PART A

INTRODUCTORY PART
Chapter 1

General Introduction

1.1 Preamble

Study of various types of wave problems in ocean constitutes an important branch of Fluid Mechanics and enables one to understand various natural wave phenomena occurring on the ocean surface and also inside the ocean. The primary task in this study is modelling of these phenomena in a manner suitable for mathematical and computational analysis, and for this purpose some simplified assumptions about the physical properties of the medium, about the motion generated, etc. have to be made.

In the mathematical study of water wave problems there exist two types of theories, namely the non-linear shallow water theory and the linearised theory. The shallow water theory was developed from the assumption that the depth of water is small compared to some characteristic length. Most of the mathematical problems in our study are based on the linearised theory of water waves in which it is assumed that the velocity components of the fluid elements and the free surface elevation or depression together with their partial derivatives are small quantities so that their products and powers of higher order can be neglected. Obviously this is valid only when the wave amplitude is small compared to the depth of the water so that the effects of the disturbance cease to be appreciable below a certain depth. If the ocean is regarded as a homogeneous fluid, and further, if the motion is assumed to be small so as to linearise the equations...
of motion and the boundary conditions, we have the so-called linearised theory of water waves, whose literature is very rich and many mathematical ideas have been developed from this theory.

However as it is well known, an ocean consists of salt water whose density increases with the increase of depth. Thus, the fluid cannot be assumed to be homogeneous strictly and it is more realistic to consider density stratification. This stratification may be assumed to occur in layers, i.e. in a multilayered fluid wherein each layer consists of a homogeneous fluid of a particular density, a lower density fluid layer being above a higher density fluid layer. This may also occur in a continuous manner, i.e. the density increases continuously with depth. The present thesis is concerned with various types of wave problems in such stratified fluids based on linearised theory.

Occurrence of waves on an ocean surface and their impact on a sea beach is a common natural phenomenon. The problems involving generation of surface waves are classical in nature and have been studied mathematically as well as experimentally by many physicists, mathematicians and engineers.

In many cases, the concerned mathematical analysis is developed in the literature within the framework of linearised theory, and homogeneous fluid. However, if the fluid is not homogeneous, then the whole scenario changes drastically even under the linear theory and the problems become rather complicated to handle mathematically.

Waves on the surface of an ocean are generated by various types of disturbances in the form of initial depression or impulse. Instead of free surface, one may consider wave generation problems at an inertial surface which is composed of a thin but uniform distribution of non-interacting floating particles such as broken ice since the ocean surface is sometimes covered by broken ice near the polar regions, or at the interface between two superposed fluids such as air and the ocean.

In this thesis, a number of wave generation problems in a stratified fluid due to initial disturbances at the inertial surface are considered. Assuming linear theory, the problems are formulated in terms of pressure function under Boussinesq approximation
and constant but small Brunt Vaisala parameter. As the stratification may also occur in layers, some interface wave generation problems are considered in two superposed fluids which are separated by an inertial interface. Also, a wave generation problem in a single homogeneous fluid moving with a constant velocity in a horizontal direction is considered assuming the presence of surface tension at the free surface. This in fact can be treated as some sort of important mathematical exercise in the study of more general problems on interface waves for two homogeneous superposed fluids moving horizontally with a constant speed. However, no such general problem could be tackled so far in the present study.

In a stratified fluid, because of the complexity arising due to stratification, it is rather difficult to handle internal wave scattering and radiation problems involving obstacles successfully. For a stratified fluid of uniform finite depth under a solid plane, the internal wave scattering and radiation problems involving plane vertical barriers, have been investigated in the literature assuming linear theory. Some internal wave scattering problems in a channel of stratified fluid of uniform finite depth under a solid plane involving plane vertical barriers of various configurations, are considered in this thesis.

The modelling of another type of internal wave scattering problem involving a semi-infinite inertial surface or solid plate covering a portion of the surface of an exponentially stratified liquid while the remaining portion of the surface is free has some physical importance. In this case the surface boundary condition becomes discontinuous in the sense that there is one condition at the covered surface and another condition at the free surface. These types of problems in a stratified fluid of infinite depth are investigated in the literature based on Wiener-Hopf technique. Here also the Wiener-Hopf technique is used to tackle the problem of internal wave scattering by a semi-infinite inertial surface in a stratified fluid of infinite depth.

With this preamble we now give a brief account of various problems presented in the thesis and related previous work.
1.2 A Brief Description of the Problems in the Thesis and Related Previous Work

The contents of the thesis are divided into four parts, namely A, B, C and D. Part ‘A’ is the introductory part which contains Chapters 1 and 2. The Chapter 1 is concerned with a general introduction, a brief description of the problems in the thesis and related previous work. This also contains a brief account of the derivation of the basic equations of motion in a stratified fluid in which the density stratification occurs in layers, e.g. a multilayered fluid where each layer consists of a homogeneous fluid of a particular constant density, or in a continuous manner, e.g. the density increases continuously with depth. Chapter 2 is devoted to a brief description about some mathematical preliminaries relevant to the problems in this thesis. This Chapter is divided into several sections describing briefly the concepts of integral transform, singular integral equations, source potentials, properties of analytic function, some properties of Fourier transform in the complex domain, Wiener-Hopf technique, method of stationary phase, method of steepest descent, derivation of the solution of a Cauchy Poisson problem.

In Part ‘B’, some wave generation problems due to initial disturbances at the inertial surface of a stratified fluid of uniform finite depth are considered. This part discusses three problems presented in Chapters 3 to 5. The Chapter 3 deals with the problem of wave generation due to initial disturbances at the inertial surface of stratified fluid. In Chapter 4, the problem of water wave generation in a stratified fluid in the presence of axi-symmetric initial disturbances is given. Chapter 5 is concerned with the three-dimensional wave generation problem in a stratified fluid of uniform finite depth involving various types of initial disturbances prescribed over a region of different geometrical shapes on the inertial surface.

Some wave generation problems in the presence of initial disturbances at the interface between two superposed fluids are considered in Part ‘C’ which consists of three problems presented in Chapters 6 to 8. In Chapter 6, an interface wave generation problem in the presence of initial disturbances at the interface between two superposed fluids is considered. Chapter 7 deals with the problem of wave generation due to axi-symmetric
initial disturbances at the inertial interface between two superposed fluids. Chapter 8 is concerned with a three-dimensional wave generation problem due to prescribed initial disturbances at the inertial interface. A wave generation problem involving the presence of surface tension at the free surface of a running stream is considered in Chapter 9. This problem is a prelude to more general problems on interface waves for two moving superposed homogeneous fluids, although no such general problem was considered here.

Some wave propagation problems in a stratified fluid are considered in Part 'D' which consists of three Chapters 10 to 12. Chapter 10 deals with the problem of internal wave scattering by a thin vertical barrier present in a channel of stratified fluid. A study on a problem of the internal wave scattering by a vertical plate or wall with a gap present in a channel of stratified fluid of uniform finite depth is presented in Chapter 11. In Chapter 12, the problem of internal wave scattering by a semi-infinite inertial surface in an exponentially stratified fluid of infinite depth is considered with the aid of Wiener-Hopf technique.

Most of the problems presented in this thesis are supplemented by appropriate numerical results.

A brief description of the various problems in parts B, C and D of this thesis and related previous works are now presented.

**Part B : Wave Generation Problems in a Continuously Stratified Fluid**

The two-dimensional Cauchy-Poisson problem concerning generation of water waves due to initial disturbances at the free surface is well studied in the literature assuming linear theory. The problem was studied in the treatise of Lamb (1945) and Stoker (1957) by the use of Fourier transform technique, and the free surface elevation was evaluated asymptotically by applying the method of stationary phase when the disturbance is concentrated at the origin. Kranzer and Keller (1959) considered the three-dimensional unsteady motion due to an arbitrary axially symmetric initial surface disturbance in an ocean of uniform finite depth and gave explicit expression for the surface elevation, and
compared the theory with experimental results. They also deduced these results for an ocean of infinite depth. Chaudhuri (1968) and Wen (1982) considered the case when the initial surface disturbance is a combination of both surface impulse and surface elevation distributed over an arbitrary region of the surface of an ocean of uniform finite depth and obtained asymptotic results by using the method of stationary phase.

In all these cases, the upper surface of the ocean was assumed to be bounded by a free surface. However there has been considerable interest to study various problems involving generation of waves in an ocean covered by an inertial surface composed of a thin layer of uniformly distributed non-interacting floating particles (e.g., broken ice, floating mat). Mandal (1988a) considered the two-dimensional unsteady motion in a deep ocean covered by an inertial surface due to initial disturbances at the inertial surface. The corresponding problem for an ocean of uniform finite depth was considered by Mandal and Ghosh (1989). Mandal and Mukherjee (1989) studied three-dimensional unsteady motion in a deep ocean covered by an inertial surface due to a prescribed axi-symmetric initial disturbance at the inertial surface while the corresponding problem for an ocean of uniform finite depth was considered by Mandal and Ghosh (1990a). Also Mandal and Ghosh (1990b) considered the problem of generation of water waves in an ocean of uniform finite depth covered by an inertial surface due to an arbitrary periodic pressure distribution on the inertial surface as well as an initial disturbances in the form of an initial surface depression or impulsive pressure acting on the inertial surface.

In all the above studies, the ocean is assumed to be homogeneous fluid. However, because of salinity, the density of the ocean increases with depth and it is thus realistic if one models an ocean as a stratified fluid.

For a weakly stratified fluid with constant Brunt Vaisala parameter, Debnath and Guha (1989) formulated the problem of wave generation due to a prescribed initial disturbances at the free surface in terms of an acceleration potential and obtained the free surface profile asymptotically for large time and distance far away from the region of disturbance. Also the problem of wave generation due to axi-symmetric disturbances
on the surface of a rotating stratified fluid of infinite depth was considered by Rollins and Debnath (1990). They also determined the free surface elevation asymptotically for large distances and times. Finally they compared these results with the results of an inviscid non-rotating non-stratified liquid.

In Chapter 3, the problem of wave generation due to initial disturbances in a weakly stratified fluid of finite depth covered by an inertial surface is considered. Assuming linear theory, the problem is formulated in terms of pressure under Boussinesq approximation and constant but small Brunt Vaisala parameter. By using Laplace transform technique, the initial value problem is reduced to a boundary value problem, which is then solved by two methods, one based on an appropriate use of Green's integral theorem and the other on Fourier transform technique. The form of the inertial surface is then obtained in terms of an integral. This integral is evaluated asymptotically for large time and distance by the method of stationary phase when the initial disturbances at the inertial surface is concentrated at a point taken as the origin. The asymptotic form of the inertial surface profile is depicted graphically and compared with the results for an ideal fluid covered by an inertial surface.

Chapter 4 deals with the problem of generation of waves due to initial axi-symmetric disturbances at the inertial surface of a weakly stratified fluid of uniform finite depth. By a similar procedure, the problem is formulated as an initial value problem which reduces to a boundary value problem by the use of Laplace transform technique. Because of axial symmetry, the solution of this boundary value problem is solved by utilizing the Hankel transform technique. Finally Laplace inversion produces the perturbed pressure and hence the depression of the inertial surface is obtained in terms of an integral involving the Bessel function, the depression of the inertial surface is obtained as an infinite double integral instead of single integral. This is also evaluated asymptotically for large distances and times using the method of stationary phase when the initial disturbances at the inertial surface is concentrated at the origin and also when prescribed over a circular area. For each case this is depicted graphically to visualize the wave motion produced. It is observed that the presence of an inertial surface has a significant effect on
the wave motion while weak stratification of the liquid does not have such a significant
effect.

