PART D

SCATTERING OF INTERNAL WAVES
Chapter 10

Internal wave scattering by a thin vertical barrier in a stratified fluid

Introduction

The problem of internal wave scattering by a bottom standing thin vertical barrier present in an exponentially stratified fluid of uniform finite depth under a solid plane was considered sometime back by Larsen (1969). He obtained the solution satisfying the radiation criteria for all the modes transmitted through and reflected by the barrier corresponding to incident internal wave of the lowest mode. Korobkin (1990) utilized Larsen’s (1969) solution in the study of the motion of a body in a weakly stratified fluid in the presence of a bottom obstacle (in the form of a submerged mountain) modelled as a thin vertical bottom standing plane barrier. However, if the obstacle is immersed onto the fluid from the top, then one needs to find the solution of the complementary problem of internal wave scattering by a partially immersed barrier.

In this chapter, we study the effect of a plane vertical barrier immersed in a weakly

\[1\]This chapter is based on the paper “Internal wave scattering by a thin vertical barrier in a stratified fluid”, accepted for presentation in 17th International Conference on OMAE 1998, 6-9 July, Lisbon, and published in the Proceedings of the Conference.
stratified fluid on a time-harmonic internal wave train propagating in the fluid. The incident waves, described by a stream function, when come across the barrier, are reflected back by the barrier with various modes of reflected internal waves and are also transmitted through the gap below the barrier also with various modes of transmitted internal waves. The reflected and transmitted internal waves are described by a scattered stream function which satisfies a boundary value problem in the fluid region. It is represented on both sides of the barrier by appropriate eigen function expansions. Use of appropriate conditions on the barrier and across the gap results in a dual series relation involving the elements of the scattering matrix. By defining a function on the barrier involving the elements of the scattering matrix as the co-efficients of its Fourier expansion series, the dual series relation reduces to a first kind singular integral equation with a Cauchy type kernel for this function, whose solution is immediate. All the elements of the scattering matrix and hence the stream function describing the resulting motion in the fluid are obtained explicitly in principle by utilizing this solution. For the lowest mode of the incoming internal wave train, the elements of the scattering matrix and the stream function are obtained. The stream lines are plotted graphically to visualise the effect of the barrier on the incoming internal wave train.

Formulation of the Problem

A brief description of the mathematical formulation of the problem is given here for the sake of completeness. We choose a rectangular cartesian co-ordinate system $x', y', z'$ in which the origin is taken at a point on the bottom with $y'$-axis vertically upwards, $x'z'$-plane as the lower rigid boundary, $y' = H$ as the surface of the solid plane below which lies an exponentially stratified fluid of uniform finite depth $H$. For two-dimensional motion in the fluid, let $u(x', y', t')$, $v(x', y', t')$ denote the velocity components along $x'$, $y'$ directions respectively for small perturbed motion in the fluid. Then the equation of continuity is

$$\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'} = 0,$$  \hspace{1cm} (1)
so that we can write

\[ u = -\frac{\partial \psi}{\partial y'}, \quad v = \frac{\partial \psi}{\partial x'}, \]  

(2)

where \( \psi(x', y', t') \) is the stream function describing the motion in the fluid. Let \( \rho_1(x', y', t') \) and \( p_1(x', y', t') \) denote respectively the density and pressure, then

\[
\begin{align*}
\rho_1(x', y', t') &= \rho_0(y') + \rho(x', y', t'), \\
p_1(x', y', t') &= g \int_{y'}^{H} \rho_0(y') dy' + p(x', y', t'),
\end{align*}
\]

(3)

\( \rho, p \) being the perturbations in density and pressure respectively. We assume that \( u, v, \rho, p \) are quantities of first order smallness and their products (including partial derivatives) and higher order powers can be neglected. In the case of stratified fluid, the linearised incompressibility condition produces

\[
\frac{\partial p}{\partial t'} - \frac{\partial \psi}{\partial x'} \frac{dp_0}{dy'} = 0.
\]

(4)

The law of stratification is given by

\[
\rho_0(y') = \rho_0(0) \exp \left( -\frac{y'}{L} \right),
\]

(5)

where \( \rho_0(0) \) is the density at the bottom and \( L \) is the linear dimension characterising stratification. The linearised forms of Euler's dynamical equations are

\[
\rho_0(y') \frac{\partial u}{\partial t'} = -\frac{\partial p}{\partial x'},
\]

(6)

\[
\rho_0(y') \frac{\partial v}{\partial t'} = -\frac{\partial p}{\partial y'} - \rho g,
\]

(7)

where \( g \) is the acceleration due to gravity.

Eliminating \( p \) from (6) and (7) and using the relation (2) we find

\[
\rho_0(y') \frac{\partial}{\partial t'} \left( \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} \right) = -g \frac{\partial \rho}{\partial x'} - \frac{dp_0(y')}{dy'} \frac{\partial^2 \psi}{\partial y' \partial t'}.
\]

(8)

Again, eliminating \( \rho \) from the equations (8) and (4) and using the relation (5) we finally obtain

\[
\frac{\partial^2}{\partial t'^2} \left( \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} \right) - \frac{1}{L} \frac{\partial^3 \psi}{\partial t'^2 \partial y'} + \frac{g}{L} \frac{\partial^2 \psi}{\partial x'^2} = 0.
\]
Using the non-dimensional variables \(x, y, t\) defined by
\[
x = \frac{\pi}{H} x', \quad y = \frac{\pi}{H} y', \quad t = \left(\frac{g'}{L}\right)^{1/2} t',
\]
and \(\psi(x, y, t) = \Psi\left(\frac{x}{\pi}, \frac{y}{\pi}, \left(\frac{L}{g'}\right)^{1/2} t\right)\), and assuming the fluid to be weakly stratified \((\frac{H}{\pi} \ll 1)\), we find that \(\psi(x, y, t)\) satisfies
\[
\nabla^2 \left(\frac{\partial^2 \psi}{\partial t^2}\right) + \frac{\partial^2 \psi}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < \pi.
\]
(9)

Since the fluid is bounded between two rigid planes \(y = 0\) and \(y = \pi\), the bottom and the top conditions are
\[
\psi = 0 \quad \text{on} \quad y = 0, \pi.
\]
(10)

For time-harmonic motion with dimensionless frequency \(w\), the time-dependence in \(\psi(x, y, t)\) can be taken as \(\psi(x, y, t) = \text{Re}\{\phi(x/y) \exp(-iwt)\}\) where \(\phi(x, y)\) now satisfies
\[
w^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < y < \pi,
\]
(11)
\[
\phi = 0 \quad \text{on} \quad y = 0, \pi.
\]
(12)

A time-harmonic internal wave propagating along positive \(x\)-direction is thus described by
\[
\phi(x, y) = \exp\left\{i \frac{kwz}{\left(1 - w^2\right)^{1/2}}\right\} \sin ky,
\]
(13)
where \(k = 1, 2, \cdots\), so that internal waves exist only if \(0 < w < 1\). In (13), \(k\) represents the mode of the internal wave. The case \(k = 1\) corresponds to the internal wave of the lowest mode.

We are now in a position to give the exact mathematical formulation of the physical problem under consideration. Let a train of internal waves described by the stream function \(\text{Re} \{\phi^{\text{inc}}(x, y) \exp(-iwt)\}\), with \(\phi^{\text{inc}}(x, y)\) representing the rightside of (13), arriving from \(x = -\infty\), be incident on a plane vertical thin barrier immersed in the fluid up to depth \(\pi - \alpha\) \((0 < \alpha < \pi)\). If this barrier is not present, the internal wave train of mode \(k\) will propagate in the fluid without any distortion. However, due to the presence of the barrier, the incident internal wave train will be scattered by the barrier,
and let $\phi(x,y)$ denote the scattered stream function, the time dependence $\exp(-i\omega t)$ being suppressed. Then $\phi(x,y)$ satisfies (11), (12) together with

$$\phi = -\sin ky \text{ on } x = 0, \quad \alpha < y < \pi, \quad (14)$$

$$r^{1/2}\nabla \phi \text{ is bounded as } r = \{x^2 + (y - \alpha)^2\}^{1/2} \to 0, \quad (15)$$

$$\phi(x,y) \text{ behaves as outgoing waves as } |x| \to \infty. \quad (16)$$

The condition (14) arises as a consequence of the fact that the barrier is a boundary of the fluid on which the stream function (incident plus the scattered) is taken to be zero. Again, as the scattered stream function $\phi(x,y)$ is symmetric about the barrier line ($x = 0$) we have

$$\phi(x,y) = \phi(-x,y). \quad (17)$$

Since $\frac{\partial \phi}{\partial x}(+0,y), \frac{\partial \phi}{\partial x}(-0,y)$ exist for all $y$, but $\frac{\partial \phi}{\partial x}(0,y)$ exists only across the gap, and as such because of (17), $\phi(x,y)$ must satisfy

$$\frac{\partial \phi}{\partial x}(0,y) = \frac{\partial \phi}{\partial x}(\pm 0,y) = 0 \text{ for } 0 < y < \alpha. \quad (18)$$

### Method of Solution

Eigen function expansions of $\phi(x,y)$ which satisfies (11), (12), (17) and (16) are given by

$$\phi(x,y) = \sum_{m=1}^{\infty} p_m^k \sin my \exp\left\{\frac{i\omega w}{(1 - \omega^2)^{1/2}|x|}\right\}, \quad (19)$$

where the unknown constants $p_m^k (m = 1, 2, \ldots)$ may be interpreted as transmission or reflection co-efficient for the $m$th transmitted ($m \neq k$) or reflected mode corresponding to the $k$th incident mode. For $m = k$, $1 + p_k^k$ is the transmission co-efficient for the $k$th transmitted mode. We call $p_m^k$ as an element of the scattering matrix for $m \neq k$.

