CHAPTER EIGHT

GROUP DIVISIBLE DESIGNS WITH VARIABLE REPLICACTIONS

§1 Introduction

Group divisible designs with variable replications (introduced here) form an important subclass of 'Balanced block designs with variable replications' (Chapter VII) and themselves include the group divisible incomplete block designs (G.D. designs) of Bose and Connor (1952) and H.C.D. designs with three associate classes of Roy (1954, 1955, 1962). They may be defined as follows.

An incomplete block design with $v$ treatments in $b$ blocks of $k$ plots each is said to be a group divisible design with variable replications if the treatments can be divided into $(\mu_1+\mu_2)$-groups satisfying the following conditions:

(i) There is a set $S_1$ of $\mu_1$ groups, each with $v_1$ treatments and a set $S_2$ of $\mu_2$ groups, each with $v_2$ treatments. Obviously,

$$v = v_1 \mu_1 + v_2 \mu_2.$$  

(ii) The treatments belonging to the set $S_1$ are replicated $r_i$ times, $i = 1, 2$.

(iii) Any two treatments taken from the same group of the set $S_1$ appear together in $\lambda_{ii}$ blocks, while any two treatments taken from two distinct groups of the same set $S_1$ appear together in $\lambda_{ij}$ blocks, $i = 1, 2$.  

(iv) Any two treatments one from the set $S_1$ and the other from the set $S_2$ appear in $\lambda_{12}$ blocks.

The following relations follow immediately,

$$bk = v_1 \mu_1 r_1 + v_2 \mu_2 r_2$$  \hspace{1cm} (8.1.1)

$$r_1(k-1) = \lambda_{11}(v_1-1) + \lambda_{11}(\mu_1-1) v_1 + \lambda_{12} \mu_2 v_2$$  \hspace{1cm} (8.1.2)

$$r_2(k-1) = \lambda_{22}(v_2-1) + \lambda_{22}(\mu_2-1) v_2 + \lambda_{12} \mu_1 v_1$$  \hspace{1cm} (8.1.3)

These designs can be obtained as a subclass of 'Balanced block designs with variable replications'.

Proof: To see this, let us consider a balanced block design with $s$ replications. Let the parameters of the balanced block design be

$$v_1, v_2, \ldots, v_s, \quad r_1, r_2, \ldots, r_s$$

$$\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1s}, \lambda_{22}, \ldots, \lambda_{ss}$$

Let $v_1 = v_2 = \ldots = v_t = v_{t+1} = v_{t+2} = \ldots = v_s = v_s$

$$r_1 = r_2 = \ldots = r_t = r_{t+1} = r_{t+2} = \ldots = r_s = r_s$$

$$\lambda_{ij} = \lambda_{1i}^t \quad \text{for} \quad i = 1, 2, \ldots, t$$

$$\lambda_{ij} = \lambda_{i2}^s \quad \text{for} \quad i = 1, 2, \ldots, s$$

$$= \lambda_{i2}^s \quad \text{for} \quad i = 1, 2, \ldots, s$$

$$= \lambda_{i2}^s \quad \text{for} \quad i = 1, 2, \ldots, t, \quad j = t+1, \ldots, s.$$
The group divisible designs with variable replications reduce to a G.D. design if either \( r_1 \) or \( r_2 \) is zero.

Proof: Consider a group divisible design with variable replications with parameters:

\[
\begin{align*}
&v_1, v_2, \mu_1, \mu_2, r_1, r_2, \lambda_{11}, \lambda_{12}, \\
&\lambda_{22}, \lambda_{22}, \lambda_{12}, b, k.
\end{align*}
\]

Let \( r_2 = 0 \). Then \( \lambda_{22}, \lambda_{12}, \lambda_{11} \), should all vanish. Hence, \( v_2 = \mu_2 = 0 \). Thus, we have a set of \( \mu_1 \) groups each containing \( v_1 \) treatments, where each treatment is replicated \( r_1 \) times. Two treatments taken from the same group occur together in \( \lambda_{11} \) blocks and two treatments taken from different groups occur together in \( \lambda_{11} \) blocks i.e. a G.D. design whose parameters are

\[
\begin{align*}
&v^* = v_1 \mu_1, \quad \mu^* = \mu_1, \quad \lambda^* = v_1, \quad r^* = r \\
&k^* = k, \quad \lambda_1^* = \lambda_{11}, \quad \lambda_2^* = \lambda_{11}.
\end{align*}
\]

H.G.D. designs (three associate classes, \( N_1=2 \)) form a subclass of group divisible designs with variable replications.

Proof: Consider a group divisible design with variable replications.

\[
\begin{align*}
&v_1, v_2, \mu_1, \mu_2, r_1, r_2, \lambda_{11}, \lambda_{11} \\
&\lambda_{22}, \lambda_{22}, \lambda_{12}, b, k.
\end{align*}
\]

Let \( r_1 = r_2 = r, \quad v_1 = v_2, \quad \mu_1 = \mu_2, \quad \lambda_{11} = \lambda_{22}, \quad \lambda_{11} = \lambda_{22} \).
Then we have an arrangement of \( v = 2v_1 \mu_1 \) treatments divided into \( N_1 = 2 \) groups of \( S_1 = v_1 \mu_1 \) treatments each such that any two treatments not belonging to the same group occur together \( \lambda_1 = \lambda_2 \) times. At the second stage, \( S_2 = v_1 \mu_1 \) treatments of either group of the first stage can be divided into \( N_2 = \mu_1 = \mu_2 \) groups of \( S_2 = v_1 = v_2 \) treatments each such that any two treatments of a group of the first stage but not belonging to the same group of the second stage occur together \( \lambda_3 = \lambda_1 = \lambda_2 \) times and any two treatments belonging to the same group of the second stage also, occur together \( \lambda_3 = \lambda_1 = \lambda_2 \) times. i.e. an R.G.D. design with three associate classes for which \( N_4 = 2 \).

### 8.2 Analysis of group divisible designs with variable replications

Let \( y_{ij} \) denote the yield of the \( i \)-th treatment from the \( j \)-th block. Then we may write

\[
y_{ij} = \mu + t_i + b_j + \epsilon_{ij}
\]

where \( \mu \) is a general effect, \( t_i \) is the effect of the \( i \)-th treatment, \( b_j \) is the effect of the \( j \)-th block and \( \epsilon_{ij} \) is a random component with expectation zero, the \( \epsilon_{ij} \)'s being independently distributed with a common variance \( \sigma^2 \).

