \( \Theta \) and \( \phi \) are two treatments which are third associates, then all the first and second associates of \( \Theta \) (including \( \Theta \) itself) are third associates of \( \phi \) and Conversely.
But if $\theta$ and $\phi$ are first or second associates, then all the third associates of $\theta$ are third associates of $\phi$ and conversely.

Parameters of both the generalisations depend on the values of $\alpha$ and $\beta$. Three major sub-classes of them will be as follows:

(A) $\alpha = 0, \quad \beta \neq 0$
(B) $\alpha \neq 0, \quad \beta \neq 0$  \hspace{1cm} (2.3.9)
(C) $\alpha \neq 0, \quad \beta = 0$

2.4. Combinatorial properties of cyclical three-associate P.B.I.B. designs

As is well-known, many combinatorial properties of P.B.I.B. designs follow from the positive semi-definiteness of the determinant $|NN'|$. Further, the Hasse-Minkowski $p$-invariant $G_p(NN')$ also helps in proving the non-existence of certain classes of designs. Accordingly we shall evaluate here the determinant $|NN'|$ and $G_p(NN')$ for both the types of generalisations.

Evaluation of $|NN'|$ (First type of generalisation).

$$
|NN'| =
\begin{vmatrix}
A & B_1 & \cdots & B_1 & B_2 & \cdots & B_2 \\
B_1 & A & \cdots & \cdots & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
B_1 & \cdots & A & \cdots & \cdots & \\
B_2 & \cdots & \cdots & A & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
B_2 & \cdots & \cdots & \cdots & A
\end{vmatrix}
$$

$$
: \frac{m_1(m_2+m_3+1)}{m_1(m_2+m_3+1)} X
$$

$$
= \frac{m_1(m_2+m_3+1)}{m_1(m_2+m_3+1)}
$$

(2.4.1)
where \( A = \begin{bmatrix} r & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & r & \cdots & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_1 & \cdots & r \end{bmatrix} : m_1 \times m_1 \)

\[ B_1 = \begin{bmatrix} \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \cdots & \cdots & \ddots & \cdots \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \end{bmatrix} : m_1 \times m_1 \]

\[ B_2 = \begin{bmatrix} \lambda_3 & \lambda_3 & \cdots & \lambda_3 \\ \cdots & \cdots & \ddots & \cdots \\ \lambda_3 & \lambda_3 & \cdots & \lambda_3 \end{bmatrix} : m_1 \times m_1 \]

and in each row and column of the above determinant \( A \) occurs in the main diagonal position and the other \((m_2+m_3)\) positions are occupied by \( m_2 \) \( B_1 \)'s and \( m_3 \) \( B_2 \)'s.

Each column of the determinant is a \( m_1(m_2+m_3+1) \times m_1 \) matrix and will be referred to as a submatrix. Consider the first submatrix

\[
\begin{bmatrix}
A \\
B_1 \\
\vdots \\
B_2
\end{bmatrix}
\]

(2.4.2)
Subtract the last column of $(2\times 4\times 2)$ from its preceding $(m_1-1)$ columns. Repeat the operation on the other $(m_2+m_3)$ sub-matrices. These operations do not affect the value of the determinant $(2.4.1)$. Finally, in the transformed determinant add

1st, 2nd, ...$(m_1-1)$-th rows to the $m_1$-th row

$(m_1+1)$-th, ... $(2m_1-1)$-th rows to the $(2m_1)$-th row

... ... ...

$m_1 (m_2+m_3)$-th ... $[m_1 (m_2+m_3+1)-1]$-th rows to $m_1 (m_2+m_3+1)$-th row.

Then $(2.4.1)$ reduces to

$$(r - \lambda_1) (m_1-1)(m_2+m_3+1) \begin{vmatrix} D \end{vmatrix}$$  \hspace{0.5cm} (2.4.3)$$

where $D = \begin{bmatrix} r + \lambda_1 (m_1-1) & m_1 \lambda_2 & \cdots & m_1 \lambda_2 & \cdots & m_1 \lambda_3 \\ m_1 \lambda_2 & r + \lambda_1 (m_1-1) & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_1 \lambda_3 & \cdots & \cdots & \cdots & \cdots & r + \lambda_1 (m_1-1) \end{bmatrix}$  \hspace{0.5cm} (2.4.4)$$

$D$ is a $(m_2+m_3+1) \times (m_2+m_3+1)$ matrix and can be easily identified as the $NN'$ matrix of a two-associate P.B.I.B. design of the following form
where \( R = r + \lambda_1(m_1-1), \lambda_{i-1} = \lambda_1 m_1, \ i = 2,3. \)

Thus
\[
|D| = (R - Z_1)^{\alpha_1} (R - Z_2)^{\alpha_2} (R + m_2 \lambda_1 + m_3 \lambda_2) \tag{2.4.5}
\]

where the process of evaluating \( Z_1 \)'s and \( \alpha_1 \)'s has been given by Connor and Clatworthy (1954).