The condition (18) and (14) when applied to (19) produces the following dual series relation for the elements $p_m^k$ as given by

$$\sum_{m=1}^{\infty} mp_m^k \sin my = 0, \quad 0 < y < \alpha, \quad (20)$$
\[ \sum_{m=1}^{\infty} p_m^k \sin my = -\sin ky, \quad \alpha < y < \pi. \]  

(21)

To solve the dual series relation (20) and (21), we construct a function \( g_k(y) \) involving the elements \( p_m^k \) as its Fourier sine series co-efficients as given by

\[ g_k(y) = -\frac{1}{k} \sum_{m=1}^{\infty} mp_m^k \sin my, \quad 0 < y < \pi, \]  

(22)

then \( g_k(y) \) is proportional to \( \frac{\partial f}{\partial x}(\pm 0, y) \), the vertical component of scattered velocity at \( x = 0 \). Because of the condition (20),

\[ g_k(y) = 0 \quad \text{for} \quad 0 < y < \alpha, \]  

(23)

and due to the edge condition (15), \( g_k(y) \) is required to have a square root singularity at \( y = \alpha \), i.e.

\[ g_k(y) = O((y - \alpha)^{-1/2}) \quad \text{as} \quad y \to \alpha + 0. \]  

(24)

Now using (23), we note from (22) that

\[ p_m^k = -\frac{2}{\pi} \int_{\alpha}^{\pi} g_k(y) \sin my \, dy. \]  

(25)

Using (25) in (21) we find that \( g_k(y) \) satisfies the integral equation

\[ \frac{1}{\pi} \int_{\alpha}^{\pi} g_k(u) \left( \sum_{m=1}^{\infty} \frac{2 \sin mu \sin my}{m} \right) du = \frac{\sin ky}{k}, \quad \alpha < y < \pi, \]  

which is equivalent to

\[ \frac{1}{\pi} \int_{\alpha}^{\pi} g_k(u) \ln \frac{\sin \left(\frac{u+y}{2}\right)}{\sin \left(\frac{u-y}{2}\right)} \, du = \frac{\sin ky}{k}, \quad \alpha < y < \pi. \]  

(26)

Differentiation of both sides with respect to \( y \) produces a first kind singular integral equation as given by

\[ \frac{1}{\pi} \int_{-1}^{a} \frac{G_k(t) \, dt}{t-s} = -\cos(k \cos^{-1} s), \quad -1 < s < a, \]  

(27)

where

\[ G_k(t) = g_k(\cos^{-1} t), \quad t = \cos u, s = \cos y, a = \cos \alpha, \]  

(28)
and the integral is in the sense of Cauchy principal value. Because of (24), \( G_k(t) \) is required to satisfy
\[
G_k(t) = O((a - t)^{-1/2}) \quad \text{as } t \to a - 0.
\] (29)

Also, near the end point \( t = -1 \),
\[
G_k(t) = 0 \quad \text{as } t \to -1,
\] (30)

since \( g_k(u) \) is zero when \( \alpha \to \pi \) (near the top of the barrier).

Now the solution of the integral equation (27) with Cauchy type kernel, which satisfies (29) and (30) is standard and is given by (cf. Mikhlin (1964))
\[
G_k(t) = \frac{1}{\pi} \left( t + \frac{1}{a - t} \right)^{1/2} \int_{-1}^{1} \frac{(a - s)^{1/2}}{s + 1} \frac{\cos(k \cos^{-1} s)}{s - t} ds.
\] (31)

Thus for \( \alpha < u < \pi \)
\[
g_k(u) = \frac{1}{\pi} \left( \frac{\cos \frac{u}{2}}{\cos \alpha - \cos u} \right)^{1/2} \int_{\alpha}^{\pi} \frac{(\cos \alpha - \cos y)^{1/2}}{\cos y - \cos u} \left\{ \sin \left( k + \frac{1}{2} \right) y - \sin \left( k - \frac{1}{2} \right) y \right\} dy.
\] (32)

Hence, the elements \( p^k_m \) \((m = 1, 2, \ldots)\) are obtained explicitly as
\[
p^k_m = -\frac{k}{m} r^k_m
\] (33)

where
\[
r^k_m = -\frac{1}{2} \left\{ P_m(\cos \alpha) + P_{m-1}(\cos \alpha) \right\} \left\{ P_k(\cos \alpha) - P_{k-1}(\cos \alpha) \right\} + D_{k,m}(\alpha).
\] (34)

In (34), \( P_m(z) \) is the Legendre polynomial of degree \( m \) and
\[
D_{k,m}(\alpha) = B_{k,m}(\alpha) + B_{k,m-1}(\alpha) - B_{k-1,m}(\alpha) - B_{k-1,m-1}(\alpha),
\] (35)

where
\[
B_{k,m}(\alpha) = \frac{1}{\pi^2} \int_{\alpha}^{\pi} \int_{\alpha}^{\pi} \left( \frac{\cos \alpha - \cos u}{\cos \alpha - \cos y} \right)^{1/2} \frac{\sin(m + \frac{1}{2}) u \sin(k + \frac{1}{2}) y}{\cos y - \cos u} dy du, \quad m = 1, 2, \ldots
\] (36)

It is easy to see that \( B_{k,m}(\alpha) \) satisfies the equalities
\[
B_{k,m}(\alpha) + B_{m,k}(\alpha) = \frac{1}{2} P_k(\cos \alpha) P_m(\cos \alpha),
\] (37)
Putting \( m = k \) in (41) and using (37) we obtain the recurrence formula

\[
B_{k,k+1}(\alpha) = B_{k-1,k}(\alpha) - \frac{k + 1}{2} \frac{P_{k-1}(\cos \alpha) - P_{k+1}(\cos \alpha)}{2k + 1} P_k(\cos \alpha).
\]

Also,

\[
B_{-1,0}(\alpha) = -B_{0,0}(\alpha) = -\frac{1}{4}.
\]

These equalities are sufficient to determine \( B_{k,m}(\alpha) \) for all integral values of \( k, m \). Thus all the elements \( p^k_m \) are obtained explicitly. Hence the stream function describing the motion in the fluid is obtained explicitly in principle.

**Discussion**

To visualise pictorially the effect of the partially immersed thin vertical barrier on the wave motion, we consider the incident internal wave train of lowest mode, i.e. the case \( k = 1 \). In this case

\[
p_m^1 = -\frac{1}{m} r_m^1,
\]

where

\[
r_m^1 = \frac{2}{\pi} \int_0^\pi g_k(y) \sin my \, dy
\]

\[
= \cos^2 \frac{\alpha}{2} [P_m(\cos \alpha) + P_{m-1}(\cos \alpha)] - \frac{1}{2} [P_{m+1}(\cos \alpha) + P_{m-2}(\cos \alpha)]
\]

\[
+ P_m(\cos \alpha) + P_{m-1}(\cos \alpha)] \quad \text{for } m \geq 2,
\]

while

\[
r_1^1 = \cos^2 \frac{\alpha}{2} (1 + \cos \alpha) - \frac{1}{2} \{P_2(\cos \alpha) + \cos \alpha\}.
\]
Thus the stream function describing the motion in the fluid is obtained as

$$\psi(x, y, t) = \text{Re} \left[ \exp \left\{ iw \left( \frac{x}{(1 - w^2)^{1/2}} - t \right) \right\} \sin y \right.$$ 

$$+ \sum_{m=1}^{\infty} p_m e^{i m y} \exp \left\{ iw \left( \frac{m|x|}{(1 - w^2)^{1/2}} - t \right) \right\} \right].$$

(47)

In the absence of the barrier, the stream lines corresponding to the incident internal wave with mode one are depicted in fig. 1 in the vicinity of $x = 0$ taking $w = .6$ and $t = 5$. When a partially immersed barrier of length (non-dimensional) $\pi - 2$ (i.e. $\alpha = 2$), is introduced along the line $x = 0$, the corresponding streamlines are depicted in fig. 2. The effect of the barrier on the motion becomes apparent from this figure.

From fig. 2, it is observed that some stream lines abruptly change their direction and the envelop of the points where these abrupt changes of direction in the stream lines occur consists of two lines which intersect at the edge. This pattern is formed due to the presence of the barrier.

**Conclusion**

The problem of internal wave scattering by a thin vertical barrier partially immersed in a weakly stratified fluid of uniform finite depth under a solid plane is considered here assuming linear theory. The stream function describing the motion in the fluid and the elements of the scattering matrix are obtained explicitly in principle for any mode of the incoming wave train. For an incoming wave train of the lowest mode, the effect of the barrier is pictorially visualised by drawing the streamlines in the absence of the barrier and also in its presence.
Chapter 11

Scattering of internal waves by a vertical barrier in a channel of stratified fluid

Introduction

In this chapter, we consider the problem of internal wave scattering by a vertical barrier in the form of a submerged plate or a wall with a submerged gap, in an exponentially stratified fluid of uniform finite depth bounded above by a rigid plane. Due to the presence of the barrier in the stratified fluid, the incident waves (described by a stream function) are reflected back by the barrier with various modes and transmitted through the gap also with various modes. The reflected and transmitted internal waves are described by a scattered stream function which satisfies a boundary value problem in the fluid region. This stream function is expressed on both sides of the barrier by appropriate eigenfunction expansions involving the elements of the scattering matrix. By the use of appropriate conditions on the barrier and the gap, a dual series relation involving the elements of the scattering matrix is obtained. By defining a function on the barrier involving the elements of the scattering matrix as the coefficients of its Fourier series, the dual series relation reduces to a Carleman integral equation in a
single interval for the submerged plate problem or in a double interval for the problem of a wall with a submerged gap. The solution of this integral equation for each case is obtained by standard technique. Using the solution of the appropriate integral equation, the elements of the scattering matrix and hence the stream function describing the resulting motion in the fluid for each case are obtained in principle, and for the lowest mode of the incoming internal wave train, these are evaluated explicitly. For both the barrier configurations, the stream lines are plotted graphically to visualise the effect of the barrier on the incoming internal wave train.

Formulation of the Problem

A train of time-harmonic progressive internal waves is propagating along the positive $x'$-direction in a channel of stratified fluid, where we choose a rectangular cartesian co-ordinate system $x', y', z'$ in which the origin is taken at a point on the bottom, $x'z'$-plane as the lower rigid boundary, $y'$-axis is vertically upwards and $y' = H$ as the upper rigid boundary. For small motion in the fluid, let $u(x', y', t')$ and $v(x', y', t')$ denote the velocity components along $x'$, $y'$ directions respectively. In the case of weakly stratified fluid, the density and pressure denoted by $\rho_1(x', y', t')$ and $p_1(x', y', t')$ are given by

$$
\rho_1(x', y', t') = \rho_0(y') + \rho(x', y', t'),
$$

$$
p_1(x', y', t') = g \int_{y'}^{H} \rho_0(y') \, dy' + p(x', y', t'),
$$

where $t'$ is the time, $\rho, p$ are perturbed density and pressure respectively and

$$
\rho_0(y') = \rho_0(0) \exp \left( -\frac{y'}{L} \right).
$$

Here, $g$ is the acceleration due to gravity, $\rho_0(0)$ is the density at the bottom and $L$ is the linear dimension characterising stratification. Again, the continuity equation is

$$
\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'} = 0,
$$
so that
\[ u = -\frac{\partial \Psi}{\partial y'} \quad \text{and} \quad v = \frac{\partial \Psi}{\partial x'}, \tag{4} \]
where \( \Psi(x',y',t') \) is the stream function describing the motion in the fluid region. We assume that \( u, v, p, \rho \) are small quantities so that their products and higher order derivatives can be neglected.