Let \( \pi = \sum_{i=1}^v t_i \), \( \sum_{i=1}^v t_i = 0 \) be an estimable treatment contrast. Then, it is known (Bose, 1950) that the best linear intra-block estimate of \( \pi \) is obtained by substituting in \( \pi \), \( t_i = \hat{t}_i \), where \( \hat{t}_i \) is the solution of the reduced normal equations.
\[ c_{1i} t_1 + c_{12} t_2 + \cdots + c_{1v} t_v = q_1, \quad i = 1, 2, \ldots, v \quad (8.2.2) \]

where \[ c_{1i} = x_i (1 - 1/k) \quad (8.2.3) \]

\[ \sigma_{1u} = -\lambda_{1u}/k, \quad i \neq u \]

Here we may write equation (8.2.2) as

\[
x_1(1 - \frac{1}{k})t_1 = \frac{\lambda_{11}}{k} \sum_{i'=k+1}^{(k+1)v_1} t_{i'} - \frac{\lambda_{11}}{k} \sum_{i'=1}^{k} S_{k'}(t_1)
- \frac{\lambda_{12}}{k} \sum_{i'=\mu_1+1}^{\mu_2+2} S_{k'}(t_1) = q_1 \quad (8.2.4)
\]

for \( i = k v_1 + 1, \ldots \) \( (k+1)v_1 \) \( k = 0, 1, \ldots, \mu_1 \) i.e. for an \( i \) belonging to the \((k+1)\)-th group of treatments of the first set and \( S_{k'}(t_1) \) sum of the treatment effects of the \( k' \)-th group of treatments

and

\[
x_2(1 - \frac{1}{k})t_1 = \frac{\lambda_{21}}{k} \sum_{i'=k+1}^{v_2(k+1)v_2} t_{i'} - \frac{\lambda_{22}}{k} \sum_{i'=1}^{v_2} S_{k'}(t_1)
- \frac{\lambda_{22}}{k} \sum_{i'=\mu_1+2}^{\mu_2+2} S_{k'}(t_1) = q_1 \quad (8.2.5)
\]

for \( i = v_1 \mu_1 + k v_2 + 1, \ldots \) \( v_1 \mu_1 + (k+1)v_2 \) \( k = 0, 1, \ldots, \mu_2 \) i.e. for an \( i \) belonging to \((k+1)\)-th group of the second set, where \( q_1 \) represents the adjusted treatment total for the \( i \)-th treatment.
In (8.2.4) and (8.2.5) we number the groups as 1, 2, ..., \( \mu_1 \), \( \mu_1 + 1 \), ..., \( \mu_1 + \mu_2 \). In numbering the treatments we follow the convention that \((k+1)\)-th group of the first set consists of treatments number \( k \, \nu_{1+1}, k \, \nu_{1+2}, \ldots, (k+1) \, \nu_{1} \) and \( k\)-th group of the second set consists of treatments number \( \nu_{1,1} + k \, \nu_{1+1}, \ldots, \nu_{1,1} + (k+1) \, \nu_{1} \).

We may rewrite equation (8.2.4) as

\[
\left[ \frac{r_1(k-1) + \lambda_{11}}{k} \right] t_1 + \frac{\lambda_{11}}{k} (k+1) \nu_{1} = \frac{\lambda_{11}}{k} \sum_{i=1}^{k+1} t_i - \frac{\lambda_{11}}{k} \sum_{k+1}^{k+1} \nu_{1,1} \left( t_1 \right) \]

or,

\[
\left[ \frac{r_1(k-1) + \lambda_{11}}{k} \right] t_1 + \frac{\lambda_{11}}{k} \sum_{i=1}^{k+1} \nu_{1,1} \left( t_1 \right) = \frac{\lambda_{11}}{k} \sum_{k+1}^{k+1} \nu_{1,1} \left( t_1 \right) - \frac{\lambda_{12}}{k} \sum_{k+1}^{k+1} \nu_{1,1} \left( t_1 \right)
\]

Summing equation (8.2.6) over the \( \nu_{1} \) equations corresponding to the \((k+1)\)-th group of treatments,

\[
\frac{r_1(k-1) + \lambda_{11}}{k} S_{\nu_{1,1}}(t_1) + \frac{\nu_{1}(\lambda_{11} - \lambda_{11})}{k} S_{\nu_{1,1}}(t_1) = \frac{\nu_{1}(\lambda_{11})}{k} \sum_{k+1}^{k+1} S_{\nu_{1,1}}(t_1)
\]

\[
\frac{\nu_{1}(\lambda_{12})}{k} \sum_{k+1}^{k+1} S_{\nu_{1,1}}(t_1) = S_{\nu_{1,1}}(Q_{1})
\]

where \( S_{\nu_{1,1}}(Q_{1}) \) represent the sum of the adjusted treatment totals of
the treatments belonging to the \((j+1)\)-th group

\[
\frac{r_1(k-1)-r_1-1}{k} \lambda_{11} \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) = \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) - \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1)
\]

\[
= \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) - \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) = s_k^*(t_1) = s_k^*(Q_1) \quad (8.2.7)
\]

Again, summing (8.2.6) over all treatments of the set \(S_1\),

\[
[r_1(k-1)+\lambda_{11}/k] \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) - r_1 \lambda_{11} \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1)
\]

\[
= \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) - \frac{v_1}{k} \lambda_{11} \frac{v_1}{k} s_k(t_1) = \sum_{k'=1}^{k} s_k^*(Q_1) \quad (8.2.8)
\]

Let us put the arbitrary restrictions as,

\[
\sum_{i=1}^{v_1} t_1 + \sum_{i=v_1+1}^{v_1+2} t_1 = 0
\]

\[
\sum_{k'=1}^{k} s_k^*(t_1) + \sum_{k'=v_1+1}^{k} s_k^*(t_1) = 0 \quad (8.2.9)
\]
or, \( \sum_{k'=\mu_1+1}^{\mu_2} S_{k'}(t_1) = -\frac{r_1}{r_2} \sum_{k'=1}^{\mu_1} S_{k'}(t_1) \) (8.2.10)

From (8.2.3),

\[
\frac{\lambda_2 \mu_2 v_2}{k} \sum_{k'=1}^{\mu_2} S_{k'}(t_1) + \frac{\lambda_2 \mu_2 v_1}{k} \sum_{k'=1}^{\mu_1} S_{k'}(t_1) = \frac{\mu_1}{k'} S_{k'}(Q_1)
\]

or, \( \frac{\lambda_2 v_2}{r_2} \left( v_2 \mu_2 + v_1 \mu_1 \right) \sum_{k'=1}^{\mu_1} S_{k'}(t_1) = \frac{\mu_1}{k'} S_{k'}(Q_1) \) (8.2.11)

or, \( \sum_{k'=1}^{\mu_1} S_{k'}(t_1) = \frac{r_2}{\lambda_2} \sum_{k'=1}^{\mu_1} S_{k'}(Q_1) \) (8.2.12)