Chapter 5 is devoted to the problem of generation of three dimensional waves due
to initial disturbances in a weakly stratified fluid of finite depth covered by an inertial
surface. By a similar procedure used in Chapter 4, the problem is reduced to a boundary
value problem, which is now solved by employing double Fourier transform. Finally
combining the inversion of double Fourier and Laplace transforms we obtain the pressure
function and hence the depression of the inertial surface is obtained in terms of a triple
integral. This is then evaluated asymptotically for large distances and times using
the method of stationary phase when the initial disturbance at the inertial surface is
prescribed over a rectangle and also over an ellipse. The asymptotic form of the inertial
surface profile is plotted graphically to show the effect of stratification as well as the
presence of inertial surface on the wave motion. It is observed that for fixed time,
when distance increases, the amplitude of the wave profile decreases and wave length
increases, and it dies out almost completely at large distances. It is also observed that
the presence of weak stratification has not much significant effect on the wave amplitude
while the presence of inertial surface has some significant effect.

Part C : Interface Wave Generation Problems

A brief survey of the literature on problems of wave generation due to initial disturbances
at the free surface of a fluid is given earlier. Possible extension of these problems to two
superposed fluids is mentioned by Wehausen and Laitone (1960) long back but no details
were given. The problem of generation of interface waves in two superposed fluids due
to initial disturbances at the interface where the upper fluid extends infinitely upwards
and the lower fluid extends infinitely downwards was considered by Dolai (1996). In the
mathematical analysis, he exploited the symmetry of the fluid region about the interface
to define a new potential function in the lower fluid region which is a linear combination
of the velocity potential for lower fluid region and another potential defined in the lower
region by reflection of the velocity potential for the upper fluid about the common interface. This new potential function satisfies an initial value problem for a single fluid, and was solved by Laplace and Fourier transform technique. Then he obtained the asymptotic form of the interface profile by the method of stationary phase. However, this method is applicable only when the interface is symmetric with respect to the two fluids. When the lower fluid is of uniform finite depth and the upper fluid extends infinitely upwards, this method will obviously not work. However, this situation can be treated mathematically by a decoupling process as described below briefly and in more detail in Chapter 6.

In Chapter 6, the wave generation problem at the interface due to initial disturbance at the interface between two superposed fluid wherein the upper fluid extends infinitely upwards while the lower fluid is of uniform finite depth is considered. Assuming linear theory, the problem is formulated by the potential functions describing the motions in the two fluids which satisfies a coupled initial value problem. By the use of Laplace transform in time reduces the initial value problem to a coupled boundary value problem. This is then decoupled into two independent boundary value problems involving the interface depression. These are solved by using Fourier transform technique and the solutions involve the interface depression. Using the interface condition to these solutions, and employing the Fourier and Laplace inversions, the interface depression is obtained in terms of an infinite integral. This is then evaluated asymptotically for large time and distance by using the method of stationary phase when the initial disturbances at the interface are in the form of initial interface depression or impulse concentrated at the origin. For both the cases, non-dimensional form of the interface depression is depicted graphically against distance for fixed time and against time for fixed distance for a number of values of the depth of the lower fluid, and compared with the case when the lower fluid is of infinite depth. From the numerical results, it is observed that if the density of the upper fluid is small, as is the case for an ocean atmosphere system, then the upper fluid has not much influence on the wave motion.

Chapter 7 deals with the problem of generation of interface waves due to initial axially
symmetric disturbances at the inertial interface between two superposed fluids wherein the lower fluid is of uniform finite depth while the upper fluid extends infinitely upwards. By an analogous method used in the study of the problem of Chapter 6, the formulation of the boundary value problem is obtained as before. Because of axial symmetry, Hankel transform technique is used in the radial co-ordinate and the coupled boundary value problem is decoupled into two independent boundary value problems, each involving the unknown interface depression. Using the interface condition in the solution of the two independent boundary wave problems, and using Hankel and Laplace inversions, the interface depression is obtained in terms of an infinite integral. This is then evaluated asymptotically for large time and distance by using the method of stationary phase for the two cases when the initial disturbances at the inertial interface are concentrated at the origin or prescribed over a circular region. To visualize the effect of the upper fluid and the effect of the inertial surface on the wave motion at the interface, the non-dimensional interface depression is plotted graphically for the two cases when the disturbance is concentrated at the origin or prescribed over a circular region. From the numerical results it is observed that the presence of the upper fluid and the inertial surface affects mainly the phase of the wave motion at the interface.

In Chapter 8, the three dimensional wave generation problem at the interface between two superposed fluids is considered. In this case the initial value problem reduces to a boundary value problem by applying Laplace transform in time and double Fourier transform in space. The inertial interface depression is obtained in terms of a double integral and is evaluated asymptotically by the method of stationary phase when the initial disturbances is concentrated over a rectangular or an elliptic region. This is then depicted graphically in a number of figures to visualize the effect of the upper fluid and also the presence of the inertial surface at the interface on the wave motion. The time evolution of the interface depression is also shown graphically and the decaying phenomena are demonstrated by drawing the phase diagram. It is observed from these figures that due to the presence of the upper fluid and the inertial interface, the basic characteristics of the wave motion remain almost the same for the case of a single fluid with a free surface, although the phase of the wave motion changes significantly.
The problem of two-dimensional unsteady motion in deep water generated due to a periodic pressure, of frequency \( w \), applied at the free surface is classical in nature and its solution is given in the treatise of Stoker (1957). Unfortunately there was an error in the analysis which was later pointed out and corrected by Miles (1962). Debnath and Rosenblant (1969) considered the initial value problem, for the generation of two-dimensional waves by an oscillatory pressure acting at the surface of an ocean of uniform finite depth in the presence of a running stream. They showed that either two or four waves may exist in the steady state case depending on the velocity of the running stream, depth of the fluid and the frequency of the applied pressure. Its extension to the two or three dimensional tsunamis in a shallow ocean due to the action of an arbitrary ocean floor or ocean surface disturbance was considered by Debnath and Basu (1978). They found that if the speed of the running stream is less than the speed of the shallow ocean wave, then the free surface elevation of surface waves decay asymptotically as \( t^{-1/2} \) for two-dimensional case and as \( t^{-1} \) for the three-dimensional case, where \( t \) denotes the time.

Chapter 9 is concerned with the two-dimensional problem of generation of surface waves in the presence of stream velocity \( (U) \) and taking into account the effect of surface tension, the initial disturbance being in the form of prescribed depression or impulse at the free surface. Assuming linear theory, the problem is formulated as an initial value problem. In the mathematical analysis use of Laplace transform technique in time, reduces the initial value problem to a boundary value problem as before. This is then solved by applying the Fourier transform technique. After Laplace and Fourier inversions, the form of the free surface elevation is obtained in terms of an integral. This integral is evaluated asymptotically for large distance and time by applying the method of stationary phase. In the presence of surface tension, there exists one stationary point in the range of integration for \( \tilde{U} > \frac{\tilde{z}}{\tilde{t}} + 1 \), where \( \tilde{U}, \tilde{z}, \tilde{t} \), represent respectively the non-dimensional stream velocity, distance from the origin and time, while in the absence of surface tension, there exists two stationary points when the stream velocity lies between in a certain range, i.e. for \( \frac{\tilde{z}}{\tilde{t}} < \tilde{U} < \frac{\tilde{z}}{\tilde{t}} + 1 \) and \( \frac{\tilde{z}}{\tilde{t}} - 1 < \tilde{U} < \frac{\tilde{z}}{\tilde{t}} \) respectively. To visualize the effects of stream velocity and the presence of surface tension on the wave motion,
the non-dimensional form of the surface elevation is plotted graphically in each case when the disturbances are concentrated at the origin. It is observed from these figures that the presence of surface tension and the running stream have some significant effect on the wave motion.

**Part D : Scattering of Internal Waves**

When a train of surface waves propagating from infinity is incident on an obstacle present in water it experiences partial reflection and transmission by and over or below the obstacle. Evaluation of reflection and transmission co-efficients is of some what mathematical and physical importance. In the literature, many researchers studied the scattering problems when the obstacle is in the form of a thin vertical plane barrier. There exists a number of basic configurations of the vertical barrier for which the wave scattering problem admits of explicit solution for normal incidence of the wave train and deep water. Some of these configurations are in the form of a partially immersed vertical barrier or submerged barrier extending down to the bottom, submerged vertical barrier extending down to some finite depth (i.e. a submerged vertical plate) and vertical wall with a submerged gap.

The problems of scattering of surface waves by obstacles of various shapes have been studied since early 1940's, when Dean (1945) for the first time investigated scattering of surface water waves by a submerged vertical barrier. Ursell (1947) considered the corresponding problem wherein a thin vertical barrier is partially immersed and utilized an integral equation procedure based on Havelock's (1929) expansion of water wave potential to obtain the solution. Evans (1968) investigated Ursell's result with the extra effect of surface tension and obtained the reflection and transmission co-efficients by applying complex variable method. The submerged vertical plate problem was considered by Evans (1970) who used the complex variable technique for its solution. Porter (1972) considered the problem of water wave diffraction by a vertical wall with a submerged gap and used a complex variable technique as well as an integral equation procedure
based on Green's integral theorem to solve it. Earlier Levine and Rodemich (1958) used an integral equation procedure based on Green's integral theorem and Williams (1966) used a reduction technique for solving the partially immersed barrier problem. Also, Goswami (1982) used an integral equation procedure based on Green's integral theorem for solving the submerged barrier problem. Mandal and Kundu (1987) demonstrated four different methods, viz., expansion method, integral equation method, reduction method and complex variable method for solving the submerged barrier problem. Mandal (1988b) considered the partially immersed and submerged barrier problems wherein the integral equation arising in both the problems is first reduced to a singular integral equation with Cauchy type kernel in a finite interval. Chakrabarti (1989) presented a direct method to solve two integral equations arising in the study of water wave diffraction by partially immersed as well as submerged vertical barriers.

The gap problem was also considered by many workers under the assumption that the gap is narrow in the sense that the length of the gap is small compared to the depth of submergence of its mid point below the free surface (cf. Tuck (1971), Guiney et. al. (1972), Owen and Bhatt (1985), Evans (1977), Packham and Williams (1972), Mandal (1987)).

In all the above problems, the ocean is assumed to be a homogeneous fluid. However because of salinity, the density of the ocean increases with depth and it becomes important to consider internal wave scattering problems in a stratified fluid.