Assuming linear theory, the incompressibility condition produces
\[ \frac{\partial \rho}{\partial t'} - \frac{\partial \Psi}{\partial x'} \frac{d \rho_0}{d y'} = 0, \tag{5} \]
and the Euler’s dynamical equations become
\[ \begin{aligned}
\rho_0(y') \frac{\partial u}{\partial y'} &= -\frac{\partial p}{\partial x'}, \\
\rho_0(y') \frac{\partial v}{\partial y'} &= -\frac{\partial p}{\partial y'} - \rho g.
\end{aligned} \tag{6} \]

Now eliminating \( p \) from the coupled equations in (6) and using the equation (5), we obtain
\[ \rho_0(y') \frac{\partial}{\partial y'} \left( \frac{\partial^2 \Psi}{\partial x'^2} + \frac{\partial^2 \Psi}{\partial y'^2} \right) = g \frac{\partial \rho}{\partial y'} - \frac{d \rho_0(y')}{d y'} \frac{\partial^2 \Psi}{\partial y' \partial t'}. \tag{7} \]

Again using the relation (2), and eliminating \( \rho \) from the equations (7) and (5), we obtain
\[ \frac{\partial^2}{\partial t'^2} \left( \frac{\partial^2 \Psi}{\partial x'^2} + \frac{\partial^2 \Psi}{\partial y'^2} \right) - \frac{1}{L} \frac{\partial^3 \Psi}{\partial t'^3} + \frac{g}{L} \frac{\partial^2 \Psi}{\partial x'^2} = 0. \tag{8} \]

Using the non-dimensional variables \( x = \frac{R}{H} x', y = \frac{R}{H} y', t = \left( \frac{L}{H} \right)^{1/2} t' \), and defining \( \psi(x, y, t) = \Psi \left( \frac{H}{R} x, \frac{H}{R} y, (\ell \frac{L}{R})^{1/2} t \right) \), and neglecting \( \frac{H}{L} \ll 1 \) (which is equivalent to the Boussinesq approximation for a weakly stratified fluid), we find that
\[ \nabla^2 \left( \frac{\partial^2 \psi}{\partial t'^2} \right) + \frac{\partial^2 \psi}{\partial x'^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < \pi, \tag{9} \]
which is the basic partial differential equation satisfied by the stream function \( \psi(x, y, t) \).

Since the fluid lies between two rigid planes \( y = 0 \) and \( y = \pi \), the bottom and top conditions satisfied by \( \psi \) are
\[ \psi = 0 \quad \text{on} \quad y = 0, \pi. \tag{10} \]
For time-harmonic motion, we may write
\[ \psi(x, y, t) = \text{Re}\{\phi^{\text{tot}}(x, y) \exp(-iwt)\}, \]
w being the dimensionless frequency, then \( \phi^{\text{tot}}(x, y) \) satisfies
\[ w^2 \nabla^2 \phi^{\text{tot}} = \frac{\partial^2 \phi^{\text{tot}}}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 \leq y \leq \pi, \]
\[ \phi^{\text{tot}} = 0 \quad \text{for} \quad y = 0, \, \pi. \tag{11} \]

In the absence of the barrier, progressive wave solutions for \( \phi(x, y) \) are
\[ \exp \left( \pm \frac{ikw}{\sqrt{1 - w^2}}x \right) \sin ky \]
where \( k \) is the mode, and these exist only when \( 0 < w < 1 \).

Let a train of progressive internal waves represented by the stream function
\[ \text{Re}\{\phi^{\text{inc}}(x, y) \exp(-iwt)\} \]
with
\[ \phi^{\text{inc}}(x, y) = \exp \left( i \frac{kw_x}{\sqrt{1 - w^2}}x \right) \sin ky \tag{13} \]
be propagating along positive \( x \)-direction, where \( k \) is the mode of the internal wave.

In the absence of barrier, the train of internal wave of mode \( k \) will propagate in the fluid without any distortion. However due to the presence of the barrier the incident internal wave train will be scattered by the barrier. Let \( \phi(x, y) \) denote the stream function of the scattered wave such that \( \phi^{\text{tot}} = \phi^{\text{inc}} + \phi \). Then \( \phi(x, y) \) satisfies (10), (11) together with
\[ \phi = -\sin ky \quad \text{for} \quad x = 0, \ y \in L, \tag{14} \]
where \( L = L_j \ (j = 1, 2) \). Here \( L_1 = (\alpha, \beta) \) corresponds to a submerged plate and \( L_2 = (0, \alpha) \cup (\beta, \pi) \) corresponds to a thin wall with a submerged gap.

Also,
\[ \phi(x, y) \] behaves as outgoing waves as \( |x| \to \infty. \tag{15} \]

Again, since the scattered stream function \( \phi(x, y) \) is symmetric about the barrier line \( (x = 0) \), we have
\[ \phi(x, y) = \phi(-x, y). \tag{16} \]

Since \( \frac{\partial \phi}{\partial x}(0, y), \frac{\partial \phi}{\partial z}(-0, y) \) exists for all \( y \), but \( \frac{\partial \phi}{\partial z}(0, y) \) exists only across the gaps and as such, because of (16), \( \phi(x, y) \) must satisfy
\[ \frac{\partial \phi}{\partial x}(0, y) = \frac{\partial \phi}{\partial z}(\pm 0, y) = 0 \quad \text{for} \quad y \in \bar{L}, \tag{17} \]
where \( \bar{L} = (0, \pi) - L \).
Method of Solution

The solution \( \phi \) satisfying the equation (8) and the conditions (12), (16) and (15) are given by

\[
\phi(x, y) = \sum_{m=1}^{\infty} p_m^k \sin my \exp \left( \frac{imw}{\sqrt{1 - u^2}} |x| \right),
\]

where \( p_m^k \) represents the \( m \)th transmitted \((m \neq k)\) or reflected coefficients corresponding to the \( k \)th incident mode and \( 1 + p_k^k \) is the transmission co-efficient when \( m = k \).

The following dual series relations for the elements \( p_m^k \), are obtained by using the conditions (17) and (14)

\[
\sum_{m=1}^{\infty} m p_m^k \sin my = 0 \text{ for } y \in L,
\]

\[
\sum_{m=1}^{\infty} p_m^k \sin my = -\sin ky \text{ for } y \in L.
\]

To solve the dual series relations (19) and (20), we construct a function \( h_k(y) \), defined by

\[
h_k(y) = -\frac{1}{k} \sum_{m=1}^{\infty} mp_m^k \sin my \text{ for } 0 < y < \pi.
\]

Then, obviously \( h_k(y) \) is proportional to the vertical component of scattered velocity at \( x = 0 \), i.e. \( \frac{\partial \phi}{\partial x}(\pm 0, y) \).

Also from the condition (19)

\[
h_k(y) = 0 \text{ for } y \in L,
\]

and \( h_k(y) \) is allowed to have at most a square root singularity near the edges of the barrier or plate.

The elements \( p_m^k \) are related to \( h_k(y) \) by

\[
p_m^k = -\frac{2}{\pi} \frac{k}{m} \int_L h_k(y) \sin my \, dy.
\]

Now by using (23) in (20), we find that \( g_k(y) \) satisfies the following integral equation

\[
\frac{1}{\pi} \int_L h_k(u) \left\{ \sum_{m=1}^{\infty} \frac{2 \sin mu \sin my}{m} \right\} du = \frac{\sin ky}{k}, \text{ } y \in L,
\]
which is equivalent to
\[ \frac{1}{\pi} \int_L h_k(y) \ln \frac{\sin(y + u)}{\sin(y - u)} \, du = \frac{\sin ky}{k}, \quad y \in L. \] (24)

This is a Carleman type integral equation and its solution can be obtained as follows.

Differentiation of both sides with respect to \( y \) produces the integral equation with a Cauchy type kernel in the form
\[ \frac{1}{\pi} \int_{L'} \frac{G_k(t)}{t - s} \, dt = -\cos(k \cos^{-1} s), \quad s \in L', \] (25)

where the integral is in the sense of Cauchy principal value, \( G_k(t) = h_k(\cos^{-1} t), \; t = \cos u, s = \cos y, \) and \( L' \) is the image of \( L \). For \( L = L_1 = (\alpha, \beta), \; L' = L'_1 = (b, a) \) with \( a = \cos \alpha, b = \cos \beta \; (b < a) \), while for \( L = L_2 = (0, \alpha) \cup (\beta, \pi), \; L' = L_2 = (-1, b) \cup (a, 1) \).

We now deal with the cases \( L = L_1 \) and \( L = L_2 \) separately.

**Case (a) \( L = L_1 = (\alpha, \beta) \).**

This corresponds to a submerged plate. In this case \( h_k(u) \) has square root singularities near \( u = \alpha \) and \( u = \beta \), the two edges of the submerged plate. Thus the solution of the integral equation (25) for \( L' = L'_1 = (b, a) \) with the requirement that
\[ G_k(t) \left[ (t - b)^{-1/2} \right. \quad \text{as} \quad t \to a - 0, \]
\[ G_k(t) \left[ (t - b)^{-1/2} \right. \quad \text{as} \quad t \to b + 0, \] (26)

is given by (cf. Cooke (1970), Gakhov (1966))
\[ G_k(t) = \frac{1}{\pi} \frac{1}{(t - b)(a - t)^{1/2}} \left\{ \int_b^a \frac{(s - b)(a - s)^{1/2}}{s - t} \cos(k \cos^{-1} s) \, ds + C \right\}, \quad b < t < a, \]

where \( C \) is an arbitrary constant and the integral is in the sense of Cauchy principal value. Thus
\[ h_k(u) = G_k(\cos u) = \frac{1}{\pi} \frac{1}{(\cos \alpha - \cos u)(\cos u - \cos \beta)} \]
\[ \times \left[ \int_\alpha^\beta \left\{ (\cos \alpha - \cos y)(\cos y - \cos \beta) \right\}^{1/2} \frac{\cos ky}{\cos y - \cos u} \, dy + C \right]. \] (27)
The constant $C$ can be evaluated by substituting the solution $h_k(u)$ given by equation (27) in the original integral equation (24) for $L = L_1 = (\alpha, \beta)$, and then evaluating the various integrals for any $y \in (\alpha, \beta)$. However, these integrals cannot be obtained analytically, but can be evaluated numerically for various values of $y \in (\alpha, \beta)$ by Gauss quadrature taking care of the singular point and integer values of $k$. The following table gives $C$ numerically for $\alpha = 1, \beta = 2, k = 1, 2, 3$ and various values of $\gamma_0$ in $(\alpha, \beta)$.