Similarly, \( \sum_{k'=\mu_1+1}^{\mu_2} S_{k'}(t_1) = -\frac{r_1}{\lambda_2} \sum_{k'=1}^{\mu_1} S_{k'}(Q_1) \)

From (8.2.7) and (8.1.2),

\[
\frac{\lambda_1 v_1 \mu_2 + \lambda_2 v_2 \mu_2}{k} S_{k'}(t_1) - \frac{\lambda_1 v_1 \mu_1}{k} \sum_{k'=1}^{\mu_1} S_{k'}(Q_1) = \frac{\mu_1}{k'} S_{k'}(Q_1)
\]

or, \( \frac{\lambda_1 v_1 \mu_2 + \lambda_2 v_2 \mu_2}{k} S_{k'}(t_1) = S_{k'}(Q_1) + \frac{v_1}{\lambda_2} \left( \lambda_1 v_2 + \lambda_2 v_1 \right) \sum_{k'=1}^{\mu_1} S_{k'}(Q_1) \)

Hence, from (8.2.6)
\[
\hat{\xi}_i = \frac{k}{r_1(k-1) + \lambda_{11}} \left[ \frac{\lambda_{11} - \lambda_{11}^i}{\lambda_{11}^{i+1} + \lambda_{12}^{i+2} v_2^{i+2}} \right] \left\{ S'(q_1) + \frac{r_1(\lambda_{11} r_2 + \lambda_{12} r_1)}{k \lambda_{12}} S'(q_1) \right\} \\
+ \frac{\lambda_{11} - \lambda_{11}^i}{k \lambda_{12}^{i+2} v_2^{i+2}} \sum_{j=1}^{\lambda_{12}} S'(q_1) + \frac{\lambda_{12}^j}{k \lambda_{12}^{i+2} v_2^{i+2}} \sum_{j=1}^{\lambda_{12}} S'(q_1) \\
= \frac{k}{r_1(k-1) + \lambda_{11}} \left[ \frac{\lambda_{11} - \lambda_{11}^i}{\lambda_{11}^{i+1} + \lambda_{12}^{i+2} v_2^{i+2}} \right] S'(q_1) \\
+ \left\{ \frac{(\lambda_{11} - \lambda_{11}^i)v_1(\lambda_{11} r_2 + \lambda_{12} r_1)}{\lambda_{11}^{i+1} + \lambda_{12}^{i+2} v_2^{i+2}} + \frac{r_2 \lambda_{11}^i}{b k \lambda_{12}^{i+2} v_2^{i+2}} \right\} \sum_{j=1}^{\lambda_{12}} S'(q_1) \\
= c_1 \left[ q_1 + c_2^i S'(q_1) + c_2^i \mu_1 \sum_{j=1}^{\lambda_{12}} S'(q_1) \right] \quad (8.2.13)
\]

for any treatment belonging to the \((k+1)\)-th group of the first set, 
\(k=0, 1, \ldots, \mu_1-1\) where \(c_1 = \frac{k}{r_1(k-1) + \lambda_{11}}\), \(c_2^i = \frac{\lambda_{11} - \lambda_{11}^i}{\lambda_{11}^{i+1} + \lambda_{12}^{i+2} v_2^{i+2}} \frac{(\lambda_{11} - \lambda_{11}^i)v_1(\lambda_{11} r_2 + \lambda_{12} r_1)}{\lambda_{11}^{i+1} + \lambda_{12}^{i+2} v_2^{i+2}} + \frac{r_2 \lambda_{11}^i}{b k \lambda_{12}^{i+2} v_2^{i+2}} \).

Similarly, \(\hat{\xi}_i = c_2 \left[ q_1 + c_2^i S'(q_1) + c_2^i \mu_1 \sum_{j=1}^{\lambda_{12}} S'(q_1) \right] \quad (8.2.14)\)

for any treatment \(i\) belonging to \((k+1)\)-th group; \(k=\mu_1, \mu_1+1, \mu_1+\mu_2-1\) where \(c_2 = \frac{k}{r_2(k-1) + \lambda_{22}}\),

\(c_2^i = (\lambda_{22} - \lambda_{22}^i)/(\lambda_{22} v_2^{i+2} + \lambda_{12} v_1^{i+2} \mu_1)\)

\(c_2^i = \frac{(\lambda_{22} - \lambda_{22}^i)(\lambda_{22} v_2^{i+2} + \lambda_{12} v_1^{i+2} \mu_1)}{\lambda_{22} v_2^{i+2} + \lambda_{12} v_1^{i+2} \mu_1} + \frac{r_2 \lambda_{22}^i}{b k \lambda_{12}^{i+2} v_2^{i+2}} \frac{r_2}{b k} \).

Now, \(\nu(q_j) = c_{jj} \sigma^2 = r_j(1 - 1/k) \sigma^2\)

\(\text{Cov}(q_j, q_j) = c_{jj} \sigma^2 = \frac{\lambda_{11}^i \mu_1 \sigma^2}{k} \).
Hence, for any two treatments belonging to the same group of the \( i \)-th set, \( \nu(\hat{t}_8 - \hat{t}_8) = 2\sigma_i^2 \), \( i = 1, 2 \).

For any two treatments belonging to two different groups of the \( i \)-th set,
\[
\nu(\hat{t}_8 - \hat{t}_8) = \frac{2\sigma_i^2}{\kappa} \left[ \hat{d}_1 +(v_1 v_1' + 2)(\lambda_{11} - \lambda_{11}') \right]
\]  
(8.2.15)

where \( \hat{d}_1 = r_i(k-1) + \lambda_{1i}' \), \( i = 1, 2 \).

Finally, for any two treatments belonging to two different sets,
\[
\nu(\hat{t}_8 - \hat{t}_8) = \frac{2\sigma_i^2}{\kappa} \left[ r_1(k-1) + c_i(c_1 v_1 - 2) \left\{ r_1(k-1) - \lambda_{11}(v_1-1) \right\} \right.
\]
\[
+ \lambda_2 v_1 \mu_2 s_2 \left( c_1 v_1' - c_1 v_1 - 2 \right) \]
(8.2.16)

\[
- \frac{2\sigma_i^2}{\kappa} [r_2(k-1) + c_i(c_2 v_2 - 2) \left\{ r_2(k-1) - \lambda_{22}(v_2-1) \right\}]
\]
\[
+ \frac{\sigma_i^2}{\kappa} \left[ v_1 v_1' + c_i v_1 v_2 + v_1 v_2' \right] (8.2.17)
\]