Since
\[
R + m_2 \lambda_1 + m_3 \lambda_2 = r + (m_1-1)\lambda_1 + m_1 m_2 \lambda_2 + m_1 m_3 \lambda_3 = rk,
\]

So,
\[
|NN'| = rk(R-Z_1)^{\alpha_1}(R-Z_2)^{\alpha_2}(r-\lambda_1)^{(m_1-1)(m_2+m_3+1)} \tag{2.4.6}
\]

For a cyclical design, following the method of Connor and Clatworthy (1954) we calculate
\[
y = \alpha - \beta + 1, \quad \Delta = (a-\beta+1)^2+2(2m_2-a+\beta-1)+1 \tag{2.4.7}
\]
\[ Z_u = \frac{1}{2}[(\Lambda_i - \Lambda_u)(-\gamma \pm \sqrt{\Delta}) + (\Lambda_i + \Lambda_u)] \quad u = 1, 2 \]  

\[ \alpha_1 = \frac{[(m_2 + m_3)(-\gamma + \sqrt{\Delta} + 1) - 2m_2]}{2\sqrt{\Delta}} \]  

\[ \alpha_2 = \frac{[(m_2 + m_3)(\gamma + \sqrt{\Delta} + 1) - 2m_3]}{2\sqrt{\Delta}} \]  

where \( Z_1 \) is the larger root if \( \lambda_2 < \lambda_3 \) and \( Z_2 \) is the larger root if \( \lambda_2 > \lambda_3 \).

These values depend on the values of \( \alpha \) and \( \beta \). Hence substituting \( \alpha \) and \( \beta \) we can evaluate the determinant. In subclass \( C \), \( \beta = 0 \). Hence from (2.1.2), \( \alpha = m_2 - 1 \). But in subclass \( A \), \( \alpha = 0 \). So from (2.1.2) \( \beta = m_3(m_2 - 1)/m_3 \). We only know that \( \beta \) is an integral number. We consider the following integral values of \( \beta \) and \( \alpha \)

\begin{align*}
A(I) & \quad \beta = m_2, \quad \alpha = 0 \\
A(II) & \quad \beta = m_3, \quad \alpha = 0 \\
A(III) & \quad \beta = m_2 - 1, \quad \alpha = 0 \\
A(IV) & \quad \beta = 1, \quad \alpha = 0
\end{align*}

Similarly, for subclass \( B \) we take the following values of \( \alpha \) and \( \beta \).

\begin{align*}
B(I) & \quad \alpha = m_3, \quad \beta = m_2 \\
B(II) & \quad \alpha = m_2, \quad \beta \text{ unspecified} \\
B(III) & \quad \beta = m_2 - \alpha - 1, \quad \alpha \neq m_2 - 1. \\
B(IV) & \quad \beta = m_2, \quad \alpha = m_2 - m_3 - 1 \neq 0
\end{align*}

A(1). From (2.1.2) we get \( m_3 = m_2 - 1 \).
From (2.4.8), \( z_1 = m_1 \lambda_3 \), if \( \lambda_2 < \lambda_3 \)
\[ = m_1 m_2 \lambda_2 - m_1 (m_2 - 1) \lambda_3 \] if \( \lambda_2 > \lambda_3 \)
\[ z_2 = m_1 m_2 \lambda_2 - m_1 (m_2 - 1) \lambda_3 \] if \( \lambda_2 < \lambda_3 \)
\[ = m_1 \lambda_3 \] if \( \lambda_2 > \lambda_3 \).

From (2.4.9) \( \alpha_1 = m_2 + m_3 = 1, \quad \alpha_2 = 1 \)

\[ |NN'| = \text{rk}(r - \lambda_1) \quad \frac{(m_1 - 1)(m_2 + m_3 + 1)}{[r + \lambda_1 (m_1 - 1) - m_1 \lambda_3] m_2 + m_3 - 1} \]

\[ = \text{rk}[(r - \lambda_1)(m_2 + m_3 + 1)] \quad [r + \lambda_1 (m_1 - 1) - m_1 \lambda_3 - 1] \quad m_2 + m_3 - 1 \]