<table>
<thead>
<tr>
<th>$\gamma_0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>1.353287</td>
<td>0.061550</td>
<td>-0.267210</td>
</tr>
<tr>
<td>1.5</td>
<td>1.353327</td>
<td>0.061484</td>
<td>-0.267138</td>
</tr>
<tr>
<td>1.7</td>
<td>1.353302</td>
<td>0.061437</td>
<td>-0.267208</td>
</tr>
<tr>
<td>1.9</td>
<td>1.353098</td>
<td>0.061594</td>
<td>-0.267473</td>
</tr>
</tbody>
</table>

Thus $C$ is obtained numerically with an accuracy of 3 decimal places. It may be noted that $C$ is independent of $y$, and the table 1 also demonstrates this fact.

The scattered elements $p^k_m$ are now obtained in principle by using equation (23) for $L = (\alpha, \beta)$ and $h_k(u)$ given by equation (27) after obtaining the constant $C$ as explained above. The scattered stream function $\phi$ is then obtained by using the relation (18).

Case (b) $L = L_2 = (0, \alpha) \cup (\beta, \pi)$

This corresponds to a vertical wall with a submerged gap between $y = \alpha$ to $y = \beta$. Here $L' = L'_2 = (-1, b) \cup (a, 1)$ is a double interval and $G_k(u)$ has square root singularities near $u = b, u = a$ and is zero near the end points $u = -1, 1$ corresponding to the vicinity of the two ends of the wall. The appropriate solution to the integral equation (24) in this, satisfying the conditions at $t = \pm 1, b, a$ is given by (cf. Gakhov (1966))

\[
G_k(t) = \frac{1}{\pi} \frac{(1 - t^2)^{1/2}}{|R(t)|^{1/2}} \int_{L'_2} \frac{|R(s)|^{1/2}}{(1 - s^2)^{1/2}} \cos(k \cos^{-1} s) \frac{\cos(k \cos^{-1} s)}{s - t} ds, \quad t \in L'_2, \tag{28}
\]

where

\[
R(t) = (t - a)(t - b). \tag{29}
\]
Thus $h_k(u)$ in this case is given by

$$h_k(u) = \frac{1}{\pi} \frac{\sin u}{|R(\cos u)|^{1/2}} \int_{L_2} \frac{|R(\cos y)|}{\cos y - \cos u} \cos ky \, dy, \quad x \in L_2. \quad (30)$$

Now the scattering co-efficients $p^k_m$ is obtained from the relation given by equation (23) and hence the scattering stream function $\phi(x,y)$ is obtained by using the relation (18).

Discussion

Scattering Co-efficients

For a submerged plate, we have chosen $\alpha = 1, \beta = 2$ so that the vertical length (non-dimensional) of the plate is 1. A representative set of the values of the scattering co-efficients $p^k_m (m = 1, 2, 3, k = 1, 2, 3)$ are given in the following table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.857988</td>
<td>-0.039243</td>
<td>.171077</td>
</tr>
<tr>
<td>2</td>
<td>-0.078591</td>
<td>-0.402869</td>
<td>-.043459</td>
</tr>
<tr>
<td>3</td>
<td>0.515209</td>
<td>-0.065256</td>
<td>-.236441</td>
</tr>
</tbody>
</table>

It may be noted that if we make $\beta \rightarrow \pi$, then the wall becomes nonexistent and it assumes the form of a bottom standing barrier considered by Larsen (1969). The expression for $h_k(u)$ in (30) then becomes

$$h_k(u) = \frac{2}{\pi} \frac{\sin \frac{u}{2}}{\sqrt{\cos u - \cos \alpha}} \int_0^\alpha \frac{\sqrt{\cos y - \cos \alpha}}{\cos y - \cos u} \cos \frac{y}{2} \cos ky \, dy.$$

For $k = 1$, this becomes

$$h_1(u) = \frac{\sqrt{2} \sin \frac{u}{2}}{\sqrt{\cos u - \cos \alpha}} \left\{ \cos u + \frac{1 - \cos \alpha}{2} \right\}, \quad 0 < \alpha < u,$$
which coincides with Larsen's (1969) result for a bottom standing barrier. The scattering co-efficients for \( k = 1 \) can be obtained explicitly in terms of Legendre polynomials and these are also given in Larsen's (1969) paper. For \( \alpha = \frac{\pi}{2} \), the co-efficients \( p_k^k \) \( (k = 1, 2, 3, \ldots) \) have been obtained analytically by Korobkin (1990). The scattering co-efficients \( p_k^k \) \( (k = 1, 2, 3) \) are also obtained here directly as a limiting process from the plate problem or the wall 'with a gap' problem by a numerical procedure. These coincide with the numerical values of \( p_k^k \) given by Korobkin (1990) obtained analytically. This is discussed in some detail.

Again, if we make \( \alpha \to 0 \), then the lower part of the wall becomes non-existent and the wall assumes the form of a top-piercing barrier considered recently by Ghosh and Basu (1998). In this case the expression for \( h_k(u) \) in (30) becomes

\[
h_k(u) = 2 \frac{\cos u}{\cos \beta - \cos u} \int_\beta^\pi \sqrt{\cos \beta - \cos y} \sin \frac{y}{2} \cos ky \, dy, \quad \beta < u < \pi,
\]

which coincides with the result given by Ghosh and Basu (1998) for a top-piercing barrier. For \( k = 1 \), the scattering co-efficients can be obtained explicitly in terms of Legendre polynomials and are given in Ghosh and Basu (1998).

To check the validity of our numerical scheme, we have made \( \alpha \to 0 \) and \( \beta = \pi/2 \) so that the plate assumes the form of a bottom-standing barrier. In this case the following table depicts \( p_k^m \), \( m = 1, 2, 3 \), \( k = 1, 2, 3 \) for \( \alpha = 0, \beta = \pi/2 \), computed from the results of submerged plate here.

### Table 3

\[ p_k^m \] (for a bottom-standing barrier obtained as a limiting case of plate)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-0.747489</td>
<td>-0.250445</td>
<td>0.061265</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-0.502668</td>
<td>-0.437156</td>
<td>-0.248718</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.183744</td>
<td>-0.374258</td>
<td>-0.560472</td>
</tr>
</tbody>
</table>
The values of $p_1^1, p_2^2$ and $p_3^3$ have been respectively given explicitly by Korobkin (1990) as $-3/4, -7/16, -9/16$ i.e. $-0.75, -0.4375, -0.5625$. He calculated these directly from the bottom standing barrier problem, and these coincide with our results up to two decimal places. This shows the correctness of the numerical scheme used here.

The scattering coefficients for a top-piercing barrier can also be obtained from the results of the plate problem by making $\beta \to \pi$. In this case, the following table depicts $p_m^k, m = 1, 2, 3; k = 1, 2, 3$ for $\alpha = \pi/2$.

**Table 4**

$p_m^k$ (for top-piercing barrier obtained as a limiting case at a plate)

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.747309</td>
<td>-0.250355</td>
<td>0.061296</td>
</tr>
<tr>
<td>2</td>
<td>-0.502932</td>
<td>-0.437289</td>
<td>-0.248763</td>
</tr>
<tr>
<td>3</td>
<td>0.183478</td>
<td>-0.374392</td>
<td>-0.560517</td>
</tr>
</tbody>
</table>

It may be noted that the two geometrical configurations for which tables 3 and 4 are depicted are complementary to each other. If $p_m^{(1)k}$ and $p_m^{(2)k}$ correspond to bottom standing and top piercing plates whose vertical length are equal to half the channel depth, then it can be shown that

$$p_m^{(1)k} = (-1)^{m+k} p_m^{(2)k}.$$  

This is also reflected in the tables 3 and 4.

**Wall with a Submerged Gap**

In this case $L = L_2 = (0, \alpha) \cup (\beta, \pi)$.

For a wall with a gap, we have also chosen $\alpha = 1, \beta = 2$ so that the length (non-dimensional) of the vertical expanse of the gap is unity. A representative set of the
values of the scattering co-efficients $p^k_m$ ($m = 1, 2, 3; k = 1, 2, 3$) are given in table 5.

### Table 5

$p^k_m$ (for wall with a gap with $\alpha = 1, \beta = 2$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-.933026</td>
<td>-.092683</td>
<td>-.214593</td>
</tr>
<tr>
<td>2</td>
<td>-.033524</td>
<td>-1.050250</td>
<td>-.047305</td>
</tr>
<tr>
<td>3</td>
<td>-.618495</td>
<td>-.366265</td>
<td>-.645788</td>
</tr>
</tbody>
</table>

Again to check the validity of the numerical scheme followed here, we have made $\beta \to \pi$ and $\alpha = \pi/2$ so that the wall assumes the form of a bottom standing barrier. In this case the table 6 depicts $p^k_m$ for $m, k = 1, 2, 3$.

### Table 6

$p^k_m$ (for bottom standing barrier obtained as a limiting case of a wall with gap $\alpha = \pi/2, \beta \to \pi$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-.746568</td>
<td>-.250057</td>
<td>-.061365</td>
</tr>
<tr>
<td>2</td>
<td>-.503799</td>
<td>-.437415</td>
<td>-.248697</td>
</tr>
<tr>
<td>3</td>
<td>.182792</td>
<td>-.374831</td>
<td>-.560893</td>
</tr>
</tbody>
</table>

Comparison of the table 6 with table 3 shows that the elements in table 6 coincide with corresponding elements in table 3 upto 2 to 3 decimal places in most cases. Also the diagonal elements agree with Korobkin's (1990) explicit values.

Again, if we make $\alpha \to 0$ then the wall assumes the form of a top-piercing barrier. In this case the table 7 depicts $p^k_m$ for $m, k = 1, 2, 3$ and $\beta = \pi/2$. 
Table 7

$p^k_m$ (for a top-piercing barrier as a limiting case of wall with a gap with $\alpha \to 0$, $\beta = \pi/2$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-.746568</td>
<td>-.250057</td>
<td>-.061265</td>
</tr>
<tr>
<td>2</td>
<td>-.503799</td>
<td>-.437415</td>
<td>-.248697</td>
</tr>
<tr>
<td>3</td>
<td>.182791</td>
<td>-.374831</td>
<td>-.560893</td>
</tr>
</tbody>
</table>

The entries in table 7 coincides with the entries in table 4 almost 2 to 3 decimal places. It may be noted that while the entries in table 7 are calculated as a limiting process from the problem of wall with a gap the entries in table 4 are calculated as a limiting process from the plate problem. This again confirms the correctness of the numerical method utilised here.