<table>
<thead>
<tr>
<th>Source</th>
<th>D.F.</th>
<th>S.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks (unadjusted)</td>
<td>( b - 1 )</td>
<td>( \Sigma \hat{d}_i^2 / k - \sigma^2 / n )</td>
</tr>
<tr>
<td>Treatments (adjusted for blocks)</td>
<td>( v - 1 )</td>
<td>( \Sigma \hat{t}_s^2 )</td>
</tr>
<tr>
<td>Error</td>
<td>( n - b - v + 1 )</td>
<td>By subtraction</td>
</tr>
<tr>
<td>Total</td>
<td>( n - 1 )</td>
<td>( \Sigma y_{ij}^2 = \sigma^2 / n )</td>
</tr>
</tbody>
</table>
In the above analysis, we have considered the block effects to be constant. If instead of considering the blocks to be fixed, we assign the treatments to a set of randomly chosen blocks then \( b_j \)'s becomes a random variable with zero mean and variance \( \sigma^2_{bj} \). In this case, any contrast between the block totals will estimate a linear function of the treatment effect (Nair, 1944). It has been shown by Rao (1947), Bose and Shimamoto (1952) and Chakraborti (1962) that the combined inter and intra-block estimates are easily obtained by replacing \( Q_s, r_s \) and \( \lambda_{ss} \) by \( P_s, R_s \) and \( \Lambda_{ss} \) where

\[
P_s = w Q_s + w' Q_s', \quad R_s = r_s [w + w'/(k-1)], \quad \Lambda_{ss} = (w - w') \lambda_{ss},
\]

and

\[
Q_s' = Q_s - r_s \sigma / bk; \quad w = 1/\sigma^2, \quad w' = 1/(\sigma^2 + k \sigma^2).
\]

Hence, we find the combined estimates as

\[
\hat{t}_s = c_s [P_s + c_s' S_f(P_s) + c_s'' \Sigma S_f(P_s)]
\]

(8.2.18)

for any treatment \( i \) of \((k+1)\)-th group, \( i = 0, \ldots, \mu_1-1 \) and

\[
\hat{t}_s = c_s [P_s + c_s' S_f(P_s) + c_s'' \Sigma S_f(P_s)]
\]

(8.2.19)

for any treatment \( i \) of \((k+1)\)-th group, \( i = \mu_1, \mu_1+1, \ldots, \mu_1+\mu_2-1 \)

where

\[
c_s = \frac{1}{[w(1-1/k) + w'/k] + (w - w'/k) \lambda_{ii}}
\]

\[
c_s' = c_s', \quad i = 1, 2.
\]

\[
c_s'' = [w + w'/(k-1)] c_s'', \quad i = 1, 2.
\]

For any two treatments belonging to the same group of the \( i \)-th set

\[
V(\hat{t}_s - \hat{t}_s') = \frac{2c_s^2}{k} [(w^2 - w'^2) \{r_1(k-1) + \lambda_{ii}\} + k w^2 r_1]
\]

(8.2.19)
For any two treatments belonging to two different groups of the $i$-th set,

$$v(t_i^* - t_j^*) = 2c_i^2/k \left\{ (w^2-w_l^2) \left\{ c_i + (v_1 c_1 + 2)v^{(1)}_1 c_1 \right\} \right\}$$

$$+ kr_1 w_l^2 (1 + c_i v_1 + 2c_i^2)$$

Finally, for any two treatments belonging to two different sets

$$v(t_i^* - t_j^*) = \frac{c_i^2}{k} \left\{ 1 + (v_1 c_1 + 2)v^{(1)}_1 c_1 \right\} \left\{ (w^2-w_l^2) \right\}$$

$$\left( r_1(k-l) + w^2 r_1 k \right)$$

$$(v_1^2 - 1) \left\{ (v_1^2 + 2)c_1^{(1)} + (v_1^2 + 2)c_1^{(1)} v^{(1)}_1 c_1 \right\} \left( w^2-w_l^2 \right) \lambda_{11}$$

$$- v_1 (v_1^2 - 1) \left\{ (v_1^2 + 2)c_1^{(1)} + 2v_1^2 c_1^{(1)} \right\} \left( w^2-w_l^2 \right) \lambda_{11}$$

$$- 2c_i c_1^{(1)} v_1 v_2 c_2^{(1)} + c_i^{(1)} v_1 c_2^{(1)} c_3^{(1)} v_1 v_2$$

$$+ c_i^{(1)} c_2^{(1)} c_3^{(1)} c_4^{(1)} v_1 v_2 c_2^{(1)} + c_i^{(1)} c_2^{(1)} c_3^{(1)} v_1 v_2$$

$$+ v_1 c_1^{(1)} c_2^{(1)} c_3^{(1)} c_4^{(1)} v_1 v_2$$

$$+ \frac{c_i^2}{k} \left\{ 1 + (v_2^2 + 2)c_2^{(1)} + (v_2^2 + 2)v_2^2 c_2 \right\} \left\{ (w^2-w_l^2) \right\}$$

$$\left( r_2(k-l) + w^2 r_2 k \right)$$

$$- (v_2^2 - 1) \left\{ (v_2^2 + 2)c_2^{(1)} + 2v_2^2 c_2^{(1)} \right\} \left( w^2-w_l^2 \right) \lambda_{22}$$

$$- v_2 (v_2^2 - 1) \left\{ (v_2^2 + 2)c_2^{(1)} + 2v_2^2 c_2^{(1)} \right\} \left( w^2-w_l^2 \right) \lambda_{22}$$

$$- v_3 (v_3^2 - 1) \left\{ (v_3^2 + 2)c_3^{(2)} + 2v_3^2 c_3^{(2)} \right\} \left( w^2-w_l^2 \right) \lambda_{33}$$

(8.2.21)

### 3.3. Combinatorial properties of group divisible designs with variable replications

Let $N$ be the incidence matrix of the design. Let $N'$ denote the transpose of $N$. Then
where $A = \begin{bmatrix} r_1 & \lambda_{11} & \cdots & \lambda_{11} \\ \lambda_{11} & r_1 & \cdots & \lambda_{11} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{11} & \lambda_{11} & \cdots & r_1 \end{bmatrix}$ : $v_1 \times v_1$

$B = \begin{bmatrix} \lambda_{11} & \lambda_{11} & \cdots & \lambda_{11} \\ \lambda_{11} & \lambda_{11} & \cdots & \lambda_{11} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{11} & \lambda_{11} & \cdots & \lambda_{11} \end{bmatrix}$ : $v_1 \times v_1$

$c = \begin{bmatrix} r_2 & \lambda_{22} & \cdots & \lambda_{22} \\ \lambda_{22} & r_2 & \cdots & \lambda_{22} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{22} & \lambda_{22} & \cdots & r_2 \end{bmatrix}$ : $v_2 \times v_2$

$d = \begin{bmatrix} \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \\ \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \end{bmatrix}$ : $v_2 \times v_2$

$E = \begin{bmatrix} \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \\ \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{12} & \lambda_{12} & \cdots & \lambda_{12} \end{bmatrix}$ : $v_1 \times v_2$

$E' = \text{transpose of } E$.