Hence,

**Theorem 2.4.1.** A set of necessary conditions for the existence of a design belonging to this class is

1) \( g_1 = r - \lambda_1 \geq 0 \)
2) \( g_2 = r + \lambda_1 (m_1 - 1) - m_1 m_2 \lambda_2 + m_1 (m_2 - 1) \lambda_3 \geq 0 \)
3) \( g_3 = r + \lambda_1 (m_1 - 1) - m_1 \lambda_3 \geq 0 \)

In particular, if the design is symmetrical then

4) \( g_1 = \frac{(m_1 - 1)(m_2 + m_3 + 1)}{m_2 + m_3 + 1} \) \( g_2 \) \( g_3 \) (or \( g_1 \) \( g_2 g_3 \) \( m_2 + m_3 - 1 \))
if \( \lambda_2 < \lambda_3 \) must be a perfect square.

Example. The designs \( r = 4, \lambda_1 = 1, \lambda_2 = 3, m_1 = 2 \) can not exist.

A(II). Here \( \alpha = 0, \beta = m_3 \). So from (2.1.2) \( m_3^2 = m_2(m_2 - 1) \) which equation cannot have integral solution for \( m_2 \) and \( m_3 \). Since \( m_2 \) and \( m_2 - 1 \) are prime to each other, so they cannot have a common factor. Hence both \( m_2 \) and \( m_2 - 1 \) should be integral square, which is not possible. Hence this type of design is not possible.

A(III). Here \( \alpha = 0, \beta = m_2 - 1 \). So from (2.1.2) \( m_2 = m_3 \). From (2.4.7) we calculate \( \Delta = m_2^2 - 2m_2 - 3 \). In order that \( \Delta \) be a perfect square we should have \( 16 = 0 \), which is a contradiction. Hence from theorem 5.3 of Connor and Olante (1954) \( \eta = (m_2 + m_3)(m_2 - 1) - 2m_2 \) should be zero which imply \( m_2 = m_3 = 2 \) and this cannot be considered as a general case.

A(IV). Here \( \alpha = 0, \beta = 1 \). From (2.1.2) \( m_3 = m_2(m_2 - 1) \). From (2.4.7) \( \Delta = 2m_2 - 1 \). So for the existence of this class of design, either (a) \( \Delta \) be a perfect square or (b) \( \eta = 0 \).

(a) The possible values of \( m_2 \) are 5, 13, 25, ... which gives odd values of \( m_2 \). Now, if \( m_2 \) is odd, then \( m_3 = m_2(m_2 - 1) \) is even and \( m_2 + m_3 + 1 \) is also even. Consider, next, the group \( G \) formed by \( a \) and its different powers when \( a^2 = 1 \). Since \( A = A^{-1} \), and \( m_2 \) is odd, so there is one element \( a \in A \) such that
\[ a^2 = a^{n-1}. \] If \( \beta \in A \), then \( \beta^{-1} \in A \). Consider the ratios \( \alpha/\beta, \beta/\alpha, \alpha/\beta^{-1}, \beta^{-1}/\alpha \). Now, \( \alpha/\beta = \alpha \beta^{-1} \). Also, \( \beta^{-1}/\alpha = \alpha \beta^{-1} \).

Hence the element \( \alpha \beta^{-1} \) appears twice contradicting \( \beta = 1 \).

Thus, here also, \( \eta = 0 \) which again imply (as in \( \Lambda(III) \)) \( m_2 = m_3 = 2 \) and can not be considered as a general class.

**B(I).** From (2.1.2) we get \( 2m_3 = m_2-1 \).

From (2.4.9) \( a_1 = m_2+m_3-m_2/(1+m_3); \ a_2 = m_2/(1+m_3) \)

Since \( a_1 \) and \( a_2 \) are integral, so \( \frac{m_2}{1+m_3} \) should be an integer.

or, \( (1+2m_3)/(1+m_3) = \) an integer or \( 2-1/(1+m_3) = \) an integer; i.e. \( 1/(1+m_3) = \) an integer; which is not possible, because of the fact that \( m_3 \) is greater than zero. So, this class of design do not exist.

**B(II).** Here \( \alpha = m_2 \). Then \( p_{2}^{2} = \alpha(-1) = m_1 \) which is not admissible. Hence this class do not exist.

**B(III).** Here \( m_2 = m_3, \ \beta = m_2-\alpha-1 \). In this class we can not evaluate \( |MN'| \) unless \( \alpha \) and \( \beta \) are known. An important sub-class of this class is \( m_2 = m_3 = 2t, p_{23}^{2} = p_{23}^{3} = m_1t \) i.e. \( \alpha = m_2-t-1, \ \beta = t \). This sub-class has been discussed by Cannor and Clatwerthy (1954) in details. This class exists and will be considered again in Chapters (V) and (VI).
B(IV). 