Stream Function

To visualize the effect of the vertical barrier on the wave motion, the incident internal wave field of the lowest mode (i.e. $k = 1$) is chosen for simplicity. In this case the stream function describing the motion in the fluid is obtained as

$$
\psi^\text{total}(x, y, t) = \text{Re} \left\{ \exp \left( i w \left( \frac{x}{\sqrt{1 + w^2}} - t \right) \right) \sin y \right. 
+ \sum_{m=1}^{\infty} p^k_m \sin my \exp \left\{ i w \left( \frac{|m|x|}{\sqrt{1 - w^2}} - t \right) \right\},
$$

where the scattering co-efficients $p^k_m$ ($m = 1, 2, \ldots$) can be obtained at least numerically. For the two configurations of the barrier, as well as the bottom-standing and top-piercing barriers as limiting cases of the plate as well as the wall with a gap, these co-efficients have been obtained numerically. In the absence of the barrier, the stream lines corresponding to the incident internal wave with mode one are depicted in figure 1, taking $w = .6$ and $t = 5$ in the vicinity of $x = 0$. 
When a barrier is introduced along the line $x = 0$, the corresponding stream lines are depicted in figs. 2 to 5. Fig. 2 shows the streamlines for a submerged plate of unit length ($\beta = 2, \alpha = 1$), fig. 3 shows the same for a thin wall with a gap of unit length ($\beta = 2, \alpha = 1$). Figs. 4 and 5 show the stream lines for a bottom-standing and top-piercing barrier. In all these figures $w = .6, t = 5$ have been taken. The results for the last two cases have been obtained by making $\beta \rightarrow \pi$ (with $\alpha = 1$) and $\alpha \rightarrow 0$ (with $\beta = 2$) in the results of the wall with a gap configuration. The same results are also obtained by making $\alpha \rightarrow 0$ (with $\beta = 2$) and $\beta \rightarrow \pi$ (with $\alpha = 1$) in the results of the plate configuration. In all the cases, it is observed that some stream lines abruptly change their directions, and the points where the changes in the direction of stream lines occur roughly lie on straight lines which intersect near the edges of the barriers. Different patterns in the stream line contours are formed for each of the barriers.

**Conclusion**

Scattering of internal waves by thin vertical barrier in the form of a submerged plate or a wall with a gap submerged in exponentially stratified fluid of uniform finite depth under a rigid plane is studied here under the assumption of linear theory and Boussinesq approximation. The problem is formulated in terms of the stream function describing the motion in the fluid. The elements of the scattering matrix and the stream function are obtained through the solution of a Carleman integral equation for any mode of the incident internal wave. The scattering co-efficients are also obtained numerically and some results are compared with the results available in the literature. Good agreement is seen to have been achieved. For the lowest mode of the incident internal waves, the streamlines are drawn to visualise pictorially the effect of the barrier for its various configurations.
Fig. 1
Chapter 12

Scattering of internal waves by a semi-infinite inertial surface

Introduction

In the mathematical modelling of wave phenomena in a deep liquid, a part of which is covered by an inertial surface while the remaining part is free, the surface boundary condition becomes discontinuous in the sense that there is one condition at the inertial surface and another condition at the free surface. For a homogeneous liquid (e.g. water), Perters (1950), Weitz and Keller (1950) developed mathematical models to investigate scattering of surface waves travelling from the free surface region and normally or obliquely incident on the line separating the free surface and the inertial surface. Gabov et. al. (1989) generalised these problems for two immiscible homogeneous liquids for which half the interface is covered by an inertial surface and the other half is a free separating boundary of the two liquids. Recently Kanoria et. al. (1999) investigated two mixed boundary value problems involving surface water wave in deep water (or interface wave in two superposed homogeneous liquids) arising due to one or two discontinuities in the surface (or interface) boundary conditions. The governing partial differential equation in these problems is the Laplace equation which, together with the boundary conditions, is generalised to the Helmholtz’s equation together with slightly different boundary conditions by introducing a complex parameter to facilitate
the use of Wiener-Hopf technique in the mathematical analysis.

Instead of a homogeneous liquid, if we have a stratified liquid in which the density varies exponentially along the vertical direction, then the governing PDE describing the propagation of steady state internal waves becomes the Klein-Gordon equation, as mentioned in section 1.3 of chapter 1. However, for the sake of completeness, this is shown here in some detail.

Gabov (1982), Gabov and Sveshnikov (1982) investigated diffraction of two-dimensional steady-state internal waves described by the Klein-Gordon equation in an exponentially stratified incompressible liquid by the boundary of a solid half plane. They used the Wiener-Hopf technique in the mathematical analysis. Several researchers investigated a number of variations of these problems by using the Wiener-Hopf technique.

In this chapter, the problem of internal wave scattering by a semi-infinite inertial surface partly covering an exponentially stratified liquid is investigated. We consider an incompressible inviscid exponentially stratified liquid occupying the region \( y \leq 0 \) when at rest, wherein the \( y \)-axis is chosen vertically upwards so that the upper surface of the liquid at the rest position coincides with the plane \( y = 0 \). The liquid is exponentially stratified along the \( y \)-direction so that its density in the unperturbed state is assumed to be of the form \( \rho_0(0) \exp(-2\beta y) \) (\( \beta > 0 \)) where \( \rho_0(0) \) is the density at the top of the liquid. If \( v_x \) and \( v_y \) denote the velocity components in the \( x \) and \( y \) directions respectively, \( \rho \) denotes the perturbed density due to the motion in the liquid, \( p \) denotes the dynamic pressure, then the linearised equations of motion are given by

\[
\begin{align*}
\rho_0(y) \frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x}, \\
\rho_0(y) \frac{\partial v_y}{\partial t} &= -\frac{\partial p}{\partial y} - \rho g, \\
\frac{\partial \rho}{\partial t} + v_y \frac{\partial \rho_0(y)}{\partial y} &= 0,
\end{align*}
\]
while the equation of continuity is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \quad (4)$$

If $\psi(x,y,t)$ denotes the stream function describing the motion in the liquid, then

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}. \quad (5)$$

Substituting

$$\psi(x,y,t) = u(x,y,t) \exp(\beta y), \quad (6)$$

it is found that the equation (1) to (4), after using the Boussinesq approximation, ultimately produce the partial differential equation

$$\frac{\partial^2}{\partial t^2} \left( \nabla^2 u - \beta^2 u \right) + w_0^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (7)$$

where $\nabla^2$ denotes the two-dimensional Laplacian, and $w_0 = (2\beta g)^{1/2}$ is the so-called Brunt Vaisala frequency.

For plane wave solutions of the PDE (7), the time dependence can be chosen to be harmonic so that $u(x,y,t)$ can be written as $\text{Re}\{u(x,y) \exp(-i\omega t)\}$ where $u(x,y)$ is now complex valued, $\omega$ is the circular frequency, and the same notation $u$ is used without any confusion. If $\exp(ik_1x + ik_2y - i\omega t)$ represents a plane wave solution of the PDE (7), then the dispersion relation is

$$\omega^2 = \frac{u_0^2 k_1^2}{k_1^2 + k_2^2 + \beta^2}. \quad (8)$$

The group velocity $v_g = \left( \frac{\partial \omega}{\partial k_1}, \frac{\partial \omega}{\partial k_2} \right)$ is then obtained as

$$v_g = \frac{w_0 \text{sgn}(k_1)}{(k_1^2 + k_2^2 + \beta^2)^{3/2}} \left( k_2^2 + \beta^2, -k_1 k_2 \right). \quad (9)$$

Thus the direction of the wave vector $k = (k_1, k_2)$ and the group velocity vector $v_g$ do not coincide unless $k_2 = 0$. Since the direction of $v_g$ determines the direction of energy flow in the wave, the direction of wave propagation is to be taken as the direction of $v_g$ rather than that of $k$. Also, the dispersion relation (8) ensures that plane wave type solutions are possible only when $\omega < w_0$, and this will be assumed all throughout here.
The complex valued function $u(x,y)$, which is related to the stream function, now satisfies the Klein-Gordon equation
\[
\frac{\partial^2 u}{\partial y^2} - \beta^2 u = \frac{1}{a^2} \frac{\partial^2 u}{\partial x^2},
\tag{10}
\]
where
\[
\frac{1}{a^2} = \frac{w_0^2}{w^2} - 1.
\tag{11}
\]

Thus the governing differential equation (10) here is a PDE of hyperbolic type in contrast to PDEs of elliptic type encountered in the classical diffraction theory.

Thus the problem is formulated as a boundary value problem involving the Klein-Gordon equation with discontinuous surface boundary conditions. The problem is handled for its solution with the aid of Wiener-Hopf technique after introducing a small positive imaginary part in the parameter $a$ defined by the relation (11), as well as by slightly generalising the surface boundary conditions, the edge conditions and the infinity requirements, and ultimately passing on to the limit as this small imaginary part of $a$ tends to zero. The diffracted field is obtained in terms of integrals which are evaluated asymptotically for large distances from the edge of the inertial surface by the method of steepest descent and interpreted physically.

**Formulation of the Problem**

Let an incompressible inviscid exponentially stratified liquid occupy the half space $y \leq 0$ when at rest, and the half-plane $y = 0, x < 0$ be the rest position of the free surface while the remaining half plane $y = 0, x > 0$ be the rest position of the inertial surface with area density $\sigma$. For small two-dimensional motion in the liquid, let $y = \eta(x,t)$ denote the elevation of the upper surface of the liquid above its rest position $y = 0$ so that $\eta$ and its derivatives are small quantities. The linearised conditions at the free surface are
\[
p(x,0,t) = \rho_0(0)g\eta(x,t),
\tag{12}
\]
\[ \frac{\partial \eta}{\partial t} = v_y \text{ on } y = 0, x < 0. \]  

Elimination of \( \eta \) from the conditions (12) and (13) and utilization of the equations (5), (1) and (6) produce the free surface condition for the complex valued function \( u(x,y) \) as

\[ \frac{g}{w^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + \beta u = 0 \text{ on } y = 0, x < 0. \]  

The linearised condition at the inertial surface can be obtained as (cf. Peters (1950))

\[ \frac{c}{\rho_0 w^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + \beta u = 0 \text{ on } y = 0, x > 0, \]  

where \( \rho_0 \equiv \rho_0(0) \) and

\[ c = g \rho_0 - \sigma w^2. \]

We can assume that the constant \( c \) is a positive quantity so long as \( w < w_0 \). This is due to the fact that under actual conditions concerning a stratified ocean, \( w_0^2 \approx 10^{-4} Hz^2 \) and when the ocean is covered with ice which is modelled as a thin elastic plate of surface density \( \sigma \), \( \frac{\rho_0}{\sigma} \approx 10Hz^2 \) (cf. Varlamov (1985)). In the present case when the ocean is covered with broken ice (inertial surface) we can assume that \( \frac{\rho_0}{\sigma} \) is also in the same range. Thus \( w_0^2 < \frac{\rho_0}{\sigma} \), and hence \( c > 0 \) since \( w < w_0 \) has already been assumed.