The problem of internal wave scattering by a bottom-standing thin vertical barrier present in an exponentially stratified fluid of uniform finite depth under a solid plane was considered quite some time back by Larsen (1969). He obtained the solution satisfying the radiation criteria for all the modes transmitted through and reflected by the barrier corresponding to incident internal waves of the lowest mode. Sandstrom (1966) (cf. Larsen (1969)) conducted a series of experiments illustrating the behaviour of internal waves incident upon sloping and vertical barriers. One of his experiments is concerned with wave of the first mode incident upon a vertical knife edge partially blocking the channel. The related problem of small horizontal oscillation of a barrier as a whole
was investigated by Krutitsky (1988). Later Korobkin (1990) utilized Larsen’s (1969) solution in the study of the motion of a body in a weakly stratified fluid in the presence of a bottom obstacle modeled as a thin vertical bottom standing plane barrier.

In Chapter 10, the problem of internal wave scattering by a partially immersed thin vertical barrier in a channel of stratified fluid is considered. The incident internal waves, described by a stream function, when come across the barrier, are reflected back by the barrier with various modes of reflected internal waves and are transmitted through the gap below the barrier also with various modes of transmitted internal waves. The reflected and transmitted internal waves are described by a scattered stream function which satisfies a boundary value problem in the fluid region. It is represented on both sides of the barrier by appropriate eigenfunction expansions. Use of appropriate conditions on the barrier and across the gap results in a dual series relation involving the elements of the scattering matrix. By defining a function on the barrier involving the elements of the scattering matrix as the co-efficients in its Fourier expansion series, the dual series relation reduces to a first kind singular integral equation with a Cauchy type kernel for this function, whose solution is immediate. All the elements of the scattering matrix and hence the stream function describing the resulting motion in the fluid are obtained explicitly in principle by utilizing this solution. For the lowest mode of the incoming internal wave train, the elements of the scattering matrix and the stream function are obtained. The stream lines are plotted graphically to visualize the effect of the barrier on the incoming internal wave train. It is observed that some stream lines abruptly change their directions, and the envelop of the points where these abrupt changes of direction in the stream lines occur consists of two lines which intersect at the edges.

Chapter 11 deals with the problem of internal wave scattering by a vertical barrier in the form of a thin wall with a submerged gap or a submerged plate in an exponentially stratified fluid of uniform finite depth bounded above by a rigid plane. Due to the presence of the barrier in the stratified fluid, the incident waves are reflected back by the barrier with various modes and transmitted through the gap or gaps also with
various modes as in the previous problem. The reflected and transmitted internal waves are described by a scattered stream function which satisfies a boundary value problem in the fluid region. This stream function is expressed on both sides of the barrier by appropriate eigenfunction expansions involving the elements of the scattering matrix. As in the previous problem, use of appropriate conditions on the barrier and the gap, results in a dual series relation involving the elements of the scattering matrix. By defining a function on the barrier involving the elements of the scattering matrix as the co-efficients of its Fourier series, the dual series relation reduces to a Carleman integral equation in a single interval for the submerged plate problem or in a double interval for the problem of a wall with a submerged gap. The solution of this integral equation for each case is obtained by standard technique. Using the solution of the appropriate integral equation, the elements of the scattering matrix and hence the stream function describing the resulting motion in the fluid for each case are obtained in principle, and for the lowest mode of the incoming internal wave train, these are evaluated explicitly. For both the barrier configurations, the stream lines are plotted graphically to visualize the effect of the barrier on the incoming internal wave train.

In all the cases, it is observed that some stream lines abruptly change their directions, and the points where the changes in the direction of stream lines occur roughly lie on straight lines which intersect near the edges at the barriers and different patterns in the stream line contours are formal for each of the barriers.

Various important and interesting methods of handling mixed boundary value problems, associated with Laplace equation, arising in the study of scattering of surface water waves have been developed and utilized by a large number of workers (cf. Ursell (1947), Stoker (1957), Peters (1950), Weitz and Keller (1950), Newman (1965), Evans (1994), Evans and Linton (1994), Gabov et. al. (1989), Goldshtein and Marchenko (1989) and others). The problems of scattering of two-dimensional surface water waves, by a discontinuity in the surface boundary conditions, constitute a special class, and methods involving the use of the powerful Wiener-Hopf technique have been utilized by Weitz and Keller (1950). Gabov et. al. (1989) generalised these problems for two
immiscible homogeneous liquids for which half the interface is covered by an inertial
surface and the other half is a free separating boundary of the liquids. It may be noted
that the problem for a single homogeneous liquid (e.g. deep water) and the problem for
two superposed immiscible homogeneous liquids are mathematically equivalent, the only
difference being that two sets of different physical constants are to be employed while
tackling the problems mathematically. Chakrabarti (2000) considered the mixed bound­
dary value problem arising in the study of scattering of two-dimensional time harmonic
surface water waves by a discontinuity on the surface boundary conditions, separating
the clean surface and an ice-covered surface, of a water of infinite depth. Kanoria et. al.
(1999) investigated two mixed boundary value problems involving surface water wave
in deep water (or interface wave in two superposed homogeneous liquids) arising due
to one or two discontinuities in the surface (or interface) boundary conditions. The
governing partial differential equation in these problems is the Laplace equation which
together with the boundary conditions, is generalised to the Helmholtz’s equation to­
gether with slightly different boundary conditions by introducing a complex parameter
to facilitate the use of Wiener-Hopf technique in the mathematical analysis. Ultimately
this parameter is made to tend to zero to obtain the solutions of the original problems.

Instead of a homogeneous liquid, if there is a stratified liquid in which the density
varies exponentially along the vertical direction, then the governing partial differential
equation describing the propagation of steady state internal waves becomes the Klein-
Gordon equation. Thus the governing differential equation here is a partial differential
equation of hyperbolic type in contrast to partial differential equations of elliptic type
encountered in the classical diffraction theory. The initial boundary value problem
for the equation of a stratified liquid in the Boussinesq approximation modelling the
oscillations of a stratified liquid under an ice field was considered by Gabov and Simakov
(1989). They constructed the solution in terms of eigenfunctions of some operator and
investigated its behaviour for large time.

Gabov (1982a, 1982b), Gabov and Sveshnikov (1982) investigated scattering of two-
dimensional steady-state internal waves described by the Klein-Gordon equation in an
exponentially stratified incompressible liquid by the boundary of a solid half plane. This models interaction of waves with an ice field covering half the surface of an infinitely deep ocean whose density varies along the vertical direction in an exponential manner. They used the Wiener-Hopf technique in the mathematical analysis. Several researchers, mostly Russians, soon afterwards investigated a number of variations of these problems by using the same technique. For example, Varlamov (1983, 1985) investigated internal wave scattering by a semi-infinite horizontal wall present inside the liquid and by a semi-infinite elastic half plate present on the surface of the liquid. The problem of diffraction of internal waves by an inclined half plane in the stratified fluid was considered by Gabov et. al. (1983).

In Chapter 12, the problem of internal wave scattering by a semi-infinite inertial surface partly covering an exponentially stratified liquid is investigated. This may be regarded as a generalisation of the classical scattering problem considered by Peters (1950) for surface water waves in the presence of an inertial surface (e.g. broken ice, floating mat) to internal wave scattering by an inertial surface covering an exponentially stratified liquid. Assuming linear theory and under Boussinesq approximation with constant Brunt-Vaisala frequency, the problem is formulated as a boundary value problem involving the Klein-Gordon equation with discontinuous surface boundary conditions. The problem is handled for its solution with the aid of Wiener-Hopf technique after introducing a small positive imaginary part in a certain parameter occurring in the Klein-Gordon equation as well as by slightly generalising the surface boundary conditions, the edge conditions and the infinity requirements, and ultimately passing on to the limit as this small imaginary part introduced earlier, tends to zero. The scattered field is obtained in terms of integrals which are evaluated asymptotically for large distances from the edge of the inertial surface by the method of steepest descent and interpreted physically.

We now present a brief description of the basic equations in a fluid starting with the case when the fluid is homogeneous. As two types of density stratification have been considered, viz., stratification in layers and continuous density stratification, we give the
basic equations for two superposed homogeneous fluids and then for a stratified fluid, all based on the linearised theory.

1.3 Basic Equations in the Linearised Theory of Water Waves

(a) Homogeneous Fluid

We consider the motion in an incompressible, inviscid, homogeneous fluid of volume density $\rho$ under the action of gravity only, the fluid being bounded above by a free surface. The fluid may be of uniform finite or infinite depth. A rectangular cartesian co-ordinate system is chosen in which the $y$-axis is taken vertically downwards and $y = 0$ is the undisturbed position of the free surface. For deep fluid, the fluid occupies the region $y \geq 0$ and for a fluid of uniform finite depth, occupies the region $0 \leq y \leq h$.

In the linearised theory, the basic equations in a homogeneous fluid are derived from the equation of continuity (equation of conservation of mass) and the Euler's equation of motion (equation of conservation of momentum) (cf. Lamb (1945), Stoker (1957), Landau and Lifschitz (1989)).

The equation of continuity in the fluid region is

$$\frac{d\rho}{dt} + \rho (\nabla \cdot q) = 0,$$

(1)

where $\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, $q \equiv (u, v, w)$ is the fluid velocity.

Since the fluid is incompressible, so that $\frac{d\rho}{dt} = 0$. Hence the equation (1) modifies to

$$\nabla \cdot q = 0.$$  

(2)

The Euler's equation of motion in the fluid region is

$$\frac{\partial q}{\partial t} + (q \cdot \nabla) q = \nabla \left( gy - \frac{p}{\rho} \right),$$  

(3)

where $p$ is the pressure and $g$ is the acceleration due to gravity.
If the motion starts from rest then the motion is irrotational so that there exists a velocity potential $\Phi(x, y, z; t)$ such that

$$ q = \nabla \Phi. $$  \hspace{1cm} (4)

Hence the continuity equation modifies in this case to

$$ \nabla^2 \Phi = 0 \text{ in the fluid region,} $$  \hspace{1cm} (5)

where $\nabla^2 \equiv (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$ is the Laplacian operator.

After linearisation and integration, the equation (3) gives rise to the linearised Bernoulli’s equation

$$ \frac{\partial \Phi}{\partial t} = g \eta - \frac{p}{\rho} \text{ in the fluid region.} $$  \hspace{1cm} (6)

At the free surface $y = \eta(x, z; t)$, where $\eta(x, z; t)$ is the free surface depression, the pressure $p$ is constant and may be taken to be equal to zero.

By Taylor’s series expansion about $y = 0$ and neglecting higher order terms, the equation (6) reduces to the linearised dynamical boundary condition at the free surface as given by

$$ \frac{\partial \Phi}{\partial t} = g \eta \text{ on } y = 0. $$  \hspace{1cm} (7)

The linearised kinematic condition at the free surface produces

$$ \frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial y} \text{ on } y = 0. $$  \hspace{1cm} (8)

Elimination of $\eta$ between (7) and (8) produces the linearised free surface condition

$$ \frac{\partial^2 \Phi}{\partial t^2} = g \frac{\partial \Phi}{\partial y} \text{ on } y = 0. $$  \hspace{1cm} (9)

The condition of no motion at the bottom gives

$$ \nabla \Phi \to 0 \text{ as } y \to \infty, $$  \hspace{1cm} (10)

for a fluid of infinite depth, and

$$ \frac{\partial \Phi}{\partial y} = 0 \text{ on } y = h, $$  \hspace{1cm} (11)

for a fluid of uniform finite depth $h$. 
If we include the effect of surface tension at the free surface, then the linearised free surface condition is to be replaced by

$$\frac{\partial^2 \Phi}{\partial t^2} - g \frac{\partial \Phi}{\partial y} - \frac{T_1}{\rho} \frac{\partial^3 \Phi}{\partial y^3} = 0 \text{ on } y = 0,$$

(12)

where $T_1$ is the co-efficient of surface tension.