We shall refer to each column of the matrix $NN'$ as a submatrix.

Consider the first submatrix.
This is a $v \times v_1$ matrix. Let us subtract the last column of (8.3.2) from its proceeding $(v_1-1)$ columns. Repeat the operation on the other $(\mu_1 + \mu_2 - 1)$ sub-matrices. The value of the determinant $|M'|$ is unaltered by these operations. In the transformed determinant add

1st, 2nd, ... $(v_1-1)$th row to the $v_1$-th row
$(v_1+1)$-th, ...$(2v_1-1)$th row to the $2v_1$-th row
... ... ...

$[(\mu_1-1)v_1+1]$-th, ... $(\mu_1v_1-1)$-th row to the $\mu_1v_1$-th row
$(\mu_1v_1+1)$-th ...$(\mu_1v_1+\mu_2-1)$-th row to the $(\mu_1v_1+\mu_2)$-th row
... ... ...
$[\mu_1v_1+(\mu_2-1)v_1+1]$-th ...$(\mu_1v_1+\mu_2v_2-1)$-th row to the $(\mu_1v_1+\mu_2v_2)$-th row

Then (8.3.1) reduces to

$$(r_1 - \lambda_{11})^{\mu_1(v_1-1)} (r_2 - \lambda_{22})^{\mu_2(v_2-1)} |P| \quad (8.3.3)$$

where $|P| = 
\begin{vmatrix}
^11^12 \ldots ^12^2 \ldots ^12^3 \\
^11^1 \ldots ^12 \ldots ^12 \\
^11^1 \ldots R_1^12 \ldots ^12 \\
\ldots \ldots \ldots \\
^11^1 \ldots R1^12 \ldots ^12 \\
^12^1 \ldots ^12 \ldots ^12^2 \ldots ^22 \\
\ldots \ldots \ldots \\
^12^1 \ldots ^12 \ldots ^12^2 \ldots ^22 \\
\end{vmatrix} : (\mu_1+\mu_2)x(\mu_1+\mu_2) \quad (8.3.4)$
where \( R_1 = x_1 + \lambda_{11}(v_1 - 1), R_2 = x_2 + \lambda_{22}(v_2 - 1), \) \( \lambda_{11} = \lambda_{11}/v_1 \)

\[
\begin{align*}
\lambda_{12} &= \lambda_{12}/v_2, \quad \lambda_{12} = \lambda_{12}/v_1, \quad \lambda_{22} = \lambda_{22}/v_2
\end{align*}
\]

Let us subtract the \( \mu_1 \)-th column of \( |B| \) from the first \( (\mu_1 - 1) \) columns and subtract the last column of \( |B| \) from \( \mu_1 \)-th, \( \ldots \) \( (\mu_1 + \mu_2 - 1) \)-th columns. In the transformed determinant add

let, 2nd, \ldots (\( \mu_1 - 1 \))-th row to the \( \mu_1 \)-th row and \( (\mu_1 + 1) \)-th row to the \( (\mu_1 + \mu_2) \)-th row. Then (8.3.4) reduces to

\[
\begin{align*}
R_1 + \lambda_{11}(\mu_1 - 1) & \quad \lambda_{12} \mu_1 \\
\lambda_{12} \mu_2 & \quad R_2 + \lambda_{22}(\mu_2 - 1)
\end{align*}
\]

Now, \( |D'| = \begin{vmatrix} R_1 + \lambda_{11}(\mu_1 - 1) & \lambda_{12} \mu_1 \\
\lambda_{12} \mu_2 & R_2 + \lambda_{22}(\mu_2 - 1) \end{vmatrix} \]

\[
= \begin{vmatrix} R_1 + \lambda_{11}(\mu_1 - 1) + \lambda_{12}\mu_2 & \lambda_{12} \mu_2 + R_2 + \lambda_{22}(\mu_2 - 1) \\
\lambda_{12} \mu_2 & R_2 + \lambda_{22}(\mu_2 - 1) \end{vmatrix}
\]

\[= kR_1 - kR_2 \]

\[= \lambda_{12}v_2\mu_2 - \lambda_{12}v_1\mu_1 \]

\[= k[r_1r_2k - \lambda_{12}(v_1\mu_1 + v_2\mu_2)] = k^2(r_1r_2 - \lambda_{12}b) \]

Hence

\[|NN'| = k^2(r_1 - \lambda_{11})(v_1 - 1)(r_2 - \lambda_{22})(v_2 - 1)(r_1r_2 - \lambda_{12}b) (r_1k - \lambda_{11}v_1\mu_1 - \lambda_{12}v_2\mu_2) \]

\[= k^2(r_1 - \lambda_{11})(v_1 - 1)(r_2 - \lambda_{22})(v_2 - 1)(r_1r_2 - \lambda_{12}b) \]

(8.3.5)
On substituting \( r_1 \) for \( r \) in \( |HN'| = 0 \) the characteristic equation becomes

\[
(r - \lambda_{11} - z) \mu_1(v_1-1) \quad (r_2 - \lambda_{22} - z) \mu_2(v_2-1) \quad (r_1 - 11 - z) \mu_1^{-1}
\]

\[
(r_2 - 22 - z) \mu_2^{-1}
\]

Thus, the characteristic roots of the equation and their multiplicities are

1) \( \theta_1 = r_1 - \lambda_{11} \) with multiplicity \( \alpha_1 = \mu_1(v_1-1) \)

2) \( \theta_2 = r_2 - \lambda_{22} \) with multiplicity \( \alpha_2 = \mu_2(v_2-1) \)

3) \( \theta_3 = r_1(v_1-1)\lambda_{11} - 1 \) with multiplicity \( \alpha_3 = \mu_1^{-1} \)

4) \( \theta_4 = r_2(v_2-1)\lambda_{22} - \lambda_{22} v_2 \) with multiplicity \( \alpha_4 = \mu_2^{-1} \)

5) \( \theta_5 \) with multiplicity 1

6) \( \theta_6 \) with multiplicity 1

when \( \theta_5 \) and \( \theta_6 \) are the two roots of the quadratic equation

\[
z^2 - z[(r_1 + r_2)k - \lambda_{12}(v_1\mu_1 + v_2\mu_2) + (r_1 r_2 - \lambda_{12} k)^2] = 0 \quad (8.3.5)
\]

Thus,

\textbf{Theorem 8.3.1.} A set of necessary conditions for a design belonging to this class are

1) \( \theta_1 = r_1 - \lambda_{11} \geq 0 \)

2) \( \theta_2 = r_2 - \lambda_{22} \geq 0 \)

3) \( \theta_3 = r_1 + (v_1-1) \lambda_{11} - \lambda_{11} v_1 \geq 0 \)

4) \( \theta_4 = r_2 + (v_2-1) \lambda_{22} - \lambda_{22} v_2 \geq 0 \)
v) Both the roots of equation 3.3.5 must be positive.