Let from the region \( x < 0, y < 0 \), a plane wave field represented by

\[ \phi_0(x,y) = \exp(-iby + ikx), \]

where

\[ k^2 = a^2(b^2 + \beta^2), \]

and \( b \) and \( k \) are taken to be positive, propagate from infinity and be incident on the edge of the inertial surface separating the free surface. The group velocity, given by the relation (9), for this wave is directed towards the edge of the inertial surface while the phase velocity is directed away from it. The total wave field \( u \) can be represented in the form

\[ u(x,y) = \phi_0(x,y) + \phi_1(x,y) + \phi(x,y), \]
where
\[ \phi_1(x, y) = R \exp(ixy + ikx), \tag{20} \]
with
\[ R = \frac{ib - \beta + \frac{ik^2}{\omega^2}}{ib + \beta - \frac{ik^2}{\omega^2}}, \tag{21} \]
so that it represents the wave reflected from the free surface, and \( \phi(x, y) \) is the diffracted field. \( \phi(x, y) \) satisfies the boundary value problem described by the Klein-Gorden equation
\[ \frac{\partial^2 \phi}{\partial y^2} - \beta^2 \phi = \frac{1}{a^2} \frac{\partial^2 \phi}{\partial x^2}, \quad y < 0, \tag{22} \]
and the boundary conditions
\[ \frac{g}{w^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y} + \beta \phi = 0 \text{ on } y = 0, x < 0, \tag{23} \]
\[ \frac{c}{\rho_0 \omega^2} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial y} + \beta \phi = A \exp(ikx) \text{ on } y = 0, x > 0, \tag{24} \]
where
\[ A = -\frac{2ibk^2\sigma}{\rho_0 (ib + \beta - \frac{ik^2}{\omega^2})}. \tag{25} \]

To apply the Wiener-Hopf technique for finding the solution for \( \phi \), we assume that the constant \('a'\) occurring in the PDE (22) has a small positive imaginary part \( \epsilon \) so that the constant \('k'\) defined by (18) has a positive imaginary part \( \delta(\epsilon) = (k^2 + \beta^2)^{1/2} \epsilon \) which tends to zero as \( \epsilon \to 0+ \).

Also \( \phi \) satisfies the edge conditions
\[ |\phi| = O(1), |\nabla \phi| = O(1), |\nabla^2 \phi| = O(1) \text{ as } r = (x^2 + y^2)^{1/2} \to 0, \tag{26} \]
and the condition at infinity, as given by,
\[ |\phi| + |\nabla \phi| + \left| \frac{\partial^2 \phi}{\partial x^2} \right| \leq \text{const.} \exp(-\chi(\epsilon)r) \text{ as } r = (x^2 + y^2)^{1/2} \to \infty, \tag{27} \]
where \( 0 < \chi(\epsilon) \leq \min(\epsilon \beta, \delta(\epsilon)) = \epsilon \beta \) so that \( \chi(\epsilon) \to 0 \) as \( \epsilon \to 0+ \).

The conditions (26) follow from the fact that the energy flux through an arbitrary closed surface encompassing the edge of the inertial surface is equal to zero while the
condition (27) follows from the requirement that the diffracted waves carry energy away to infinity.

In the next section, the Wiener-Hopf technique is applied to the generalised BVP satisfying the Klein-Gordon equation (22) involving the complex parameter \( a = a_1 + i \epsilon \), the surface boundary conditions (23) and (24), the edge conditions (26) and the infinity requirement (27).

**Solution of the Problem**

Let \( \Phi(\alpha, y) \) denote the Fourier transform of \( \phi(x, y) \) defined by

\[
\Phi(\alpha, y) = \int_{-\infty}^{\infty} \phi(x, y) \exp(iax) dx
\]

where \( \alpha = \sigma + i \tau, \sigma \) and \( \tau \) being real. Then

\[
\Phi(\alpha, y) = \Phi_-(\alpha, y) + \Phi_+(\alpha, y),
\]

where

\[
\Phi_+, \Phi_- = \int_{0}^{\infty}, \int_{-\infty}^{0} \phi(x, y) \exp(iax) d\alpha.
\]  

(28)

Now using the condition (27) we find that \( \Phi_+(\alpha, y) \) is regular in the upper half plane \( \tau > -\chi(\epsilon) \) and \( \Phi_-(\alpha, y) \) is regular in the lower half plane \( \tau < \chi(\epsilon) \) of the complex \( \alpha \)-plane. The edge conditions (26) alongwith the Abelian theorem (cf. Noble (1958)) ensure that

\[
|\Phi_+(\alpha, 0)| = O(|\alpha|^{-1}) \text{ as } |\alpha| \to \infty \text{ in } \tau \geq -\chi(\epsilon). \tag{29}
\]

To use the Wiener-Hopf procedure, the boundary conditions (23) and (24) are rewritten as

\[
\frac{g}{w^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y} + \beta \phi = \begin{cases} 0 & \text{on } y = 0, x < 0, \\ f(x) & \text{on } y = 0, x > 0, \end{cases}
\]  

(30)

and

\[
\frac{c}{\rho_0 w^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y} + \beta \phi = \begin{cases} g(x) & \text{on } y = 0, x < 0, \\ A \exp(ikx) & \text{on } y = 0, x > 0, \end{cases}
\]  

(31)
where \( f(x) \) (for \( x > 0 \)) and \( g(x) \) (for \( x < 0 \)) are unknown functions. The edge conditions (26) ensure that

\[
\begin{align*}
|f(x)| &= O(1) \text{ as } x \to 0^+, \\
|g(x)| &= O(1) \text{ as } x \to 0^-.
\end{align*}
\]  

(32)

Now, use of Fourier transform to the PDE (23) produces the ODE

\[
\frac{d^2 \Phi}{dy^2} + \frac{\gamma^2(\alpha)}{a^2} \Phi = 0, \quad y < 0,
\]

(33)

where

\[
\gamma^2(\alpha) = \alpha^2 - a^2 \beta^2,
\]

and we choose that branch of the function \( \gamma(\alpha) = (\alpha^2 - a^2 \beta^2)^{1/2} \) for which \( \gamma(0) = -ia\beta \) in the complex \( \alpha \)-plane cut along the line joining the points \( -(a_1 + i\epsilon)\beta \) and \( (a_1 + i\epsilon)\beta \) through infinity. A solution of the equation (33) is

\[
\Phi(\alpha, y) = D(\alpha) \exp \left( \frac{i\gamma(\alpha)}{a} y \right), \quad y < 0,
\]

(34)

where \( D(\alpha) \) is an arbitrary function of \( \alpha \). Using Fourier transform to the conditions (30) and (31) we find that \( \Phi(\alpha, y) \) must satisfy

\[
\left( \beta - \frac{g \alpha^2}{w^2} \right) \Phi(\alpha, 0) + \frac{d\Phi}{dy}(\alpha, 0) = F_+(\alpha),
\]

(35)

and

\[
\left( \beta - \frac{c \alpha^2}{\rho_0 w^2} \right) \Phi(\alpha, 0) + \frac{d\Phi}{dy}(\alpha, 0) = -\frac{A}{i(\alpha + k)} + G_-(\alpha),
\]

(36)

where the unknown functions

\[
F_+(\alpha) \equiv \int_0^\infty f(x) \exp(iax)dx \quad \text{and} \quad G_-(\alpha) \equiv \int_0^- g(x) \exp(iax)dx
\]

are regular in the two overlapping half planes \( \tau > -\chi(\epsilon) \) and \( \tau < \chi(\epsilon) \) respectively with \( |F_+(\alpha)| = O(|\alpha|^{-1}) \) as \( |\alpha| \to \infty \) in \( \tau > -\chi(\epsilon) \) and \( |G_-(\alpha)| = O(|\alpha|^{-1}) \) as \( |\alpha| \to \infty \) in \( \tau < \chi(\epsilon) \). Using (34) in the conditions (35) and (36) and eliminating \( D(\alpha) \), we obtain the following Wiener-Hopf relation, for the determination of the two functions \( F_+(\alpha) \) and \( G_-(\alpha) \), as given by
$$\frac{F_+ (\alpha)}{K (\alpha)} + G_- (\alpha) = \frac{A}{i (\alpha + k)}, \quad (37)$$

valid in the strip $\tau_- < \tau < \tau_+$ where $\tau_{\pm}$ are chosen such that $-\chi (e) < \tau_- < 0 < \tau_+ < \chi (e)$, and

$$K (\alpha) = \frac{-g \alpha^2 + \beta + i \gamma (\alpha)}{c \alpha^2 - \beta - i \gamma (\alpha)} = -\frac{g \rho_0}{c} \frac{\gamma (\alpha) - ia \beta \frac{1 - a^2}{1 + a^2}}{\gamma (\alpha) - ia \beta \frac{2 \rho_0 - (1 + a^2) c}{(1 + a^2) c}} \quad (38)$$

To solve the Wiener-Hopf problem described by the equation (37), it is necessary to factorize the function $K (\alpha)$ as $K (\alpha) = K_+ (\alpha) K_- (\alpha)$ where $K_+ (\alpha)$ is regular in the half plane $\tau > \tau_-$ and $K_- (\alpha)$ is regular in the half plane $\tau < \tau_+$. For this purpose, the cases $a^2 < 1$ and $a^2 > 1$ are to be considered separately. Here of course $a^2$ is considered as a real quantity. We note that for $a^2 < 1$, $w < w_0 / \sqrt{2} \equiv w_*$. In the case the quantity $2 \rho_0 g - (1 + a^2) c$ occurring in the denominator in (38) is always positive. However, for $a^2 > 1, 2 \rho_0 g - (1 + a^2) c$ is positive so long as $w_* < w < w_p$ where

$$w_p^2 = w_*^2 \left(1 - \frac{\sigma w_0^2}{2 \rho_0 g}\right)^{-1} \quad (39)$$

and this is negative when $w > w_p$. These observations are to be kept in mind while factorizing $K (\alpha)$.