If the liquid moves with uniform velocity $U$ horizontally along the $x$-direction, then the velocity vector $q$ in this case is given by

$$q = \nabla \Phi + U i,$$

(13)

where $i$ is the unit vector along the $x$-direction.

In this case the linearised dynamic condition (7) modifies to

$$\frac{\partial \Phi}{\partial t} + U \frac{\partial \Phi}{\partial x} = g \eta,$$

(14)

and the kinematic condition (8) is to be replaced by

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial y} - U \frac{\partial \eta}{\partial x} \text{ on } y = 0.$$

(15)

The free surface condition (9) modifies to

$$\frac{\partial^2 \Phi}{\partial t^2} + U \frac{\partial^2 \Phi}{\partial x \partial t} = g \frac{\partial \Phi}{\partial y} - U g \frac{\partial \eta}{\partial x}.$$

(16)

In the presence of surface tension at the free surface, the linearised free surface condition (16) further modifies to

$$\frac{\partial^2 \Phi}{\partial t^2} + U \frac{\partial^2 \Phi}{\partial x \partial t} - \frac{T_1}{\rho} \frac{\partial^3 \Phi}{\partial y^3} = g \frac{\partial \Phi}{\partial y} - U g \frac{\partial \eta}{\partial x}.$$

(17)

If the surface is covered by an inertial surface composed of a thin but uniform distribution of non-interacting floating material of area density $\rho \epsilon$ ($\epsilon > 0$), then the free surface condition (9) is to be replaced by

$$\frac{\partial^2 \Phi}{\partial t^2} - \epsilon \frac{\partial^3 \Phi}{\partial t^2 \partial y} = g \frac{\partial \Phi}{\partial y} \text{ on } y = 0.$$

(18)
(b) Superposed Homogeneous Fluids

In a stratified fluid, the density may vary discontinuously in the sense that the fluid region consists of layers of homogeneous fluids, the density being constant in each layer. For simplicity we consider two superposed homogeneous fluids only and obtain the basic equations.

We consider the motion in two incompressible, inviscid and homogeneous fluids with densities \( \rho_1 \) and \( \rho_2 \) \((\rho_1 > \rho_2)\) of the lower and upper fluids respectively separated by a common interface. We consider \( y\)-axis to be vertically downwards into the lower fluid and \( xz\)-plane as the mean interface, so that \( y > 0 \) is the lower fluid region and \( y < 0 \) is the upper fluid region, and \( y = 0 \) is the mean position of the surface of separation.

The basic equations in the linearised theory are derived, as before, from the equations of continuity and the Euler’s equation of motion (cf. Wehausen and Laitone (1960)).

The equations of continuity in the two-fluid regions are

\[
\nabla \cdot \mathbf{q}_1 = 0 \quad \text{in} \quad y > 0, \\
\nabla \cdot \mathbf{q}_2 = 0 \quad \text{in} \quad y < 0,
\]

The Euler’s equations of motion in the two-fluid regions are given by

\[
\frac{\partial \mathbf{q}_1}{\partial t} + (\mathbf{q}_1 \cdot \nabla) \mathbf{q}_1 = \nabla \left( gy - \frac{p_1}{\rho_1} \right) \quad \text{for} \quad y > 0, \\
\frac{\partial \mathbf{q}_2}{\partial t} + (\mathbf{q}_2 \cdot \nabla) \mathbf{q}_2 = \nabla \left( gy - \frac{p_2}{\rho_2} \right) \quad \text{for} \quad y < 0,
\]

where \( \mathbf{q}_1, \mathbf{q}_2 \) are the fluid velocities and \( p_1, p_2 \) are the fluid pressures in the lower and upper fluid regions respectively.

We assume that the motion starts from rest so that it is irrotational and can be described by the velocity potentials \( \Phi_1(x, y, z; t) \) and \( \Phi_2(x, y, z; t) \) in the lower and upper fluids respectively. Thus

\[
\mathbf{q}_1 = \nabla \Phi_1, \\
\mathbf{q}_2 = \nabla \Phi_2,
\]

Hence, the continuity equations (1) produce

\[
\nabla^2 \Phi_1 = 0 \quad \text{in} \quad y > 0, \\
\nabla^2 \Phi_2 = 0 \quad \text{in} \quad y < 0.
\]
The equations of motion (2) after integration and linearisation give the Bernoulli's equations in the two layers as

\[ \begin{align*}
\frac{\partial \Phi_1}{\partial t} &= gy - \frac{p_1}{\rho_1} \quad \text{in } y \geq 0, \\
\frac{\partial \Phi_2}{\partial t} &= gy - \frac{p_2}{\rho_2} \quad \text{in } y \leq 0.
\end{align*} \]  

Since the pressure is continuous at the interface, the coupled equations in (5) produce the interface condition

\[ \rho_1 \left( \frac{\partial \Phi_1}{\partial t} - g \eta \right) = \rho_2 \left( \frac{\partial \Phi_2}{\partial t} - g \eta \right) \quad \text{on } y = \eta, \]  

where \( y = \eta(x, z; t) \) is the interface depression. Thus the linearised dynamic condition at the interface is

\[ \rho_1 \left( \frac{\partial \Phi_1}{\partial t} - g \eta \right) = \rho_2 \left( \frac{\partial \Phi_2}{\partial t} - g \eta \right) \quad \text{on } y = 0. \]  

obtained by using Taylor's series expansion about \( y = 0 \) and neglecting higher order terms.

Now \( F(x, y, z; t) = \eta(x, z; t) - y = 0 \) represents the interface, so that

\[ \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0 \quad \text{on } y = \eta. \]  

After linearisation, this gives

\[ \frac{\partial \eta}{\partial t} = v = \frac{\partial \Phi_1}{\partial y} = \frac{\partial \Phi_2}{\partial y} \quad \text{on } y = 0. \]  

This is known as the linearised kinematic condition at the interface.

Elimination of \( \eta \) between (7) and (9) produces the linearised interface condition as

\[ \frac{\partial^2 \Phi_1}{\partial t^2} - g \frac{\partial \Phi_1}{\partial y} = s \left( \frac{\partial^2 \Phi_2}{\partial t^2} - g \frac{\partial \Phi_2}{\partial y} \right) \quad \text{on } y = 0, \]  

where \( s = \frac{\rho_1}{\rho_2} \) (0 \( \leq s \leq 1 \)).

It may be noted that in the absence of the upper fluid (i.e. for \( s = 0 \)) the condition (10) reduces to the free surface condition given in (9) in the previous subsection (a).

The interface depression \( \eta(x, z; t) \) is obtained from (7) as

\[ \eta = \frac{1}{g(1 - s)} \left( \frac{\partial \Phi_1}{\partial t} - s \frac{\partial \Phi_2}{\partial t} \right) \quad \text{on } y = 0. \]  

(ii)
If the interface is composed of a thin but uniform distribution of non-interacting floating material of area density \((\rho_1 - \rho_2) \epsilon (\epsilon \geq 0)\), then the interface condition (10) is to be replaced by

\[
\left( \frac{\partial^2 \Phi_1}{\partial t^2} - s \frac{\partial^2 \Phi_2}{\partial t^2} \right) - \epsilon \frac{\partial^2}{\partial t^2} \left( \frac{\partial \Phi_1}{\partial y} - s \frac{\partial \Phi_2}{\partial y} \right) = g \left( \frac{\partial \Phi_1}{\partial y} - s \frac{\partial \Phi_2}{\partial y} \right) \text{ on } y = 0.
\] (12)

If the lower fluid is of infinite depth and the upper fluid is of infinite height, then the condition of no motion at infinite depth and height gives

\[
\nabla \Phi_1 \to 0 \quad \text{as } y \to \infty,
\]
\[
\nabla \Phi_2 \to 0 \quad \text{as } y \to -\infty.
\] (13)

However, if the upper fluid is of infinite height and the lower fluid is of uniform finite depth below the interface, then the condition of no motion at the bottom and the top gives

\[
\frac{\partial \Phi_1}{\partial y} = 0 \quad \text{on } y = h,
\]
\[
\nabla \Phi_2 \to 0 \quad \text{as } y \to -\infty.
\] (14)

If we consider the effect of interfacial tension at the interface then the condition (10) reduces to

\[
\left( \frac{\partial^2 \Phi_1}{\partial t^2} - \frac{g}{\partial t^2} \frac{\partial \Phi_1}{\partial y} \right) - s \left( \frac{\partial^2 \Phi_2}{\partial t^2} - \frac{g}{\partial t^2} \frac{\partial \Phi_2}{\partial y} \right) = -\frac{T_1}{\rho_1} \left\{ \begin{array}{l}
\frac{\partial^3 \Phi_1}{\partial y^3} \\
\frac{\partial^3 \Phi_2}{\partial y^3}
\end{array} \right\} \text{ on } y = 0.
\] (15)

(c) Stratified Fluid

In this section we derive the basic equation for a stratified fluid whose density varies continuously in the vertical direction. We consider the motion in an incompressible inviscid weakly stratified fluid of uniform finite depth \(h\) under the action of gravity only and bounded by a free surface. We choose a rectangular cartesian co-ordinate system in which the \(y\)-axis is taken vertically downwards into the fluid region, \(zz\)-plane is taken to be horizontal so that the plane \(y = 0\) is the undisturbed free surface and the fluid
occupies the region $0 \leq y \leq h$. If $\rho_1(x, y, z; t)$ and $P(x, y, z; t)$ denote respectively the density and pressure at any point in the stratified fluid region, then we can write

$$\rho_1(x, y, z; t) = \rho_0(y) + \rho(x, y, z; t),$$

(1)

and

$$P(x, y, z; t) = g \int_0^y \rho_0(y) \, dy + P(x, y, z; t),$$

(2)

where $\rho_0(y)$ denotes the density of the fluid at rest at depth $y$, $\rho(x, y, z; t)$ and $P(x, y, z; t)$ respectively denote the perturbed density and pressure and $g$ is the acceleration due to gravity. The quantities $\rho$ and $p$ are assumed to be small.

Assuming linear theory, the basic equations in stratified fluid are obtained from the equation of continuity, the Euler's equation of motion and the incompressibility condition of the fluid (cf. Chia-Yeh-Shun (1980), Kundu (1990), Baines (1995)).

The equation of continuity in the fluid region is

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \mathbf{q}) = 0.$$

(3)

The Euler's equation of motion in the fluid region is

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = \nabla \left( g y - \frac{p}{\rho_1} \right),$$

(4)

and the compressibility condition of the fluid is

$$\frac{\partial \rho_1}{\partial t} + (\mathbf{q} \cdot \nabla) \rho_1 = 0.$$

(5)

Using the equation (5), the equation (3) produces

$$\nabla \cdot \mathbf{q} = 0.$$

(6)

After linearisation, equation (4) produces

$$\begin{aligned}
\rho_0(y) \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x}, \\
\rho_0(y) \frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} + g \rho, \\
\rho_0(y) \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z},
\end{aligned}$$

(7)
and
\[ \frac{\partial \rho}{\partial t} + \nu \frac{d \rho_0(y)}{dy} = 0, \]  
(8)
wherein \( u, v, w \) are also assumed to be small in the linearisation process.