Example. The class of designs with parameters $\nu_2=5$, $r_2=3$, $\lambda_{22}=1$, $N_{22}^2=2$ is impossible.

8.4. Methods of construction

A) Method of symmetrically repeated differences.

Consider a module $M$ consisting of $n$ elements. To each element $u$ of $M$ let us associate a set of $\mu_1+\mu_2$ treatments. Let us correspond the treatments having the suffixes $1, 2, \ldots, \mu_1$ with the $\mu_1$ groups of the first set and the treatments having the suffixes $\mu_1+1, \ldots, \mu_1+\mu_2$ with the $\mu_2$ groups of the second set.

Let us select a set of $t$ blocks each of size $k$ such that

1) Among the $kt$ treatments occurring in the $t$ blocks, each of the suffixes $1, 2, \ldots, \mu_1$ are represented $r_1$ times and each of the suffixes $\mu_1+1, \ldots, \mu_1+\mu_2$ are represented $r_2$ times.

ii) Among the different types of differences that arise from these blocks, the non-zero elements of the module occur $\lambda_{11}$ times among the pure differences of the type $(i, i)$, $i = 1, 2, \ldots, \mu_1$ and $\lambda_{22}$ times among the pure differences of the type $(i, i)$, $i = \mu_1+1, \ldots, \mu_1+\mu_2$; the elements of the module occur $\lambda_{11}$ times among the mixed differences of the type $(i, j)$, $i, j = 1, 2, \ldots, \mu_1$ and $\mu_2$ times among the mixed differences of the type $(i, j)$, $i, j = \mu_1+1, \ldots, \mu_1+\mu_2$; while the non-zero elements appear $\lambda_{12}$ times among the mixed differences of the type $(i, j)$, $i = 1, 2, \ldots, \mu_1$, $j = \mu_1+1, \ldots, \mu_1+\mu_2$.
Then on developing these blocks by adding the elements of $M$ we get the solution of the design

$$v_1 = v_2 = n, \quad x_1, x_2, \quad \lambda_{11}, \lambda_{12}, \lambda_{22}, \lambda_{12},$$

Examples. Consider the module of residue classes mod 2. So, there are two element of $M$, viz., 0 and 1. To each element $u$ of $M$ let us associate a set of 5 treatments $u_1, u_2, u_3, u_4$ and $u_5$. Let the treatments corresponding to $u_1$ and $u_2$ correspond to groups 1 and 2 of the set $S_1$, while those corresponding to $u_3, u_4$ and $u_5$ correspond to groups 3, 4 and 5 of the set $S_2$.

i) Consider the set of 6 blocks $(l_1 0_3 l_4 l_5), (l_1 l_3 0_4 l_5), (0_1 0_3 0_4 l_5), (l_2 0_3 l_4 l_5), (l_2 l_3 0_4 l_5)$ and $(0_2 0_3 0_4 0_5)$.

On developing these blocks (i.e., adding 0 and 1 in succession) we get the solution of the design

$$v_1 = v_2 = 2, \quad \mu_1 = 2, \quad \mu_2 = 3, \quad b = 12, \quad k = 4,$$

$$\lambda_{11} = 0, \quad \lambda_{12} = 0, \quad \lambda_{22} = 1, \quad \lambda_{22} = 2,$$

whose lay-outs are

$$(l_1 0_3 l_4 l_5), (l_1 l_3 0_4 l_5), (0_1 0_3 0_4 l_5), (l_2 0_3 l_4 l_5), (l_2 l_3 0_4 l_5), (0_2 0_3 0_4 0_5)$$

ii) Consider the set of 7 initial blocks $(0_1 l_1 0_2 l_2), (l_1 0_3 l_4 l_5), (l_1 l_3 0_4 l_5), (0_1 0_3 0_4 l_5), (l_2 0_3 l_4 l_5), (l_2 l_3 0_4 l_5)$ and $(0_2 0_3 0_4 l_5)$.

These blocks on developing gives the solution of the design

$$v_1 = v_2 = 2, \quad \mu_1 = 2, \quad \mu_2 = 3, \quad b = 14, \quad k = 4,$$

$$\lambda_{11} = \lambda_{12} = 2, \quad \lambda_{22} = 1, \quad \lambda_{12} = 0, \quad \lambda_{22} = 2.$$
1) Consider the set of 8 initial blocks \((1_1 0_3 1_4 1_5); (1_1 1_3 0_4 1_5), (0_1 0_3 0_4 1_5), (0_1 1_3 1_4 1_5), (0_2 0_3 1_4 1_5), (0_2 0_3 0_4 1_5)\) and \((0_2 1_3 1_4 1_5)\).

On developing these blocks over the elements of \(M\) we get the solution of the design

\[ n_1 = n_2 = 2, \quad \mu_1 = 2, \quad \mu_2 = 3, \quad b = 18, \quad k = 4, \]

\[ \lambda_{11} = \lambda_{11} = 0, \quad \lambda_{22} = 0, \quad \lambda_{22} = 2, \quad \lambda_{12} = 2. \]

iv) Consider the set of 9 blocks \((0_1 1_1 0_2 1_2), (1_1 0_3 1_4 1_5), (1_1 1_3 0_4 1_5), (0_1 0_3 0_4 1_5), (0_1 1_3 1_4 1_5), (1_2 0_3 1_4 1_5), (1_2 1_3 0_4 1_5), (0_2 0_3 0_4 1_5)\) and \((0_2 1_3 1_4 1_5)\).

On developing these blocks by successively adding the elements of \(M\), we get the solution of the design

\[ n_1 = n_2 = 2, \quad \mu_1 = 2, \quad \mu_2 = 3, \quad b = 18, \quad k = 4, \]

\[ \lambda_{11} = \lambda_{11} = 2, \quad \lambda_{22} = 0, \quad \lambda_{22} = 2, \quad \lambda_{12} = 2. \]

B) Construction of group divisible designs with variable replications by using the blocks of two G.D. designs together.

Consider a G.D. design \(D_1\) with the parameters \(v, b, r, k, \lambda_1, \lambda_2, m, n\) and another G.D. design with parameters \(v', b', r', k, \lambda_1', \lambda_2', m' (> m), n' = n\) where the second design is such that the first \(m\) groups of treatments 1, 2, ..., \(v\) of the latter design are the same as the \(m\) groups of \(D_1\). Then by taking the blocks
of the two designs together we get a $G\cdot D$ design with variable replications with the following parameters:

$$v^* = v', b^* = b + b', r_1 = r + r', r_2 = r', k = k, v_1 = v_2 = n,$$

$$\lambda_1 = m; \mu_2 = m' = m, \lambda_1' = \lambda_2 + \lambda_2', \lambda_1' = \lambda_2 + \lambda_2', \lambda_1 = \lambda_2', \lambda_2 = \lambda_2', \lambda_2' = \lambda_2.$$ 

**Proof:** $r_1$: The treatments which are common to both the designs, appear $r$ times in the blocks of the first design and $r'$ times in the $b'$ blocks of the second design. Hence $r + r'$ times in all.