(a) $0 < a^2 < 1$

$K (\alpha)$ can be expressed as

$$K (\alpha) = -\frac{g \rho_0}{c} \frac{L (\alpha)}{N (\alpha)}, \quad (40)$$

where

$$L (\alpha) = \gamma (\alpha) - ia \beta \frac{1 - a^2}{1 + a^2}, \quad (41)$$

and

$$N (\alpha) = \gamma (\alpha) - ia \beta \frac{2 \rho_0 g - (1 + a^2) c}{(1 + a^2) c}. \quad (41)$$
It is obvious that for the above choice of the branch of \( \gamma(\alpha) \), both \( L(\alpha) \) and \( N(\alpha) \) have no zeros in the strip \( \tau_- < \tau < \tau_+ \), and these can be factorised as (cf. Noble (1958))

\[
L(\alpha) = L_+(\alpha)L_-(\alpha), \quad N(\alpha) = N_+(\alpha)N_-(\alpha), \tag{42}
\]

where \( L_-(\alpha) = L_+(-\alpha) \), \( N_-(\alpha) = N_+(-\alpha) \), \( |L_+(\alpha)| = O(|\alpha|^{1/2}) \) as \( |\alpha| \to \infty \) in \( \tau > \tau_- \), \( |N_+(\alpha)| = O(|\alpha|^{1/2}) \) as \( |\alpha| \to \infty \) in \( \tau < \tau_+ \), and

\[
L_+(\alpha) = \left( \frac{-2ia\beta}{1 + a^2} \right)^{1/2} \exp \left[ \int_0^\alpha \left( \frac{\xi - \alpha_0}{2} + \frac{1 - a^2}{1 + a^2} \{ \alpha_0 \wedge_+ (\alpha_0) - \xi \wedge_+ (\xi) \} \right) \frac{d\xi}{\xi^2 - \alpha_0^2} \right], \tag{43}
\]

\[
N_+(\alpha) = \left( \frac{-2ia\beta \rho_0}{(1 + a^2)c} \right)^{1/2} \exp \left[ \int_0^\alpha \left( \frac{\xi - \alpha_0}{2} + \frac{2g\rho_0 - (1 + a^2)c}{(1 + a^2)c} \right) \frac{d\xi}{\xi^2 - \alpha_0^2} \right] \times (\alpha_0 \wedge_+ (\alpha_0) - \xi \wedge_+ (\xi)) \frac{d\xi}{\xi^2 - \alpha_0^2}, \tag{44}
\]

with

\[
\alpha_0^2 = a^2 \beta^2 \left[ 1 - \left( 1 - \frac{2g\rho_0}{(1 + a^2)c} \right)^2 \right], \quad \alpha_+^2 = \frac{4a^4 \beta^2}{(1 + a^2)^2}, \tag{45}
\]

and

\[
\wedge_+(\xi) = \frac{a\beta}{\pi (\xi^2 - a^2 \beta^2)^{1/2}} \ln \left( \frac{\gamma(\alpha) + \xi - a\beta}{\gamma(\alpha) - \xi + a\beta} \right), \quad \wedge_-(\xi) = \wedge_+(-\xi). \tag{46}
\]

Thus

\[
K_+(\alpha) = \left( \frac{-g\rho_0}{c} \right)^{1/2} \frac{L_+(\alpha)}{N_+(\alpha)}
\]

\[
= i \frac{\exp \left[ \int_0^\alpha \left( \frac{\xi - \alpha_0}{2} + \frac{2g\rho_0 - (1 + a^2)c}{(1 + a^2)c} (\alpha_0 \wedge_+ (\alpha_0) - \xi \wedge_+ (\xi)) \right) \frac{d\xi}{\xi^2 - \alpha_0^2} \right]}{\exp \left[ \int_0^\alpha \left( \frac{\xi - \alpha_0}{2} + \frac{2g\rho_0 - (1 + a^2)c}{(1 + a^2)c} (\alpha_0 \wedge_+ (\alpha_0) - \xi \wedge_+ (\xi)) \right) \frac{d\xi}{\xi^2 - \alpha_0^2} \right]}, \tag{47}
\]

and

\[
K_-(\alpha) = K_+(-\alpha), \quad |K_+(\alpha)| = O(1) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \tau > \tau_-.
\]

Now the relation (37) is rewritten as

\[
\frac{F_+(\alpha)}{K_+(\alpha)} = -K_-(\alpha)G_-(\alpha) + \frac{AK_-(\alpha)}{i(\alpha + k)},
\]
which is further rewritten as
\[
\frac{F_+(\alpha)}{K_+(\alpha)} - \frac{AK_-(-k)}{i(\alpha + k)} = -K_-(\alpha)G_-(\alpha) + \frac{A}{i(\alpha + k)}(K_-(\alpha) - K_-(\alpha)), \quad \tau_- \leq \tau < \tau_+.
\]  
(48)

The left side of (48) is analytic in the half plane \(\tau > \tau_-\) and the right side is analytic in the half plane \(\tau < \tau_+\), and as \(|\alpha| \to \infty\) in the respective half planes, each side is of the order \(O(|\alpha|^{-1})\). Applying the principle of analytic continuation and Liouville's theorem, we find that each side of (48) vanishes identically. Thus we find the unknown function \(F_+(\alpha)\) as given by
\[
F_+(\alpha) = \frac{AK_-(-k)}{i(\alpha + k)} K_+(\alpha).
\]  
(49)

Now the use of (34) in (35) gives \(D(\alpha)\) as
\[
D(\alpha) = \begin{cases} 
\frac{F_+(\alpha)}{i(\alpha + k)}, & 
\frac{1}{\sqrt{\pi}} \{\gamma(\alpha) - ia\} \text{L}(\alpha) \\
BK_+(k), & 
\frac{1}{\sqrt{\pi}} \{\gamma(\alpha) - ia\} \text{L}_-(\alpha) \text{N}_+(\alpha),
\end{cases}
\]  
(50)

with
\[
B = \left(\frac{\rho_s}{gc}\right)^{1/2} w^2 A.
\]

Thus by Fourier inversion, \(\phi(x, y)\) \((y < 0)\) is obtained in this case as
\[
\phi(x, y) = \frac{BK_+(k)}{2\pi} \int_{\Gamma} \frac{\exp\left(\frac{(\alpha + k)\gamma}{\alpha} - iax\right)}{(\alpha + k) \{\gamma(\alpha) - ia\} \text{L}_-(\alpha) \text{N}_+(\alpha)} d\alpha, \quad 0 < w \leq w_s,
\]  
(52)

where \(\Gamma\) is a line parallel to the real axis lying in the strip \(\tau_- < \tau < \tau_+\).

(b) \(a^2 > 1\) \((w > w_s)\)

For the choice of the branch of \(\gamma(\alpha)\) made here, the function \(\text{L}(\alpha)\) has zeros at \(\alpha = \pm \alpha_s\), for \(a^2 > 1\) where \(\alpha_s\) is defined in (45) while the function \(\text{N}(\alpha)\) has zeros at \(\alpha = \pm \alpha_0, \alpha_0\) being defined also in (45), only when \((1 + a^2)c > 2g\rho_0\), i.e. only when \(w > w_p\) where
\[
w_p^2 = w_s^2 \left(1 - \frac{\sigma w_s^2}{2g\rho_0}\right)^{-1}.
\]  
(53)

Thus it follows that for \(a^2 > 1\), two situations arise according as \(w < w_p\) and \(w > w_p\). These are also dealt with separately.
In this case $L(\alpha)$ has zeros at $\alpha = \pm \alpha_s$ while $N(\alpha)$ does not have any zero in the strip $r_- < r < r_+$ for the aforesaid choice of the branch of $\gamma(\alpha)$. We write $K(\alpha)$ in this case as

$$K(\alpha) = -\frac{g_0}{c} \frac{\alpha^2 - \alpha_s^2}{M(\alpha)N(\alpha)},$$

(54)

where

$$M(\alpha) = \gamma(\alpha) - ia\beta \frac{\alpha^2 - 1}{a^2 + 1}. \quad (55)$$

We note that $M(\alpha)$ is analytic in the strip $r_- < r < r_+$ and can be factorized as

$$M(\alpha) = M_+(\alpha)M_-(\alpha)$$

where

$$M_+(\alpha) = \left( -\frac{2ia^3}{1+a^2} \right)^{1/2} \exp \left[ \int_0^\alpha \left\{ \frac{\xi - \alpha_s}{2} + \frac{a^2 - 1}{a^2 + 1} (\alpha_s \wedge_+ (\alpha_s) - \xi \wedge_+ (\xi)) \right\} \frac{d\xi}{\xi^2 - \alpha_s^2} \right],$$

(56)

and $M_-(\alpha) = M_+(-\alpha), |M_\pm(\alpha)| = O(|\alpha|^{1/2})$ as $|\alpha| \to \infty$ in $r_- < r < r_+$, $M_+(\alpha)$ is analytic in the half-plane $r_+ < r$. Finally, $K_+(\alpha)$ in this case is obtained as

$$K_+(\alpha) = -i(\alpha + \alpha_s) \exp \left[ -\int_0^\alpha \left\{ \frac{\xi - \alpha_0}{2} + \frac{2g_0}{(1+a^2)c} \frac{1+a^2}{(1+a^2)c} (\alpha_0 \wedge_+ (\alpha_0) - \xi \wedge_+ (\xi)) \right\} \frac{d\xi}{\xi^2 - \alpha_0^2} - \int_0^\alpha \left\{ \frac{\xi - \alpha_s}{2} + \frac{a^2 - 1}{a^2 + 1} (\alpha_s \wedge_+ (\alpha_s) - \xi \wedge_+ (\xi)) \right\} \frac{d\xi}{\xi^2 - \alpha_s^2} \right].$$

(57)

$\phi(x, y)$ in this case is found to be

$$\phi(x, y) = \frac{BK_+(k)}{2\pi} \int_{r_-} \frac{M_-(\alpha) \exp \left( \frac{ir_0(\alpha - \alpha_s)}{a} y - iax \right)}{(\alpha + k)(\alpha - \alpha_s)\{\gamma(\alpha) - ia\beta\} N_+(\alpha)} d\alpha, \ \ \ \ w_0 < w < \infty,$$

(58)

where $\Gamma$ is the same contour as in (52).

(II) $w_0 < w < w_0$

In this case $L(\alpha)$ has zeros at $\alpha = \pm \alpha_s$ and $N(\alpha)$ has zeros at $\alpha = \pm \alpha_0$ in the strip $r_- < r < r_+$, so that $K(\alpha)$ can be written as

$$K(\alpha) = -\frac{g_0}{c} \frac{(\alpha^2 - \alpha_s^2) P(\alpha)}{(\alpha^2 - \alpha_0^2) M(\alpha)},$$
where

\[ P(\alpha) = \gamma(\alpha) - ia\beta \frac{(1 + a^2)c - 2g\rho_0}{(1 + a^2)c}, \]  

which is analytic in the strip \( \tau_- < \tau < \tau_+ \). \( P(\alpha) \) can be factorized as

\[ P(\alpha) = P_+(\alpha)P_-(-\alpha), \]

where

\[ P_+(\alpha) = \left\{ -2i\alpha \left( 1 - \frac{g\rho_0}{(1 + a^2)c} \right) \right\}^{1/2} \exp \left[ \int_0^\alpha \left\{ \frac{\xi - \alpha_0}{2} + \frac{(1 + a^2)c - 2g\rho_0}{1 + a^2} \left( \alpha_0 + \alpha - \xi \right) \right\} \frac{d\xi}{\xi^2 - \alpha_0^2} \right], \]

and \( P_-(\alpha) = P_+(-\alpha), |P_\pm(\alpha)| = O(|\alpha|^{1/2}) \) as \( |\alpha| \rightarrow \infty \) in \( \tau_- \leq \tau \leq \tau_+ \), \( P_+(\alpha) \) is analytic for \( \tau > \tau_- \) while \( P_-(\alpha) \) is analytic for \( \tau < \tau_+ \). Thus, we find that in this case

\[ K_+(\alpha) = \frac{4\pi}{\alpha + \alpha_0} \exp \left[ \int_0^\alpha \left\{ \frac{\xi - \alpha_0}{2} + \frac{(1 + a^2)c - 2g\rho_0}{1 + a^2} \left( \alpha_0 + \alpha - \xi \right) \right\} \frac{d\xi}{\xi^2 - \alpha_0^2} \right]. \]

Finally, \( \phi(x, y) \) in this case is obtained as

\[ \phi(x, y) = \frac{BK_+(k)}{2\pi} \int_\Gamma \frac{P_+(\alpha)M_-(\alpha)\exp \left( \frac{iy}{\alpha} - i\alpha x \right)}{(\alpha + k)(\alpha + \alpha_0)(\gamma(\alpha) - i\alpha\beta)} \, d\alpha, \quad w_p < w < w_0, \tag{62} \]

where \( \Gamma \) is the same contour mentioned earlier.