It may be noted that because of stratification of the fluid, the ensuing motion is essentially rotational and as such the problem has to be formulated in terms of the pressure function or an appropriate velocity component.

Under Boussinesq approximation, \( \rho_0'(y) \) can be neglected except when it is multiplied by \( g \), the gravity, and thus the continuity equation (6) can be approximately expressed as
\[ \frac{\partial}{\partial x} (\rho_0(y)u) + \frac{\partial}{\partial y} (\rho_0(y)v) + \frac{\partial}{\partial z} (\rho_0(y)w) = 0. \]  
(9)
If we make further assumption about the variation of density to be in the form of an exponential function, then we have the case of the so-called constant Brunt-Vaisala frequency, and in this case, we obtain the governing partial differential equation with the help of equations (7) and (8), as given by
\[ \nabla^2 (\rho_0(y)u) + N^2 \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial z^2} \right) = 0 \] in the fluid region,  
(10)
where \( \nabla^2 \) is the Laplacian, \( N^2 = \frac{-g}{\rho_0'0} \) is the constant Brunt-Vaisala frequency.

Let \( \eta(x, y, z; t) \) denote the free surface depression, then the linearised kinematic condition at the free surface is
\[ \frac{\partial \eta}{\partial t} = v \text{ on } y = 0. \]  
(11)

At the free surface the pressure \( P(x, y, z; t) \) must be equal to the atmospheric pressure which is a constant and may be taken to be zero. Hence equation (2) becomes
\[ g \int_0^\eta \rho_0(y) \, dy + p(x, y, z; t) = 0 \text{ on } y = \eta(x, z; t). \]  
(12)

This condition is known as the dynamical boundary condition at the free surface. Expanding by Taylor’s series about \( y = 0 \) and neglecting higher order terms, the dynamical boundary condition (12) reduces to
\[ p(x, 0, z; t) = -g \rho_0(0) \eta. \]  
(13)
The free surface depression is obtained as

$$\eta = -\frac{p}{g\rho_0(0)} \text{ on } y = 0. \quad (14)$$

Elimination of \( r \) between (11) and (13), produces the free surface condition

$$\frac{\partial p}{\partial t} + g\rho_0(0)v = 0 \text{ on } y = 0. \quad (15)$$

The condition of no motion at the bottom gives

$$v = 0 \text{ on } y = h. \quad (16)$$

If the free surface is covered by an inertial surface composed of a thin but uniform distribution of disconnected floating materials of area density \( \rho_0(0)\epsilon (\epsilon \geq 0) \), where \( \rho_0(0) \) is the density of the fluid at the top, then the free surface condition (15) is to be replaced by

$$\frac{\partial p}{\partial t} + g\rho_0(0)v = -\rho_0(0)\epsilon \frac{\partial^2 v}{\partial t^2} \text{ on } y = 0. \quad (17)$$

For two-dimensional motion, the equation (6) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (18)$$

while (7) become

$$\rho_0(y) \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x},$$

$$\rho_0(y) \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + g\rho. \quad (19)$$

If we assume the motion to be described by a stream function \( \Psi(x,y;t) \) so that \( u = -\frac{\partial \Psi}{\partial y} \) and \( v = \frac{\partial \Psi}{\partial x} \) then the equations (19) reduce to

$$\rho_0(y) \frac{\partial^2 \Psi}{\partial y \partial t} = \frac{\partial p}{\partial x},$$

$$\rho_0(y) \frac{\partial^2 \Psi}{\partial x \partial t} = -\frac{\partial p}{\partial y} + g\rho, \quad (20)$$

and the incompressibility condition (8) produces

$$\frac{\partial \rho}{\partial t} + \frac{\partial \Psi}{\partial x} \frac{\rho_0(y)}{\partial y} = 0. \quad (21)$$
Now eliminating $p$ from the two equations in (20) and using the equation (21), we obtain

$$\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + N^2 \frac{\partial^2 \Psi}{\partial x^2} + \frac{N^2}{g} \frac{\partial^3 \Psi}{\partial t \partial y^2} = 0. \tag{22}$$

It may be noted that if $\frac{N^2}{g} << 1$, then equation (22) produces

$$\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + N^2 \frac{\partial^2 \Psi}{\partial x^2} = 0, \tag{23}$$

which is known as the Sobolev partial differential equation.

If we assume the motion to be time-harmonic and described by the stream function $\Psi(x, y; t) = \text{Re}\{\psi(x, y) \exp(-i\omega t)\}$ then equation (22) reduces to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{N^2}{Kg} \left( \frac{\partial^2 \psi}{\partial x^2} - K \frac{\partial \psi}{\partial y} \right) \text{ in the fluid region}, \tag{24}$$

where $K = \frac{g}{\sigma^2}$.

For the two-dimensional case, the coupled equations of (20) and the linearised free surface condition produce

$$K \frac{\partial \psi}{\partial y} - \frac{\partial^2 \psi}{\partial x^2} = 0 \text{ on } y = 0, \tag{25}$$

and the bottom condition is

$$\psi = 0 \text{ on } y = h. \tag{26}$$

In the presence of inertial surface instead of the free surface, the condition (25) is to be replaced by

$$K \frac{\partial \psi}{\partial y} - (1 - \epsilon K) \frac{\partial^2 \psi}{\partial x^2} = 0 \text{ on } y = 0. \tag{27}$$

Again eliminating $p$ from coupled equations in (20) and using the equation (21), we find that $\Psi$ satisfies

$$\frac{\partial^2}{\partial t^2} (\nabla^2 \Psi) + \frac{\rho_0(y)}{\rho_0(y)} \frac{\partial^3 \Psi}{\partial y \partial t^2} = -g \frac{\rho_0(y)}{\rho_0(y)} \frac{\partial^3 \psi}{\partial x^2}, \tag{28}$$

where $\nabla^2$ denotes the two-dimensional Laplacian.

If we substitute $\Psi(x, y; t) = u(x, y; t) \exp(\beta y)$, then equation (28) reduces to the following partial differential equation

$$\frac{\partial^2}{\partial t^2} (\nabla^2 u - \beta^2 u) + \omega_0^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{29}$$
where \( \omega_0 = (2\beta g)^{1/2} \). In this case, \( \omega_0 \) is called Brunt-Vaisala frequency.

If we assume the motion to be time-harmonic and described the stream function in this form

\[
u(x, y; t) = u(x, y) \exp(\imath \omega t).
\] (30)

Then the complex valued function \( u(x, y) \) satisfies the Klein-Gordon equation

\[
\frac{\partial^2 u}{\partial y^2} - \beta^2 u = \frac{1}{a^2} \frac{\partial^2 u}{\partial x^2},
\] (31)

where

\[
\frac{1}{a^2} = \frac{\omega_0^2}{\omega^2} - 1.
\] (32)

Thus the governing differential equation in a stratified fluid is of hyperbolic type.
Chapter 2

Mathematical Preliminaries

A brief description of the different concepts of mathematical ideas and properties which have been used in the mathematical study of the various physical problems presented in this thesis, is given in this chapter.

2.1 Integral Transform

In the problems of applied mathematics as well as mathematical physics there are many applications of integral transforms. In the mathematical analysis, some well known integral transforms such as Fourier, Laplace and Hankel transforms are utilized in this thesis. Most of the integral transforms and their inversion formula are available in standard books on integral transform, e.g. Sneddon (1972), Davies (1977), Titchmarsh (1937) and others.

Fourier Transform

If \( f(x) \) is piece-wise continuously differentiable and absolutely integrable on the whole real line \( R \), then at a point of continuity

\[
f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(\xi) \exp(-i\xi x) d\xi,
\]

(1)
where
\[ F(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) \exp(i\xi x) dx. \] (2)

The function \( F(\xi) \) is called the Fourier transform of \( f(x) \) and the pair of equations (1) and (2) are called the Fourier inversion theorem.

When \( f(x) \) is given only for positive values of \( x \), there are two forms of Fourier integral transform and are known as Fourier cosine transform and Fourier sine transform.

In the first case, when \( f(x) \) is an even function of \( x \) i.e., \( f(-x) = f(x) \ (x > 0) \), then
\[ f(x) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} F_c(\xi) \cos \xi x \ d\xi, \] (3)

where
\[ F_c(\xi) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} f(x) \cos \xi x \ dx. \] (4)

The function \( F_c(\xi) \) is called the Fourier cosine transform of the function \( f(x) \) and the formula (3) and (4) are together called Fourier cosine inversion theorem.

If \( f(x) \) is an odd function, i.e., \( f(-x) = -f(x) \ (x > 0) \), then
\[ f(x) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} F_s(\xi) \sin \xi x \ d\xi, \] (5)

where
\[ F_s(\xi) = \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} f(x) \sin \xi x \ dx. \] (6)

The function \( F_s(\xi) \) is called the Fourier sine transform of the function \( f(x) \) and the formula (5) and (6) are together called the Fourier sine inversion theorem.

Laplace Transform

If \( f(x) \) has a continuous derivative and if \( f(x) = O(\exp(\nu x)) \) for large positive values of \( x \), where \( \nu \) is real, then
\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p) \exp(px) dp, \ (c > \nu), \] (7)
where
\[ F(p) = \int_0^\infty f(x) \exp(-px) dx, \quad (\text{Re } p > 0). \tag{8} \]

The constant \( c \) is real and is such that all the singularities of \( F(p) \) lie to the left of the line \( \text{Re } p = c \) in the complex \( p \)-plane. The function \( F(p) \) is called the Laplace transform of \( f(x) \) and equation (7) is the inverse Laplace transform.

**Hankel Transform**

If \( x^{1/2}f(x) \) is piece-wise continuous and absolutely integrable on the positive real line, then, at the point of continuity, for \( \nu > -\frac{1}{2} \)
\[
f(x) = \int_0^\infty tF_\nu(t)J_\nu(xt) \, dt, \tag{9}\]
where
\[
F_\nu(t) = \int_0^\infty x f(x)J_\nu(xt) \, dx. \tag{10}\]

The formulation (10) is the Hankel transform of \( f(x) \) and the result expressed by formula (9) and (10) is called Hankel inversion theorem.

### 2.2 Singular Integral Equation

Various problems in mathematical physics can be reduced to a singular integral equation of first kind with Cauchy type kernel of the form
\[
\frac{1}{\pi} \int_L \frac{\phi(t)}{x - t} \, dt = \psi(x), \quad x \in L, \tag{1}\]
where the integral is in the sense of Cauchy principal value, \( \psi(x) \) is a known continuous function and \( L \) may consist of either a single interval or several disjoint intervals. The solution of this integral equation depends on the conditions satisfied by \( \phi(t) \) at the end points dictated by the physics of the problem.
Solution of the Integral Equation for $L = (a, b)$

In this case the integral equation (1) becomes

$$\frac{1}{\pi} \int_a^b \frac{\phi(t)}{x-t} \, dt = \psi(x), \quad a < x < b,$$  \hspace{1cm} (2)

The solution of (2) can be obtained for three cases viz., $\phi(x)$ has integrable singularities at both ends, $\phi(x)$ is bounded at one end but unbounded at the other end and $\phi(x)$ is bounded at both the ends.