From (2.4.8), \( Z_1 = m_1 \lambda_3 \) if \( \lambda_2 < \lambda_3 \)
\[
= m_1 (1+m_3)^{\lambda_2} - m_1 m_3^{\lambda_3} \text{ if } \lambda_2 > \lambda_3
\]
\( Z_2 = m_1 (1+m_3)^{\lambda_2} - m_1 m_3^{\lambda_3} \) if \( \lambda_2 < \lambda_3 \)
\[
= m_1 \lambda_3 \text{ if } \lambda_2 > \lambda_3
\]

From (2.4.9) \( a_1 = \frac{m_2}{1+m_3} \),
\[
a_2 = \frac{m_2 m_3 - m_2}{1+m_3}
\]

.. \( W'W = cx(r-\lambda_1) (m_1-1)(m_2+m_3+1) \frac{[r+\lambda_1(m_1-1)-m_1 \lambda_3] m_2}{(1+m_3)} \)

.. \( [r+\lambda_1(m_1-1)-m_1(1+m_3) \lambda_2 + m_1 m_3 \lambda_3] \text{ if } \lambda_2 < \lambda_3 \)
\[
= cx(r-\lambda_1) (m_1-1)(m_2+m_3+1) \frac{[r+\lambda_1(m_1-1)-m_1 \lambda_3] m_2 + m_3 - m_2}{(1+m_3)}
\]

.. \( [r+\lambda_1(m_1-1)-m_1(1+m_2) \lambda_2 + m_1 m_3 \lambda_3] \frac{m_2}{(1+m_3)} \text{ if } \lambda_2 > \lambda_3 \)

Hence,

Theorem 2.4.2. A set of necessary conditions for the existence of a design belonging to this class is

i) \( g_1 = r - \lambda_1 \geq 0 \)

ii) \( g_2 = r+\lambda_1(m_1-1)-m_1 \lambda_3 \geq 0 \)

iii) \( g_3 = r+\lambda_1(m_1-1)-m_1(1+m_3) \lambda_2 + m_1 m_3 \lambda_3 \geq 0 \)
In particular, if the design is symmetrical then we must have

\[
\begin{align*}
\text{iv) } & q_1^2 (m_1-1)(m_2+m_3+1) + m_2^2 + m_3^2 = \frac{m_2}{1+m_3} \frac{m_3}{1+m_3} \\
& q_2^2 \quad \text{or,} \\
& q_3^2 \quad \text{if } \lambda_2 < \lambda_3
\end{align*}
\]

\[m_2 = m_2^2 + m_3^2 = \frac{m_2}{1+m_3} \]

\[m_3 = m_3^2 = \frac{m_3}{1+m_3} \]

a perfect square.

**Example**: The class of designs \( r = 4, \lambda_1 = 1, \lambda_2 = 3, m_1 = 2 \) do not exist.

\[z_1 = m_1 \lambda_2 \text{ if } \lambda_2 < \lambda_3 \]

\[= m_1 (1+m_2) \lambda_3 - m_1 m_2 \lambda_2 \text{ if } \lambda_2 > \lambda_3 \]

\[z_2 = m_1 (1+m_2) \lambda_3 - m_1 m_2 \lambda_2 \text{ if } \lambda_2 < \lambda_3 \]

\[= m_1 \lambda_2 \text{ if } \lambda_2 > \lambda_3 \]

From (2.4.9) \( \alpha_1 = \rho, \alpha_2 = m_2^2 + m_3^2 = \rho, \beta = m_3/(1+m_2) \)

\[|N_{L'}| = \operatorname{rk}(r-\lambda_1) \left[ (m_1-1)(m_2+m_3+1) \right] \]

\[\left[ r+\lambda_1 (m_1-1) m_2 m_3 \lambda_2 \right]^p \]

\[\left[ r+\lambda_1 (m_1-1) m_2 m_3 \lambda_2 \right] \]

\[= \operatorname{rk}(r-\lambda_1) \left[ (m_1-1)(m_2+m_3+1) \right] \left[ r+\lambda_1 (m_1-1) m_2^2 m_3 \lambda_2 \right]^p \]

\[\left[ r+\lambda_1 (m_1-1) m_2 \lambda_2 \right]^m \] if \( \lambda_2 < \lambda_3 \)

\[(2.4.14)\]
Hence, Theorem 2.4.3. A set of necessary conditions for the existence of this type of design is

i) $g_1 = r - \lambda_1 \geq 0$

ii) $g_2 = r + \lambda_1(m_1-1) - m_1 \lambda_2 \geq 0$

iii) $g_3 = r + \lambda_1(m_1-1) - m_1(1+m_2)\lambda_3 + m_1m_2\lambda_2 \geq 0$

If in particular, the design is symmetrical, then

iv) $g_1 \cdot (m_1-1)(m_2+m_3+1) \geq 0 \quad g_2 \cdot (m_2+m_3+p) \geq 0 \quad g_3 \cdot (m_2+m_3+p) \geq 0$ (if $\lambda_2 < \lambda_3$)

must be a perfect square.