$r_2$: The treatments $v_1, \ldots, v'$ appear only in $b'$ blocks of the second design. So, $r_2 = r'$.

$\lambda_1$: Any two treatments which belong to the same group of the design $D_1$ belong to the same group of $D_2$ by assumption. Hence $\lambda_1 = \lambda_2 + \lambda_2'$.

Similarly $\lambda_{11}' = \lambda_2 + \lambda_2'$.

Other parameters follow easily.

**Example 1.** Consider the following two $G\cdot D$ designs of Bose, Shrikhande and Bhattacharyya (1953):

$D_1$: $v = 10, b = 20, r = 8, k = 4, m = 5, n = 2, \lambda_1 = 0, \lambda_2 = 3$,

whose solution is obtained by developing the initial blocks

$$(0_2 1_2 2_2 4_2 4_2), (0_2 1_2 2_2 4_2 4_2), (0_2 2_2 3_2 4_2 4_2) \mod 5$$

and $D_2$: $v = b = 14, r = k = 4, m = 7, n = 2, \lambda_1 = 0, \lambda_2 = 1$ whose solution is obtained by developing the initial blocks
Then by considering the blocks of the two designs together we get the solution of the G.D. design with variable replications whose parameters are

\[ D_1: v_1 = v_2 = 2, \mu_1 = 5, \mu_2 = 2, b_1 = 34, v_3 = 14, r_1 = 12, r_2 = 4, \]
\[ k^* = 4, \lambda^*_1 = 0, \lambda^*_1 = 4, \lambda^*_2 = 3, \lambda^*_2 = 0, \lambda^*_2 = 1 \]

whose solution is

\[
\begin{align*}
(0_1 1_2 2_2 3_2), & (1_1 2_2 3_2 0_2), & (2_1 3_2 4_2 1_2), & (3_1 4_2 0_2 2_2), \\
(4_1 0_2 1_2 3_2), & (0_2 1_1 2_1 4_1), & (1_2 2_1 3_1 0_1), & (2_2 3_1 4_1 1_1), \\
(3_2 4_1 0_1 2_1), & (4_2 0_1 1_1 3_1), & (0_1 2_2 3_2 4_2), & (1_1 3_2 4_2 0_2), \\
(2_1 4_2 0_2 1_2), & (3_1 0_2 1_2 3_2), & (4_1 1_2 2_2 4_2), & (0_2 2_1 3_1 4_1), \\
(1_2 3_1 4_1 0_1), & (2_2 4_1 0_1 1_1), & (3_2 0_1 1_1 2_1), & (4_2 1_1 2_1 3_1), \\
(0_1 2_2 1_2 4_2), & (1_1 2_2 3_2 5_2), & (2_1 3_2 4_2 6_2), & (3_1 4_2 5_2 0_2), \\
(4_1 5_2 6_2 1_2), & (5_1 6_2 0_2 2_2), & (6_1 0_2 1_2 3_2), & (0_2 1_2 2_1 4_1), \\
(1_2 2_1 3_1 5_1), & (2_2 3_1 4_1 6_1), & (3_2 4_1 5_1 0_1), & (4_2 5_1 6_1 1_1), \\
(5_2 6_1 0_1 2_1) \text{ and } (6_2 0_1 1_1 3_1). \\
\end{align*}
\]

The groups are \( S_1 : (0_1 0_2), (1_1 1_2), (2_1 2_2), (3_1 3_2), (4_1 4_2) \)
\[ S_2 : (5_1 5_2), (6_1 6_2). \]

**Example 2.** Consider the following two G.D. designs

\[ D_1: v=14, b=28, r=6, k=\bar{k}, m=7, n=2, \lambda_1 = 0, \lambda_2 = 1 \]
and \[ D_2: v'=18, b'=56, r'=9, k'=3, m'=9, n'=3, \lambda_1 = 2, \lambda_2 = 1. \]

These two designs will give the solution of the design
Example 2. Consider the following two G.D. designs

\[ D_1: \nu=16, \rho=7, m=2, \lambda_1=0, \lambda_2=1, r=k=4 \]
\[ D_2: \nu=26, b=52, r=8, k=4, m=13, n=2, \lambda_1=0, \lambda_2=1. \]

By adjoining the blocks of these two designs we get the solution of the design

\[ D^*: v_1^* = v_2^* = 2, \nu^* = 26, \rho^* = 66, \nu_1^* = 12, \nu_2^* = 8, \]
\[ k^* = 4, \mu_1^* = 0, \mu_2^* = 6, \lambda_1^* = 0, \lambda_1^* = 2, \lambda_2^* = 1, \]
\[ \lambda_2^* = 1, \lambda_2^* = 1. \]

C) Group divisible designs with variable replications obtained by omitting a block of a singular G.D. design.

If from a singular group divisible design with parameters \( v'=nv, b', r', k'=nk, \lambda_1'=\lambda, \lambda_2'=\lambda, n', m'=v \) we omit a block, then we get a G.D. design \( D^* \) with two replications with the following parameters:

\[ D^*: v^*=nv, b^*=b-1, r^*_1=r-1, r^*_2=r, k^*=nk, \lambda^*_1=r-1, \lambda^*_1'=\lambda-1, \lambda^*_2=r, \]
\[ \lambda^*_2 = \lambda, \mu^*_1 = k, \mu^*_2 = v-k. \]