Now by passing on to the limit as \( \epsilon \rightarrow +0 \) in the results (52), (58) and (62), we obtain the final result in the compact form

\[ \phi(x, y) = \frac{BK_+(k)}{2\pi} \int_0^\infty \frac{\Omega_1(\alpha)\exp \left( \frac{iy}{\alpha} - i\alpha x \right)}{(\alpha + k)(\gamma(\alpha) - i\alpha\beta)\Omega_2(\alpha)} \, d\alpha, \tag{63} \]

where

\[ \Omega_1(\alpha) = \begin{cases} 
1, & 0 < w \leq w_s, \\
M_-(\alpha), & w_s < w \leq w_p, \\
P_+(\alpha)M_-(\alpha), & w_p < w < w_0, 
\end{cases} \tag{64} \]
\[ \Omega_2(\alpha) = \begin{cases} 
L_-(\alpha)N_+(\alpha), & 0 < w \leq w_s, \\
(\alpha - \alpha_4)N_+(\alpha), & w_s < w \leq w_p, \\
(\alpha - \alpha_3)(\alpha + \alpha_0), & w_p < w < w_0, 
\end{cases} \] (65)

and \( K_{-}(-k) \) having appropriate values in the different ranges of \( w \). The integration in (63) is taken along the real axis of the \( \alpha \)-plane with indentation above the negative poles and below the positive poles. In the next section we analyse the solutions (63) asymptotically.

**Asymptotic Analysis of the Solutions**

For asymptotic analysis of the integrals in (63), we introduce the polar co-ordinates \((r, \theta)\) defined by \( x = r \cos \theta, \ y = -r \sin \theta, \ 0 \leq \theta \leq \pi \), where \( \theta \) is measured in a clockwise sense from the \( x \)-axis. We note that the characteristic equations of the PDE (22) represent a pair of straight lines \( y = \pm ax \) passing through the origin and form a characteristic cone. These straight lines are inclined at angles \( \theta_c \) and \( \pi - \theta_c \) where \( \theta_c \) is defined by \( \tan \theta_c = a \).

Let \( \theta_0 \) be defined by \( \tan \theta_0 = \frac{\pm 2\pi}{k} \), then \( \theta_0 \) is the angle which the group velocity vector of the incident wave field \( \phi_0 \) makes with the \( x \)-axis.

From the representations (63) it follows that \( \phi(x, y) \) is continuous in the region \( y \geq 0 \) together with its gradient, but the second order derivatives are logarithmically divergent everywhere on the boundaries of the characteristic cone given by the lines \( \theta = \theta_c, \ \theta = \pi - \theta_0 \) which pass through the origin. This means that the logarithmic singularity of the second order derivatives of the diffracted fields on the edge of the inertial surface propagates along the characteristics of the governing PDE.

Asymptotic estimates of the integrals in the representations (63) in different regions are obtained by the method of steepest descent. The final result for the total wave field \( u \) is obtained in the following form
\[ u = \phi_0 + \begin{cases} 
\phi_2 + \phi_1^{(1)} + \phi_{IS}, & 0 \leq \theta < \theta_0, \\
\phi_1 + \phi_1^{(1)} + \phi_{IS}, & \theta_0 < \theta < \theta_c, \\
\phi_1 + \phi_1^{(2)} + \phi_s, & \theta_c < \theta < \pi - \theta_c, \\
\phi_1 + \phi_1^{(1)} + \phi_{s}, & \pi - \theta_c < \theta \leq \pi,
\end{cases} \tag{66} \]

where \( \phi_1 \) is given by (20),

\[ \phi_2 = \frac{ck^2 + \rho_0 w^2 (ib - \beta)}{ck^2 - \rho_0 w^2 (ib + \beta)} \exp \left( iby + ikx \right), \]

\[ \phi_{IS} = \frac{iBK_+ (k) \Omega_1 (-\alpha_0)}{(k - \alpha_0) \{ \gamma (\alpha_0) - i \alpha \beta \} \Omega_2 (-\alpha_0)} \exp \left( \frac{\beta \{(1 + a^2) \gamma - 2 \rho \rho_0 \}}{(1 + a^2) \gamma } y + i \alpha \beta x \right), \]

\[ \phi_s = \frac{iBK_+ (k) \Omega_1 (\alpha_s)}{(k + \alpha_s) \{ \gamma (\alpha_s) - i \alpha \beta \} \Omega_2 (\alpha_s)} \exp \left( \frac{\beta}{a^2 + 1} \{(a^2 - 1) y - 2 \alpha \beta x \} \right), \]

\[ \phi_1^{(1)} = \frac{BK_+ (k) \Omega_1 (\alpha_1^{(1)})}{(k + \alpha_1^{(1)}) \{ \gamma (\alpha_1^{(1)}) - i \alpha \beta \} \Omega_2 (\alpha_1^{(1)})} \frac{1}{\{2 \pi \beta \gamma P (\theta) \}^{1/2}} \]

\[ \times \frac{\alpha \beta \sin \theta}{P(\theta)} \exp \left( i \left\{ \beta \gamma P (\theta) + \frac{\pi}{4} \right\} \left[ 1 + O \left( \frac{1}{\beta \gamma} \right) \right] \right), \]

\[ \phi_1^{(2)} = \frac{iBK_+ (k) \Omega_1 (\alpha_1^{(2)})}{(k + \alpha_1^{(2)}) \{ \gamma (\alpha_1^{(2)}) - i \alpha \beta \} \Omega_2 (\alpha_1^{(2)})} \frac{1}{\{2 \pi \beta \gamma Q (\theta) \}^{1/2}} \]

\[ \times \frac{\alpha \beta \sin \theta}{Q(\theta)} \exp \left( - \beta \gamma Q (\theta) \left[ 1 + O \left( \frac{1}{\beta \gamma} \right) \right] \right), \]

with

\[ P(\theta) = (a^2 \cos^2 \theta - \sin^2 \theta)^{1/2}, \quad 0 < \theta < \theta_c, \quad \pi - \theta_c < \theta < \pi, \]

\[ Q(\theta) = (\sin^2 \theta - a^2 \cos^2 \theta)^{1/2}, \quad \theta_c < \theta < \pi - \theta_c, \]

\[ \alpha_1^{(1)} = -\frac{a^2 \beta \cos^2 \theta}{P(\theta)}, \quad \alpha_1^{(2)} = \frac{i \alpha \beta \cos^2 \theta}{Q(\theta)}. \]
In (66), $\phi_0$ is the incident internal wave field, $\phi_1$ is the wave reflected from the free surface, $\phi_2$ is the wave reflected from the inertial interface, $\phi_{IS}$ is the wave due to inertial surface, which exists only when $w_p < w < w_0$, $\phi_S$ is the surface wave which exists when $w_s < w < w_0$, and $\phi_d^{(1)}, \phi_d^{(2)}$ are contributions due to diffraction. This asymptotic analysis is consistent with a similar analysis by Gabov and Sveshnikov (1982) and Varlamov (1985) in connection with their studies on internal wave scattering by the edge of an ice field and an elastic plate in the form of a half space respectively.

**Discussion**

The terms $\phi_0, \phi_1, \phi_2$ represent the zeroth approximation terms in the representation (66) of the total wave field in the sense that they occur according to the laws of geometrical optics and without considering diffraction by the inertial surface. This becomes obvious when the area density $\sigma$ of the inertial surface is made equal to zero. In that case, $\phi_2 = \phi_1$ and $\phi_d^{(1)}, \phi_d^{(2)}, \phi_{IS}$ and $\phi_S$ vanish identically as expected since no diffraction can occur in the absence of the inertial surface.

We first discuss the terms $\phi_d^{(1)}$ and $\phi_d^{(2)}$ which are due to diffraction. $\phi_d^{(1)}$ arises in the region $0 \leq \theta < \theta_e$ and $\pi - \theta_e < \theta < \pi$ i.e. in the region within the characteristic cone. It represents waves whose amplitude decays away from the edge of the inertial surface like $(\beta r)^{-1/2}$. Also, as observed by Gabov and Sveshnikov (1982), the crests of these waves represent a family of hyperbola whose asymptotes are $\theta = \theta_e, \pi - \theta_e$ i.e. coincide with the boundaries of the characteristic cone. The term $\phi_d^{(2)}$ arises in the region $\theta_e < \theta < \pi - \theta_e$ i.e. in the region outside the characteristic cone. However, unlike $\phi_d^{(1)}$, it has no wave like character and it decays exponentially away from the edge of the inertial surface in the region $\theta_e < \theta < \pi - \theta_e$.

The term $\phi_S$ exists only when $w_s < w < w_0$ and arises in the region $\pi - \theta_e < \theta \leq \pi$. It represents a surface wave near the free surface and decays exponentially away from it. As, observed by Varlamov (1985) in a similar situation, the energy of this wave remains in the vicinity of the free surface towards negative $x$-direction.
Finally, the term $\phi_{IS}$ exists only when $w_p < w < w_0$ and arises in the region $0 \leq \theta < \theta_c$. When $\theta = 0$ (i.e. $y = 0$), it describes a wave on the inertial surface propagating along the positive $x$-direction without any decay of its amplitude. When $\theta_0 < \theta < \theta_c$, $\phi_{IS}$ may also be regarded as a wave in the liquid under the inertial surface, and the energy associated with it decays exponentially with the depth of the liquid. This observation is also consistent with the result obtained by Varlamov (1985) in the case of internal wave diffraction by an elastic plate.

It may be mentioned here that when the inertial surface is in the form of a strip instead of a half space, a three part Wiener-Hopf problem will arise which can be dealt with approximately.