Example. The designs $r = 4$, $\lambda_1 = 1$, $\lambda_2 = 3$, $m_1 = 2$ do not exist.

Second type of generalisation

Here $|\mathbb{H}^{1'}|$ can be written as

$|\mathbb{H}^{1'}| = \begin{vmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & A \end{vmatrix}$

where $A = \begin{bmatrix} r & \lambda_1 & \cdots & \lambda_1 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_1 & r & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1 & \cdots & r & \cdots & \cdots & \cdots & \cdots \\ \lambda_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_2 & \cdots & \cdots & \cdots & \cdots & \cdots & r \end{bmatrix}$

: $(m_2+m_3+1) \times (m_2+m_3+1)$
\[
B = \begin{bmatrix}
\lambda_3 & \lambda_3 & \cdots & \lambda_3 \\
\lambda_3 & \lambda_3 & \cdots & \lambda_3 \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_3 & \lambda_3 & \cdots & \lambda_3 
\end{bmatrix}^t \in (m_2+m_3+1) \times (m_2+m_3+1)
\]

Hence \(|NN'| = |A-B| \times |A+(m_1-1)B|\)

Putting \(R = r-\lambda_3\), \(\Lambda_1 = \lambda_1-\lambda_3\), \(\Lambda_2 = \lambda_2-\lambda_3\)
and \(R' = r+(m_1-1)\lambda_3\), \(\Lambda'_1 = \lambda_1+(m_1-1)\lambda_3\), \(\Lambda'_2 = \lambda_2+(m_1-1)\lambda_3\)

it is found that both \(|A-B|\) and \(|A+(m_1-1)B|\) are of the same form as \(|A|\) when \((r, \lambda_1, \lambda_2)\) is replaced by \((R, \Lambda_1, \Lambda_2)\) or \((R', \Lambda'_1, \Lambda'_2)\).

But \(|A|\) is evaluated easily as it is the det. \(|NN'|\) of a two-associate P.B.I.B. design.

Here we shall discuss only those classes which were found suitable in the first type of generalisations.

\(A(I)\). Here we get

\[
|NN'| = rk[r+m_2\lambda_1+m_3\lambda_2+(m_2+m_3+1)\lambda_3]^{m_1-1} \cdot \begin{bmatrix} r-\lambda_2 \end{bmatrix}^{m_1}
\]

\[
\cdot [r-(l+m_3)\lambda_1+m_3\lambda_2]^{m_1} \begin{bmatrix} m_2 \end{bmatrix}^{m_2} \begin{bmatrix} m_3 \end{bmatrix}^{m_3} \begin{bmatrix} l \end{bmatrix}^{l-1} \text{ if } \lambda_1 < \lambda_2
\]

\[
= rk[r+m_2\lambda_1+m_3\lambda_2-(i+m_2+m_3)\lambda_3]^{m_1-1} \cdot \begin{bmatrix} r-\lambda_2 \end{bmatrix}^{m_1} \begin{bmatrix} m_2 \end{bmatrix}^{m_2} \begin{bmatrix} m_3 \end{bmatrix}^{m_3} \begin{bmatrix} l \end{bmatrix}^{l-1}
\]

\[
\cdot [r-(l+m_3)\lambda_1+m_3\lambda_2]^{m_1} \text{ if } \lambda_1 > \lambda_2. \quad (2.4.15)
\]
Theorem 2.4.4. A set of necessary conditions for the existence of this type of design is

1) $g_1 = r - m_1 \lambda_1 + m_2 \lambda_2 + (\lambda_2 - 1) \lambda_3 \geq 0$

2) $g_2 = r - \lambda_2 \geq 0$

3) $g_3 = r - (1 + m_3) \lambda_1 + m_2 \lambda_2 \geq 0$.

In particular, if the design is symmetrical then

$g_1^{-1} \cdot g_2 \cdot g_3 \cdot g_1^{-1} (m_2 + m_3 - 1) \cdot g_1^{-1} \cdot g_2 \cdot g_3 \cdot g_1^{-1} (m_2 + m_3 - 1) \cdot g_3$ if $\lambda_1 > \lambda_2$ must be a perfect square.