The solution of this integral equation will contain an arbitrary constant in the first case, no arbitrary constant will appear in the second case but in the third case, the solution will exist only when $\psi(x)$ satisfies a certain solvability criterion. The solution for these three cases are as follows (cf. Mushkhelishvile (1963), Gakhov (1966), Mikhlin (1964), Parton and Perlin (1982), Burrows (1985), Tricomi (1957), Gladwell (1980), Peters (1963), Estrada and Kanwal (1985), Mandal (1986b), Ursell (1947)).

**Case I:** If $\phi(x)$ is required to have integrable singularities at both ends, i.e.,

$$\phi(x) = \begin{cases} O \left( \frac{1}{|x-a|^{1/2}} \right) & \text{as } x \to a, \\ O \left( \frac{1}{|x-b|^{1/2}} \right) & \text{as } x \to b, \end{cases}$$

then the solution of (2) is given by

$$\phi(x) = \frac{c}{\pi (x-a)^{1/2}(b-x)^{1/2}} - \frac{1}{\pi} \int_a^b \frac{(t-a)^{1/2}(b-t)^{1/2}}{(x-a)^{1/2}(b-x)^{1/2}} \frac{\psi(t)}{x-t} \, dt,$$  \hspace{1cm} (3)

where $c$ is an arbitrary constant.

**Case II:** If $\phi(x)$ is required to be bounded at $x = b$, say, and unbounded at $x = a$, i.e.,

$$\phi(x) = O \left( \frac{1}{|x-a|^{1/2}} \right) \text{ as } x \to a,$$

then the solution of (2) is given by

$$\phi(x) = -\frac{1}{\pi} \frac{(b-x)^{1/2}}{(x-a)^{1/2}} \int_a^b \frac{(t-a)^{1/2}}{(b-t)^{1/2}} \frac{\psi(t)}{x-t} \, dt.$$  \hspace{1cm} (4)
Similarly, when \( \phi(x) \) is required to be bounded at \( x = a \) and unbounded at \( x = b \), then the solution is given by

\[
\phi(x) = -\frac{1}{\pi} \frac{(x-a)^{1/2}}{(b-x)^{1/2}} \int_a^b \frac{\psi(t)}{(t-a)^{1/2} (b-t)^{1/2} x-t} dt.
\]

**Case III:** When \( \phi(x) \) is required to be bounded at both the ends then the solution of (2) is given by

\[
\phi(x) = -\frac{1}{\pi} (x-a)^{1/2} (b-x)^{1/2} \int_a^b \frac{\psi(t)}{(t-a)^{1/2} (b-t)^{1/2} x-t} dt,
\]

provide \( \psi(t) \) satisfies the solvability criteria

\[
\int_a^b \frac{1}{(t-a)^{1/2} (b-t)^{1/2} x-t} \psi(t) dt = 0.
\]

**Solution of the Integral Equation for** \( L = (-1, -k) \cup (k, 1) \) \( (0 < k < 1) \)

In this case, the integral equation (2) is

\[
\frac{1}{\pi} \int_L \frac{\phi(t)}{x-t} dt = \psi(x), \quad x \in L,
\]

where \( L \) consists of the disjoint intervals \((-1, -k)\) and \((k, 1)\) \((0 < k < 1)\) and suppose

\[
\phi(t) = \begin{cases} 
O(|1-t^2|^{-1/2}) \text{ as } t \to 1, \\
O(|t^2-k^2|^{-1/2}) \text{ as } t \to \pm k.
\end{cases}
\]

Then the solution can be obtained as (cf. Tricomi (1951), Lewin (1975))

\[
\phi(t) = -\frac{1}{\pi} (1-t^2)^{-1/2} (t^2-k^2)^{-1/2} \int_{k<|x|<1} (1-x^2)^{1/2} (x^2-k^2)^{1/2} \frac{\psi(x)}{t-x} dx
\]

\[
+ \frac{ct + c_1}{(1-t^2)^{1/3} (t^2-k^2)^{1/2}} \text{sgn}(t),
\]

where \( c \) and \( c_1 \) are arbitrary constants.
2.3 Source Potential

Velocity potential due to various types of singularities present in water is generally known as a source potential and have wide applications in the linearised theory of water waves. For the radiation or scattering problems in the linearised theory of water waves, involving the presence of a body or a number of bodies, the resulting motion can be described by a series of singularities placed on the body or bodies. These singularities are characterized by means of a velocity potential (also known as Green's function) which gives a typical singular solution of Laplace equation. In the case of two-dimensional problems, these singularities are called line source or multipole source but in the three-dimensional case, these singularities are called point source or point multipoles. By a suitable application of Green's integral theorem to the potential function or stream function or pressure function describing the physical problem and an appropriate source potential, the problem can be reduced to an integral equation with a singular kernel. This singular integral equation can be solved analytically or numerically by appropriate methods. Thorne (1953) gave a detailed survey of source potentials due to fundamental line or point singularities submerged in deep water or water of uniform finite depth. These results were modified in the presence of surface tension (cf. Rhodes-Robinson (1970)) and when the free surface is covered by an inertial surface (cf. Rhodes-Robinson (1982, 1884), Black and Cerone (1982), Mandal and Kundu (1986, 1987), Mandal (1986a, 1988a), Mandal and Ghosh (1989)).

Source Potential in the One Fluid Medium

We consider irrotational motion under gravity due to the presence of a line source submerged at a point \((\xi, \eta)\) in the water region. It is described by the potential function \(G(x, y; \xi, \eta; t)\), whose Laplace transform in time \(t\) represented by \(G(x, y; \xi, \eta; p)\), (same notation \(G\) being used without any confusion), satisfies

\[
\nabla^2 G = 0, \quad y > 0 \text{ except at } (\xi, \eta), \quad (1)
\]
\[ p^2 G - g \frac{\partial G}{\partial y} = 0 \text{ on } y = 0, \quad (2) \]

\[ G \to \ln r \text{ as } r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \to 0, \quad (3) \]

\[ \nabla G \to 0 \text{ as } y \to \infty, \quad (4) \]

for a deep fluid, or

\[ \frac{\partial G}{\partial y} = 0 \text{ on } y = h, \quad (5) \]

for a fluid of uniform finite depth \( h \).

The solution of the boundary value problem described by (1) to (5) can be obtained as (cf. Thorne (1953))

\[ G(x, y; \xi, \eta; \rho) = \ln \frac{r}{r'} - 2 \int_0^\infty \exp\left\{-k(y + \eta)\right\} \frac{\omega^2}{\omega^2 + \mu^2} \cos k(x - \xi) \, dk, \quad (6) \]

for a deep fluid, and

\[ G(x, y; \xi, \eta; \rho) = \ln \frac{r}{r'} - 2 \int_0^\infty \frac{\exp(-kh) \sinh ky \sinh k\eta \cos k(x - \xi)}{k \cosh kh} \, dk \]

\[ -2 \int_0^\infty \frac{\cosh k(h - \eta) \cosh k(h - \rho) \cos k(x - \xi)}{k \cosh kh \cosh kh} \frac{\mu^2}{\mu^2 + \rho^2} \, dk, \quad (7) \]

for a fluid of uniform finite depth \( h \), where

\[
\begin{align*}
    r' &= \sqrt{(x - \xi)^2 + (y + \eta)^2}, \\
    \omega^2 &= gk, \\
    \mu^2 &= gk \tanh kh.
\end{align*}
\]

(8)

If the water is covered by an inertial surface composed of a thin distribution of non-interacting floating materials of surface density \( \rho \epsilon \) where \( \rho \) is the density of the liquid, then the condition (2) is to modified as

\[ p^2 G - (g + \epsilon p^2) \frac{\partial G}{\partial y} = 0 \text{ on } y = 0, \quad (9) \]
Then the source potentials \( G(x, y; \xi, \eta; p) \) can be obtained as (cf. Mandal (1988a), Mandal and Ghosh (1989)).

\[
G(x, y; \xi, \eta; p) = \ln \frac{r}{r'} - 2\varepsilon \int_0^\infty \frac{\exp\{-k(y + \eta)\}}{1 + k\varepsilon} \cos k(x - \xi) dk
\]

\[
-2 \int_0^\infty \frac{\exp\{-k(y + \eta)\}}{k(1 + k\varepsilon)} \cos k(x - \xi) \frac{\omega^2}{\omega^2 + p^2} dk,
\]

(10)

for a deep fluid,

\[
G(x, y; \xi, \eta; p) = \ln \frac{r}{r'} - 2 \int_0^\infty \left\{ \frac{\varepsilon \cosh k(h - y) \cosh k(h - \eta)}{D(k)} \right. \\
+ \frac{\exp(-kh) \sinh ky \sinh k\eta}{k} \left. \frac{\cos k(x - \xi) \cosh kh}{\cosh kh} \right\} dk
\]

\[
-2 \int_0^\infty \frac{\cosh k(h - y) \cosh k(h - \eta) \cos k(x - \xi)}{kD(k)} \frac{\mu^2}{\mu^2 + p^2} dk,
\]

(11)

for a fluid of uniform finite depth \( h \), where

\[
\omega^2 = \frac{gk}{1 + k\varepsilon}, \\
D(k) = \cosh kh + \varepsilon k \sinh kh, \\
\mu^2 = \frac{gk \sinh kh}{D(k)}.
\]

(12)

It may be noted that because of stratification of the fluid, the ensuing motion is essentially rotational and as such the problem has to be formulated in terms of the pressure function or an appropriate velocity component. If the upper surface of stratified fluid of finite depth \( h \) is covered by an inertial surface, then the source potential \( G(x, y; \xi, \eta; p) \) also satisfies equations (1), (3), (5) and the condition (2) modifies to

\[
p^2 \frac{\partial^2 G}{\partial y^2} - (g + \omega^2) \frac{\partial G}{\partial y} = 0 \text{ on } y = 0,
\]

(13)

where, \( \lambda = \frac{\mu^2}{(\omega^2 + N^2)^{1/2}} \) and \( N^2 \) be the Brunt-Vaisala parameter.
In this case, $G(x,y; \xi, \eta; p)$ can be obtained as

$$G(x,y; \xi, \eta; p) = \ln \frac{r}{r'} - 2\lambda \int_0^\infty \left[ \cosh k(h - y) \cosh k(h - \eta) \over D(k) \right. \left. + \frac{\exp(-kh) \sinh ky \sinh k\eta}{k} \frac{\cos k(x - \xi)}{\cosh kh} \right] dk$$

$$-2 \int_0^\infty \cosh k(h - y) \cosh k(h - \eta) \frac{\lambda \mu^2}{k \sinh kh} \cos k(x - \xi) dk,$$  \hspace{1cm} (14)

where

$$D(k) = \cosh kh' + \epsilon k \lambda \sinh kh', \quad \left\{ \begin{array}{l} 
\mu^2 = \frac{gk \sinh kh'}{D(k)}. 
\end{array} \right\}$$  \hspace{1cm} (15)

The source potential in stratified fluid with upper surface covered by an inertial surface is utilized in wave generation problems considered in Chapter 3 of this thesis.