**Proof:** The proof follows easily by considering the fact that in a singular group divisible design if any treatment \( \varnothing \) occurs in a block then all the first associates of \( \varnothing \) occur in that block.
Example. Consider for example, the singular $G*D$ design

$$v=14, b=7, r=3, k=6, \lambda_1=3, \lambda_2=1, n=2, m=7.$$ 

The layout is

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Treatments</th>
<th>Blocks</th>
<th>Treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(1_1^1 2_1^1 2_2^1 4_1^1 4_2^1)$,</td>
<td>2.</td>
<td>$(2_1^2 3_1^2 3_2^2 5_1^2 5_2^2)$,</td>
</tr>
<tr>
<td>3.</td>
<td>$(3_1^3 3_2^3 4_1^3 4_2^3 6_1^3 6_2^3)$,</td>
<td>4.</td>
<td>$(4_1^4 4_2^4 5_1^4 5_2^4 0_1^0 0_2^0)$,</td>
</tr>
<tr>
<td>5.</td>
<td>$(5_1^5 5_2^5 6_1^5 6_2^5 1_1^1 1_2^1)$,</td>
<td>6.</td>
<td>$(6_1^6 6_2^6 0_1^0 0_2^0 2_1^2 2_2^2)$,</td>
</tr>
<tr>
<td>7.</td>
<td>$(0_1^7 0_2^7 1_1^1 1_2^1 3_1^3 3_2^3)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If we omit block 1 then we get the solution of the design

$$v^*=14, b^*=6, r^*_1=2, r^*_2=3, k^*=6, \lambda^*_1=2, \lambda^*_2=3, \lambda^*_1=0, \lambda^*_2=1$$

$$\lambda^*_1=1, \mu^*_1=3, \mu^*_2=4, v^*_1=v^*_2=2,$$ whose groups are

$S_1: (1_1, 1_2, (2_1, 2_2), (4_1, 4_2)); S_2: (0_1, 0_2), (3_1, 3_2), (5_1, 5_2), (6_1, 6_2)$

D) Group divisible designs with variable replications obtained from balanced block designs with two replications by replacing each treatment by a group of treatments.

In a balanced block design with two replications, if we replace each treatment by a group of $n$ treatments, then we get a $G*D$ design with variable replications. If the parameters of the balanced block design are

$$v_1, \mu_2, r_1, r_2, b, k, \lambda_1, \lambda_2, \lambda_2$$

then the parameters of the $G*D$ design with variable replications are
will be

\[ n_1 = n_2, \quad \mu_1^* = n, \quad \mu_2^* = v_2, \quad \nu_1^* = r_1, \quad \nu_2^* = r_2, \quad b^* = b, \quad k^* = nk, \]

\[ \lambda_{11}^* = r_1, \quad \lambda_{12}^* = \lambda_{11}, \quad \lambda_{22}^* = r_2, \quad \lambda_{22}^* = \lambda_{22} \]

Conversely, every G.D. design with variable replications for which

\[ n_1 = v_2, \quad \lambda_{11} = r_1, \quad \lambda_{22} = r_2 \]

is obtainable this way from the corresponding balanced block design.

**Proof:** The proof follows easily by extending the argument of Bose, Shrikhande and Bhattacharjya (1953) for the singular 3-D. design from the corresponding B.I.D. design.

**Example:** Consider the following balanced block design:

\[ v_1 = 5, \quad v_2 = 5, \quad r_1 = 8, \quad r_2 = 4, \quad b = 12, \quad k = 4, \quad \lambda_{11} = 5, \quad \lambda_{12} = \lambda_{22} = 2 \]

whose solution is

1. \((1, 2, 3, 5)\), 2. \((2, 3, 4, 6)\), 3. \((3, 4, 5, 7)\), 4. \((1, 4, 5, 6)\)
5. \((2, 5, 6, 7)\), 6. \((1, 3, 5, 7)\), 7. \((1, 2, 4, 7)\), 8. \((1, 2, 3, 5)\)
9. \((1, 2, 3, 4)\), 10. \((1, 2, 4, 5)\), 11. \((1, 3, 4, 5)\) and
12. \((2, 3, 4, 5)\).

By replacing each treatment 6 by \((a_1, a_2)\) we get the solution of the group divisible design with variable replications with parameters

\[ v_1 = v_2 = 2, \quad \mu_1^* = 5, \quad \mu_2^* = 2, \quad \nu_1^* = 3, \quad \nu_2^* = 4, \quad k^* = 3, \]

\[ b^* = 12, \quad \lambda_{11}^* = 8, \quad \lambda_{11}^* = 5, \quad \lambda_{12}^* = 2, \quad \lambda_{22}^* = 4, \quad \lambda_{22}^* = 2 \]
whose groups are

\[ S_1: (1_1, 1_2), (2_1, 2_2), (3_1, 3_2), (4_1, 4_2), (5_1, 5_2) \]
\[ S_2: (6_1, 6_2); (7_1, 7_2). \]

E) If in a balanced block design with two replications, the number of treatments occurring from a group is the same for all the blocks, then by replacing each treatment of the first group by a group of \( n_1 \) treatments and each treatment of the second group by a group of \( n_2 \) treatments, we get a \( B \times B \) design with the following parameters:

\[
\begin{align*}
\nu_1 &= n_1, \quad \nu_2 = n_2, \quad \gamma_1 = \nu_1, \quad \gamma_2 = \nu_2 \\
\tau_1 &= r_1, \quad \tau_2 = r_2, \quad b^* = b, \quad k^* = n_1 \bar{\mu} + n_2(k - \bar{\mu}) \\
\lambda_{11} &= \tau_1, \quad \lambda_{11} = \lambda_{11}, \quad \lambda_{22} = \tau_2, \quad \lambda_{22} = \lambda_{22}, \quad \lambda_{12} = \lambda_{12}
\end{align*}
\]

where \( \bar{\mu} \) treatments from the first group of treatments appear in all the blocks.

**Proof:** The proof is the same as in D.

**Example:** Consider for example the balanced block design

\[ v_1 = 5, \quad v_2 = 6, \quad r_1 = 4, \quad r_2 = 5, \quad b = 10, \quad k = 5, \quad \lambda_{11} = 1, \quad \lambda_{12} = 2, \quad \lambda_{22} = 2. \]

whose layout is

1. (1, 2, 5, 7, 8), 2. (1, 2, 6, 9, 10), 3. (1, 3, 4, 7, 9),
4. (1, 3, 6, 8, 11), 5. (1, 4, 5, 10, 11), 6. (2, 3, 4, 8, 11),
7. (2, 3, 5, 9, 11), 8. (2, 4, 6, 7, 11), 9. (3, 5, 6, 7, 10),
10. (4, 5, 6, 8, 9)
Here $\bar{\mu} = 2$, $k - \bar{\mu} = 3$. Let us replace the treatment $\emptyset$ from the first group of treatments (i.e., treatments 7, 8, 9, 10, 11) by three treatments $\emptyset_1$, $\emptyset_2$, $\emptyset_3$ and the treatment $\emptyset$ from the second group of treatments by two treatments $\emptyset_1$ and $\emptyset_2$. Then we get the solution of the design

$$v_1^* = 3, \quad v_2^* = 2, \quad \mu_1^* = 5, \quad \mu_2^* = 6, \quad r_1 = 4, \quad r_2 = 5, \quad b^* = 10, \quad k^* = 12,$$

$$\lambda_1 = 4, \quad \lambda_1^* = 1, \quad \lambda_2 = 5, \quad \lambda_1^* = 2, \quad \lambda_2 = 2.$$
REFERENCES