Example. The class of designs $r=4$, $\lambda_1=5$, $\lambda_2=1$, $m_3=2$ are impossible.

B(IV). Here

$|\Omega_2| = r \left[ r + m_2 \lambda_1 + m_2 \lambda_2 -(1+ m_2 + m_2) \lambda_3 \right]^{m_1 - 1}$

$\cdot \left[ r - (1 + m_3) \lambda_1 + m_2 \lambda_2 \right] \cdot m_2 + m_2 - m_2 / (1 + m_3)$

$\cdot \left[ r - \lambda_2 \right]^{m_2 / (1 + m_3)}$ if $\lambda_1 < \lambda_2$

$= r \left[ r + m_2 \lambda_1 + m_3 \lambda_2 -(1+ m_2 + m_2) \lambda_3 \right]^{m_1 - 1}$

$\cdot \left[ r - (1 + m_3) \lambda_1 + m_2 \lambda_2 \right]^{1 + m_3}$

$\cdot \left[ r - \lambda_2 \right]^{m_2 + m_2 - m_2 / (1 + m_3)}$ if $\lambda_1 > \lambda_2$ (2.4.16)

Hence,

Theorem 2.4.5. A set of necessary conditions for the existence of this class of designs is
i) \( g_1 = r + m_2 \lambda_1 + m_3 \lambda_2 - (1 + m_2 + m_3) \lambda_3 \geq 0 \)

ii) \( g_2 = r - (1 + m_3) \lambda_1 + m_3 \lambda_2 \geq 0 \)

iii) \( g_3 = r - \lambda_2 \geq 0 \)

In particular, if the design be symmetrical, then

\[
\begin{align*}
iv) \quad g_1 &= \frac{m_1-1}{m_1} g_2 + m_1 \frac{m_2 + m_3 - m_2}{(1 + m_3)} \frac{m_1 m_2}{(1 + m_3)} \quad \text{[or, } g_1 = \frac{m_1-1}{m_1} g_2 + m_1 \frac{m_2 + m_3 - m_2}{(1 + m_3)}] \\
&\text{if } \lambda_1 > \lambda_2 \text{ must be a perfect square.}
\end{align*}
\]

Example. The class of designs \( r=6, \lambda_1=1, \lambda_2=0, \lambda_3=2, m_2=m_3=2 \) are impossible.

\( G. \) Here

\[
\begin{align*}
|NN^*| &= rk[r + m_2 \lambda_1 + m_3 \lambda_2 - (1 + m_2 + m_3) \lambda_3]^{m_1-1} [r-\lambda_2]^{m_1 m_3/(1 + m_2)} \\
&\cdot [r + m_2 \lambda_1 - (1 + m_2) \lambda_2]^{m_1 [m_2 + m_3 - m_2]/(1 + m_3)} \quad \text{if } \lambda_1 < \lambda_2 \\
&= rk[r + m_2 \lambda_1 + m_3 \lambda_2 - (1 + m_2 + m_3) \lambda_3]^{m_1-1} [r-\lambda_1]^{m_1 [m_2 + m_3 - m_3]/(1 + m_2)} \\
&\cdot [r + m_2 \lambda_1 - (1 + m_2) \lambda_2]^{m_1 m_2/(1 + m_2)} \quad \text{if } \lambda_1 > \lambda_2 \quad (2.4.17)
\end{align*}
\]

Hence,

Theorem 2.4.6. A set of necessary condition for the existence of this type of design is
1) \( g_1 = r+m_2\lambda_1 + m_3\lambda_2 - (1+m_2+m_3)\lambda_3 \geq 0 \)

ii) \( g_2 = r-\lambda_1 \geq 0 \)

iii) \( g_3 = r+m_2\lambda_1 - (1+m_2)\lambda_2 \geq 0 \)

In particular, if the design be symmetrical, then

\[
\begin{align*}
g_1 &= g_1 - \frac{m_1[m_2+m_3-m_2/(1+m_2)] m_2 m_3/(1+m_2)}{g_3} \\
g_2 &= g_2 - \frac{m_1(1+m_2-m_2)}{g_1} \\
g_3 &= g_3 - \frac{m_1 m_3}{g_2}
\end{align*}
\]

(or \( \frac{g_1}{g_2} = \frac{g_2}{g_3} \) if \( \lambda_1 > \lambda_2 \)) must be a perfect square.

Example. The class of designs \( r=4, \lambda_1=0, m_2=m_3=2, \lambda_2=1, \lambda_3=2 \) do not exist.