### 2.4 Properties of Analytic Functions Defined by Integrals

We discuss here certain properties of analytic functions to understand the Wiener-Hopf technique. This technique is used to solve the mixed boundary value problem arising in the problem of internal wave diffraction by a semi-infinite inertial surface of a stratified fluid considered in Chapter 12 of this thesis. The references here are the standard books of Noble (1958), Copson (1935), and Titchmarsh (1939).

**Theorem 1:** Let $R$ denote an open connected subset of the complex $\alpha$-plane, and $L$ denote a smooth contour in the complex $z$-plane in the following sense.

In the complex $z$-plane, the position of any point on $L$ can be represented by a parameter $t$ in such a way that $z = \xi(t) + i\eta(t)$ where $t \in [t_1, t_2]$, and both the functions $\xi(t)$ and $\eta(t)$ are continuous on $[t_1, t_2]$. 

Suppose, $g(\alpha, z) = f(z)h(\alpha, z)$ \((\alpha \in R, z \in L)\) which satisfy the following conditions.

(a) \(h(\alpha, z)\) is a continuous function of the complex variable \(\alpha\) and \(z\) where \(\alpha\) lies inside a region \(R\) and \(z\) lies on a contour \(L\).

(b) \(h(\alpha, z)\) is regular in \(R\) for each \(z \in L\).

(c) \(f(z)\) has only a finite number of finite discontinuities on \(L\) and a finite number of maxima and minima on any finite part of \(L\).

(d) \(f(z)\) is bounded except at a finite number of points on \(L\). If \(z_0\) is such a point so that \(g(\alpha, z) \to \infty\) as \(z \to z_0\) then

$$\lim_{\delta \to 0} \int_{(L-\delta)} g(\alpha, z) \, dz,$$

exists, where the notation \((L-\delta)\) denotes the contour \(L\) apart from a small length \(\delta\) surrounding \(z_0\), and \(\lim(\delta \to 0)\) denotes the limit as this excluded length tends to zero. The limit is approached uniformly when \(\alpha\) lies in any closed domain \(F\) contained in \(R\).

(e) If \(L\) goes to infinity and any bounded part of \(L\) is smooth, the conditions (a) and (b) are satisfied for any bounded part of \(L\), and the infinite integral

$$\int_L g(\alpha, z) \, dz,$$

is uniformly convergent when \(\alpha\) lies in any closed domain \(F\) contained in \(R\). Then the function

$$G(\alpha) = \int_L g(\alpha, z) \, dz,$$

is regular in \(R\).

The proof of the theorem 1 is given in the standard books by Copson (1935), Titchmarsh (1939) and Noble (1958). As a special case the complex variable \(z\) may be taken to be real so that we can deduce the following three results from theorem 1.
Result I: If \( f(x) \) is a function of the real variable \( x \) which satisfies conditions (c) and (d) of theorem 1 for \( 0 \leq x < \infty \) and if

\[
|f(x)| < A \exp(r_- x),
\]

as \( x \to -\infty \) with \( A \) and \( r_- \) as real constants \((A > 0)\), then

\[
F_+(\alpha) = \int_0^\infty f(x) \exp(iax) \, dx,
\]

is regular in the upper half plane, \( \tau > r_- \).

Result II: If \( f(x) \) is a function of the real variable \( x \) which satisfies conditions (c) and (d) of theorem 1 for \( -\infty < x \leq 0 \) and if

\[
|f(x)| < B \exp(r_+ x),
\]

as \( x \to -\infty \) with \( B \) and \( r_+ \) as real constants \((B > 0)\), then

\[
F_-(\alpha) = \int_{-\infty}^0 f(x) \exp(iax) \, dx,
\]

is regular in the lower half plane \( \tau < r_+ \).

Result III: Define

\[
F(\alpha) = \int_{-\infty+ic}^{\infty+ic} \frac{f(z)}{z-\alpha} \, dz = \int_{-\infty}^{\infty} \frac{f(\xi+ic)}{(\xi-\sigma) + i(c-\tau)} \, d\xi,
\]

where \( \alpha = \sigma + i\tau, z = \xi + ic \), and \( c \) is a real constant. Suppose \( f(\xi + ic) \) regarded as a function of \( \xi \) satisfies the conditions (c) and (d) of theorem 1, and

\[
|f(\xi + ic)| < C|x|^{-k} \text{ for } |x| > M,
\]

with \( C, k \) and \( M \) as positive constants. Then \( F(\alpha) \) as defined by equation (5) is a regular function in the upper half plane \( \tau > c \). It also represents a different regular function in the lower half plane \( \tau < c \).
Theorem 2: Let \( f(\alpha) \) be an analytic function of \( \alpha = \sigma + i\tau \), regular in the strip \( \tau_- < \tau < \tau_+ \) and
\[
|f(\sigma + i\tau)| < C|\sigma|^{-\rho}, \quad \rho > 0,
\]
for \( |\sigma| \to \infty \), the inequality holding uniformly for \( \tau_- + \delta \leq \tau \leq \tau_- - \delta, \quad \delta > 0 \). Then for \( \tau_- < c < \tau < d < \tau_+ \), we have
\[
f(\alpha) = f_+(\alpha) + f_-(\alpha),
\]
with
\[
f_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty + ic}^{\infty + ic} \frac{f(z)}{z - \alpha} \, dz, \quad (\tau_- < c < \tau), \tag{9}
\]
\[
f_-(\alpha) = \frac{1}{2\pi i} \int_{-\infty + id}^{\infty + id} \frac{f(z)}{z - \alpha} \, dz, \quad (\tau < d < \tau_+), \tag{10}
\]
where \( f_+(\alpha) \) is regular for all \( \tau > \tau_- \) and \( f_- (\alpha) \) is regular for all \( \tau < \tau_+ \).

We note that for any \( \alpha = \sigma + i\tau \), where \( \alpha \) lies between \( \tau_- < \alpha < \tau_+ \), one can consider a contour \( L \) in \( \tau_- < \tau < \tau_+ \) of the rectangular form whose vertices are \( \pm a + i\tau, \pm a + id \) \((\tau_+ > d > \tau > c > \tau_-)\), and the application of Cauchy Residue theorem to the function \( \frac{f(z)}{z - \alpha} \) having a simple pole at \( z = \alpha \) within the contour \( L \) yields the equation (8) with \( f_+(\alpha) \) and \( f_- (\alpha) \) as given by equations (9) and (10) respectively. From our assumption as regards the behaviour of \( f(\alpha) \) as \( |\sigma| \to \infty \) in the strip, the integrals on \( \sigma = \pm a \) tend to zero as \( a \to \infty \). Also the result III is directly applied to the functions \( f_\pm(\alpha) \) to show the regularity property.

Theorem 3: Let \( K(\alpha) \) be an analytic function of \( \alpha = \sigma + i\tau \), regular and non-zero in strip \( \tau_- < \tau < \tau_+ \) such that
\[
|K(\alpha) - 1| < C|\sigma|^{-\rho}, \quad \rho > 0,
\]
for \( |\sigma| \to \infty \), and the inequality (11) holds uniformly for \( \tau_- + \delta \leq \tau \leq \tau_- - \delta, \quad \delta > 0 \). Then we have
\[
K(\alpha) = K_+(\alpha)K_-(\alpha), \tag{12}
\]
where $K_+ (\alpha)$ and $K_- (\alpha)$ are regular and non-zero in $r > r_-$ and $r < r_+$ respectively, and are given by

$$K_{\pm} (\alpha) = \exp \{ f_{\pm} (\alpha) \}, \quad (13)$$

where

$$f_+ (\alpha) = \frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{\ln K(z)}{z - \alpha} dz, \quad (\tau_- < \epsilon < \tau_+),$$

$$f_- (\alpha) = -\frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{\infty + id} \frac{\ln K(z)}{z - \alpha} dz, \quad (\tau < d < \tau_+).$$

We note that the decompositions of $f(\alpha) = f_+ (\alpha) + f_- (\alpha)$ and $K(\alpha) = K_+ (\alpha) K_- (\alpha)$ are not unique.

In the presence of logarithms on the integrand in equation (14) sometime makes the integration difficulty. In such situations the following procedure is often helpful.

Suppose, we have $\ln K(\alpha) = \ln K_+ (\alpha) + \ln K_- (\alpha)$. On differentiating this relation with respect to $\alpha$ we get

$$\frac{K'(\alpha)}{K(\alpha)} = \frac{K_+'(\alpha)}{K_+(\alpha)} + \frac{K'_-(\alpha)}{K_-(\alpha)}. \quad (15)$$

Note that the first term on the right hand side is analytic in the upper half plane and the second term is analytic in the lower half plane. By the use of the theorem 2 in equation (15) to decompose $\frac{K'(\alpha)}{K(\alpha)}$ as $f_+ (\alpha) + f_- (\alpha)$. Then the factors $K_{\pm} (\alpha)$ are obtained by integrating the equations

$$\frac{K'_{\pm} (\alpha)}{K_{\pm} (\alpha)} = f_{\pm} (\alpha). \quad (16)$$

This procedure is convenient only in situations where additive decompositions of $\frac{K'(\alpha)}{K(\alpha)}$ is simple.

In practical applications, the main difficulty arise in the evaluation of the integrals given in equations (9), (10), and (14). For evaluation of this type of integrals we refer to Noble (1958), where various procedures of evaluating such integrals are discussed elaborately.
**Result IV:** Let $K(a)$ be an even function of $a$ satisfies all the conditions of theorem 3 and valid in the strip $\tau_- < r < \tau_+ (r_+ > 0)$, symmetric about the real $a$-axis. Then equation (12) holds for $K_+(a) = K_-(a)$.

The above result follows immediately from theorem 3, if in the first equation of (14) we replace $a$ by $-\alpha$ and change the integration variable $z$ to $-z$ and then use the property $K(-z) = K(z)$. This result will be very useful to find the approximate solutions of three part Wiener-Hopf problems.

### 2.5 Certain Properties of Fourier Transform in the Complex Domain

The Fourier and Bilateral Laplace transform techniques are utilized in connection with the Wiener-Hopf technique and the complete equivalence of both the transforms in the complex plane has been demonstrated by Noble (1958). We discuss the properties of real Fourier transform technique in section 2.1. Now we discuss some properties of Fourier transform in complex domain which are used to study the Wiener-Hopf technique.

If $f(x)$ denotes a continuous function of a real variable $x$ where $x \in (-\infty, \infty)$, then we introduce the real Fourier transform of $f(x)$, which are discussed in the section 2.1. If only informations regarding the infinity behaviour of the continuous function $f(x)$ are given by the growth conditions defined by (2) and (4) in previous section 2.4 then the Fourier transform of $f(x)$ may not exist. Now we consider the problem of finding $f(x)$ from a knowledge of $F_+(a)$ and $F_-(a)$ where $F_+(a)$ exist for $\tau > \tau_-$ and $F_-(a)$ exists for $\tau < \tau_+$.