Evaluation of \( C_p(NN') \). We shall derive the Hasse-Minkowski \( p \)-invariant of \( NN' \) for subclass \( C \) of both the generalisations, but it can be easily shown that the same expression holds true for the subclasses \( A \) and \( B \) as well.

We shall make use of the results of Ogawa (1959), Shrikhande and Jain (1962) and Singh and Sukla (1963) without further reference.

Let \( M = NN' \) be a positive definite matrix of order \( v \). Let \( g_0, g_1, g_2, g_3 \) be the 4 distinct and rational eigenvalues of \( M \) with multiplicities \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) respectively. Let \( X_1 \) be a \( (v \times \alpha_1) \) matrix such that the columns \( x_{ij} \) generate the eigenspace of \( M \) corresponding to \( g_i, \ i = 0, 1, 2, 3. \)

Consider the following set of \( 1+(m_1-1)(m_2+m_3+1)+\alpha_2 \)

\( v \)-vectors.
\[ x'_{01} = (1, 1, \ldots, 1) \]
\[ x'_{11} = (1_1, 0, \ldots, 0) \]
\[ x'_{12} = (1_2, 0, \ldots, 0) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ x'_{1,m_1-1} = (j_{m_1-1}, 0, \ldots, 0) \]
\[ x'_{1,m_1} = (0, 1, 0, \ldots, 0) \]
\[ x'_{1,(m_1+1)} = (0, j_2, 0, \ldots, 0) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ x'_{1,2(m_1-1)} = (0, j_{m_1-1}, 0, \ldots, 0) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ x'_{1,(m_1-1)(m_2+m_3)+1} = (0, 0, \ldots, j_1) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ x'_{1,(m_1-1)(m_2+m_3)+1} = (0, 0, \ldots, j_{m_1-1}) \]
\[ x'_{21} = (1, -1, 0, \ldots, 0) \]
\[ x'_{22} = (1, 1, -2, \ldots, 0) \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ x'_{2\alpha_2} = (1, 1, 1, \ldots, -\alpha_2, 0, \ldots, 0) \]

where \( i = (i, i, \ldots, i) \) is an \( m_1 \)-vector
\( -i = (-i, -i, \ldots, -i) \) is an \( m_1 \)-vector
\( j_1 = (i, -1, 0, \ldots, 0) \) is an \( m_1 \)-vector
\( j_1 = (1, 1, \ldots, -i, 0, \ldots, 0) \) is an \( m_1 \)-vector
It is easy to verify that \( X_0 = (X_{01}) \) is the eigen-vector corresponding to \( \varrho_0 = r \), \( X_1 = (X_{11}, X_{12}, \ldots, X_{1l}, (m_1-1)(m_2+m_3+1)) \)
is the eigen space corresponding to the eigen value \( \varrho_1 = r - \lambda_1 \),
\( X_2 = (X_{21}, X_{22}, \ldots, X_{2\alpha_2}) \) the eigen space corresponding to
\( \varrho_2 = r + (m_1-1) \lambda_1 - m_1 \lambda_2 \) for the first type of generalisation.

Let \( S = (X_0, X_{11}, \ldots, X_1, (m_1-1)(m_2+m_3+1)); \)
\( X_{21}, X_{22}, \ldots, X_{2\alpha_2}, X_{31}, \ldots, X_{3\alpha_3} \) where \( X_{31}, \ldots, X_{3\alpha_3} \) constitute
a set of \( \alpha_3 \) vectors orthogonal to the vector space generated by
\( X_0, X_{11}, \ldots, X_1, (m_1-1)(m_2+m_3+1), X_{21}, \ldots, X_{2\alpha_2} \). Then \( S \) is
a non-singular \( v \times v \) matrix with rational elements.

Putting \( Q_i = (X_i, \ldots, X_i) \), \( i = 0, 1, 2, 3 \)
we have

\[
|Q_0| = v
|Q_1| = [1, 2, 3, \ldots, (m_1-1)m_1]^{m_2+m_3+1}
= [m_1 \left\{ (m_1-1)! \right\}^2]^{m_2+m_3+1}
\sim \frac{m_2+m_3+1}{m_1} \quad (2.4.19)
\]

\[
|Q_2| = [1, 2, m_1, 2, 3, m_1, \ldots, \alpha_2(\alpha_2+1)m_1]^{\alpha_2}
= (m_1^{\alpha_2})^{\alpha_2} \left[ (\alpha_2+1)! \right]^{\alpha_2} \sim (\alpha_2+1)^{\alpha_2} \quad (2.4.20)
\]
Since \( S^*MS = \begin{bmatrix} g_0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & g_3 \end{bmatrix} \)

or \( M = \begin{bmatrix} g_0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & g_3 \end{bmatrix} \)

We find

\[
C_p(M) = (-1,-1)_p \left( g_0, -\nu g_1 g_2 g_3 \right)_p (g_1, g_2)_p (g_1 g_2, g_1 g_2)_p \\
\left( g_2, g_3 \right)_p (g_1, -1)_p (g_1, g_2)_p (g_2, g_3)_p (g_3, -1)_p (g_3, -1)_p (g_3, -1)_p
\]

\[
= \left( g_3, -1 \right)_p (g_3, -1)_p (g_3, -1)_p (g_1, -1)_p (g_1, -1)_p (g_1, -1)_p
\]

\[
= (2.4.21)
\]

As all the terms on the right hand side of \( C_p(M) \) are known, \( C_p(NN') \) can be evaluated.

For the second type of generalisation the same expression for \( C_p(NN') \) will be obtained but with different \( Q_1, Q_2, Q_3 \), as we find that the following set of \( 1+(m_1-1)+m_1(m_2+m_3-a_3') \) \( v \)-vectors form the eigen space corresponding to \( g_0 = rk \),
\[ q_1 = r + m_2 \lambda_1 + m_3 \lambda_2 - (1 + m_2 + m_3) \lambda_3 \quad \text{and} \quad q_2 = r - \lambda_1 \]

where \( a_2^i \) is different for different subclasses. (For subclass C \( a_2^i = m_3 / (1 + m_2) \)).

\[
\begin{align*}
X'_{01} &= (1, 1, \ldots, 1) \\
X'_{11} &= (1, -1, 0, \ldots, 0) \\
& \quad \ldots \\
X'_{1,(m_1-1)} &= (1, 1, 0, \ldots, -(m_1-1)) \\
X'_{21} &= (j, 0, 0, \ldots, 0) \\
& \quad \ldots \\
X'_{2,(m_2+m_3-a_2^i)} &= (j, m_2+m_3-a_2^i, 0, \ldots, 0) \\
& \quad \ldots \\
X'_{2, (m_2+m_3-a_2^i)(m_1-1)+1} &= (0, 0, \ldots, j) \\
& \quad \ldots \\
X'_{2, m_1(m_2+m_3-a_2^i)} &= (0, 0, \ldots, j, m_2+m_3-a_2^i) \\
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{i} &= (1, 1, \ldots, 1) - \text{an} (m_2 + m_3 + l) - \text{vector} \\
\mathbf{i} &= (-1, -1, \ldots, -1) - \text{a} (m_2 + m_3 + l) - \text{vector} \\
\mathbf{j} &= (1, -1, 0, \ldots, 0) - \text{a} (m_2 + m_3 + l) - \text{vector} \\
\mathbf{j} &= (1, 1, \ldots, -1, 0, \ldots, 0) - \text{a} (m_2 + m_3 + l) - \text{vector} \\
\end{align*}
\]

Here \( |Q_0| \sim v \) (2.4.23)
\[ |Q_1| \sim [m_1(m_2+m_3+1)]^{m_1-1} \quad (2.4.24) \]
\[ |Q_2| \sim [m_2+m_3+1-a_2]^{m_1} \quad (2.4.25) \]

For both the types of generalisations, for subclasses A and B, \(|Q_1|\) remains unaltered. Now, it is known that by interchanging suitably selected rows and columns, a matrix is transformed to a congruent matrix. Hence the grammian will also remain unchanged for such transformation. By writing the third associates in the positions of the second associates (in subclass 0) we can verify that the eigen vector \(\lambda_2\) corresponding to the eigen value 
\[ \phi_2 = r+(m_1-1)\lambda_1 - m_1\lambda_2 \]
for the first type of generalisation becomes the eigen vector corresponding to the eigen value 
\[ \phi_2' = r+(m_1-1)\lambda_1 - m_1\lambda_3 \]
also for the first type of generalisation. However, the order of \(\lambda_2\) may change. Second type of generalisation can be similarly discussed.

**Example.** The design \(v=b=24, r=k=6, m_1=4, m_2=2, m_3=3\)
\[ \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1 \]
\[ (p^1_{jk}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 18 \end{bmatrix} \]
\[ (p^2_{jk}) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 18 \end{bmatrix} \]
\[ (p_{ijk}^3) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 12 \end{bmatrix} \]

which belongs to second type of generalisation of subclass C is impossible.

References


Roy, P.M. (1953) : Hierarchical group divisible incomplete block designs with m-associate classes. Science and Culture, 19, 210-211.