Suppose, $c$ and $d$ be such that $c > \tau_-$ and $d < \tau_+$. Then using the results I and II in section 2.4, we can write

$$F_+(\sigma + ic) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \{f(x) \exp(-cx)\} \exp(i\sigma x) dx, \quad \text{(1)}$$

exists for a fixed $c$ and similarly

$$F_-(\sigma + id) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^0 \{f(x) \exp(-dx)\} \exp(i\sigma x) dx, \quad \text{(2)}$$
exists for a fixed \( d \). By the Fourier inversion theorem applied to \( F_+ \) and \( F_- \) which are given in equations (1) and (2), we get

\[
\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F_+(\sigma + ic) \exp(-i\sigma x) d\sigma = \begin{cases} 
  f(x) \exp(-cx), & x > 0, \\
  0, & x < 0,
\end{cases}
\]

and

\[
\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F_-(\sigma + id) \exp(-i\sigma x) d\sigma = \begin{cases} 
  0, & x > 0, \\
  f(x) \exp(-dx), & x < 0.
\end{cases}
\]

Thus,

\[
\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F_+(\sigma + ic) \exp(-i(\sigma + ic)x) d\sigma = \begin{cases} 
  f(x), & x > 0, \\
  0, & x < 0,
\end{cases}
\]

or

\[
\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ic}^{\infty+ic} F_+(\alpha) \exp(-i\alpha x) d\alpha = \begin{cases} 
  f(x), & x > 0, \\
  0, & x < 0.
\end{cases}
\]

Similarly,

\[
\frac{1}{(2\pi)^{1/2}} \int_{-\infty+id}^{\infty+id} F_-(\alpha) \exp(-i\alpha x) d\alpha = \begin{cases} 
  0, & x > 0, \\
  f(x), & x < 0.
\end{cases}
\]

Combining (4) and (5), we finally obtain

\[
f(x) = \frac{1}{(2\pi)^{1/2}} \left[ \int_{-\infty+ic}^{\infty+ic} F_+(\alpha) \exp(-i\alpha x) d\alpha + \int_{-\infty+id}^{\infty+id} F_-(\alpha) \exp(-i\alpha x) d\alpha \right].
\]

Equation (6) is called the Generalised Fourier integral formula.

If \( r_- < r_+ \), then in the common strip \( r_- < r < r_+ \), both \( F_+(\alpha) \) and \( F_-(\alpha) \) are analytic. For this we choose \( c = d = c_0 \) (say), where, \( r_- < c_0 < r_+ \). Then we obtain

\[
f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ic_0}^{\infty+ic_0} F(\alpha) \exp(-i\alpha x) d\alpha,
\]

where

\[
F(\alpha) = F_+(\alpha) + F_-(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) \exp(i\alpha x) dx,
\]
is analytic in the strip $\tau_- < \tau < \tau_+$. From this result we derive the following result.

**Result V: Bi-lateral Laplace Transform**

If $\alpha$ is replaced by $is$ in equation (7), we obtain

$$F(is) = \hat{F}(s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) \exp(-sx)dx,$$

(8)

and

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{c_0-i\infty}^{c_0+i\infty} \hat{F}(s) \exp(xs)ds.$$  

(9)

This is known as the bi-lateral Laplace transform.

**Result VI: Suppose that $F(\alpha)$, $\alpha = \sigma + i\tau$ is regular in $\tau_- < \tau < \tau_+$, and $|F(\alpha)| \to 0$ as $|\sigma| \to \infty$, uniformly for $\tau_- + \delta < \tau < \tau_+ - \delta$, $\delta > 0$. If we define a function $f(x)$ by

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty + i\alpha}^{\infty + i\alpha} F(\alpha) \exp(-i\alpha x)d\alpha,$$

(10)

with $c_0 \in (\tau_-, \tau_+)$ and $x \in (-\infty, \infty)$, then

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) \exp(i\alpha x)dx,$$

(11)

$$|f(x)| < \exp\{(\tau_- + \eta) x\} \text{ as } x \to \infty,$$

$$|f(x)| < \exp\{(\tau_+ - \eta) x\} \text{ as } x \to -\infty,$$

(12)

$\eta$ being an arbitrary positive quantity.

This result shows that $f(x)$, as defined by equation (10), is a solution of the integral equation (11).

Finally we state an Abelian theorem for the unilateral Fourier transform which is used in the last section in this thesis.
**Theorem 4: Abelian Theorem**

Let $f(x)$ be a function of a real variable $x \in (0, \infty)$ and

$$F_+(\alpha) = \int_0^\infty f(x) \exp(i\alpha x) \, dx. \quad (13)$$

Assume that the integral defining $F_+(\alpha)$ converges for all $\alpha$ belonging to a half-plane $\tau > \tau_-$ and

$$f(x) \sim Ax^\lambda, \quad (14)$$
as $x \to 0^+$ ($A$, an arbitrary complex constant, $\lambda$ complex with $\text{Re} \, \lambda > -1$). Then

$$F_+(\alpha) \sim A \frac{\Gamma(1 + \lambda)}{\alpha^{1+\lambda}} \exp \left\{ \frac{\pi i}{2} (1 + \lambda) \right\}, \quad (15)$$
as $\alpha \to \infty$ in the region $\tau > \tau_-$.

For proof of the theorem 4 we refer to Widder (1971) and Doetsch (1974). This theorem relates the behaviour of $f(x)$, for small $x$, to the behaviour of the transform $F_+(\alpha)$ as $\alpha \to \infty$ in the upper region $\tau > \tau_-$. The order “hypothesis-conclusion” of theorem 4 can not be inverted as there is no a priori reason to expect that the existence of $\alpha^{1+\lambda} F_+(\alpha)$ as $\alpha \to \infty$ will imply the existence of $x^{-\lambda} f(x)$ as $x \to 0^+$. In practical circumstances in partial differential equations, however the expectation is that $F_+(\alpha)$ satisfies the conditions (15), $f(x)$ will satisfy the condition (14) and the theorem 4 can be treated as reversible. In the section D, we shall have occasions to infer the behaviour of a function near the origin from the infinity behaviour of its transformation.

### 2.6 Wiener-Hopf Technique

The Wiener-Hopf technique is a powerful method to solve boundary value problems of practical importance involving linear partial differential equations. A typical problem in electromagnetic theory, hydrodynamics, elastodynamics, potential theory, etc. requires solution of the steady-state wave equation in free space when semi-infinite boundary or finite strip boundary are present. The boundary condition becomes mixed in the
sense that it has one form on one part and another form on the remaining part. If one discontinuity arises on the boundary condition then the governing boundary value problem is called two-part Wiener-Hopf problem while if two discontinuities are present on the boundary condition then the governing boundary value problem is called three part Wiener-Hopf problem (cf. Noble (1958), Jones (1964), Dean G. Duffy (1994), Morse and Freshbach (1953)).

We now present a brief description of Wiener-Hopf procedure to find the solution of a typical equation involving complex function. This is the problem of determination of two unknown functions, say, $\Phi_+(\alpha)$ and $\Phi_-(\alpha)$, where $\alpha = \sigma + i\tau$ which satisfies a single functional relation of the form

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Phi_-(\alpha) + C(\alpha) = 0,$$  

valid in a strip $\tau_+ < \tau < \tau_+$ where $-\infty < \sigma < \infty$ of the complex $\alpha$-plane. The unknown functions $\Phi_+(\alpha)$ and $\Phi_-(\alpha)$ are only known to be regular in two overlapping half planes $\tau > \tau_-$ and $\tau < \tau_+$ respectively and the asymptotic behaviours as $|\alpha| \to \infty$ in the corresponding half-planes are known. The functions $A(\alpha), B(\alpha), C(\alpha)$ are also regular in the strip $\tau_+ < \tau < \tau_+$. The equation (1), called the Wiener-Hopf equation, is known to be valid in the above mentioned strip $\tau_+ < \tau < \tau_+$.

We assume, for simplicity, that none of the functions $A(\alpha)$ and $B(\alpha)$ in equation (1) is zero in the strip $\tau_+ < \tau < \tau_+$. Then the technique of solving equation (1), known as the Wiener-Hopf technique, consists of the following four steps.

**Step 1:** Rewrite equation (1) as

$$\frac{A(\alpha)}{B(\alpha)}\Phi_+(\alpha) + \frac{C(\alpha)}{B(\alpha)} = 0,$$

and decompose the function $\frac{A(\alpha)}{B(\alpha)}$ with the help of theorem 3 as

$$\frac{A(\alpha)}{B(\alpha)} = \frac{K_+(\alpha)}{K_-(\alpha)},$$
where, $K_+ (\alpha)$ and $K_- (\alpha)$ are regular and non-zero in the half planes $\tau > \tau_-$ and $\tau < \tau_+$ respectively. The fundamental step of the Wiener-Hopf procedure for solution of this equation is that the factorisation of $A(\alpha)/B(\alpha)$ is to be carried out in such a way that one part is analytic in the upper half plane $\tau > \tau_-$ and another part is analytic in the lower half plane $\tau < \tau_+$.

Rewrite equation (2), by using (3) as

$$K_+ (\alpha) \Phi_+ (\alpha) + K_- (\alpha) \Psi_- (\alpha) + K_- (\alpha) \frac{C(\alpha)}{B(\alpha)} = 0.$$  \hspace{1cm} (4)

**Step 2:** Use of theorem 2 to write the function $\frac{C(\alpha)}{B(\alpha)} K_- (\alpha)$, which is regular in the strip $\tau_- < \tau < \tau_+$, as

$$\frac{C(\alpha)}{B(\alpha)} K_- (\alpha) = D_+ (\alpha) + D_- (\alpha),$$  \hspace{1cm} (5)

where $D_+ (\alpha)$ and $D_- (\alpha)$ are regular in the above two overlapping half-plane respectively.

**Step 3:** Rearrange equation (4), by using equation (5) as

$$K_+ (\alpha) \Phi_+ (\alpha) + D_+ (\alpha) = -K_- (\alpha) \Psi_- (\alpha) - D_- (\alpha).$$  \hspace{1cm} (6)

**Step 4:** Note that the left side of equation (6) is regular in the upper half plane $\tau > \tau_-$, the right side is regular in the lower half plane $\tau < \tau_+$, while the equation (6) is valid in the open strip $\tau_- < \tau < \tau_+$. The left side is therefore analytic continuation of the right side. Hence equation (6) defines an entire function $E(\alpha)$ of the complex variable $\alpha$. Thus the two unknown function $\Phi_+ (\alpha)$ and $\Psi_- (\alpha)$ are determined as follows:

$$\Phi_+ (\alpha) = \frac{-D_+ (\alpha) + E(\alpha)}{K_+ (\alpha)}, \hspace{0.5cm} \alpha \in (\tau_-, \infty),$$  \hspace{1cm} (7)

$$\Psi_- (\alpha) = \frac{-D_- (\alpha) - E(\alpha)}{K_- (\alpha)}, \hspace{0.5cm} \alpha \in (-\infty, \tau_+).$$  \hspace{1cm} (8)

The unknown entire function $E(\alpha)$ appearing in equations (7) and (8) is to be determined from the known behaviour of the unknown functions $\Phi_+ (\alpha)$ and $\Psi_- (\alpha)$ as $|\alpha| \to \infty$ in their respective half planes of regularity. For example, if it so happens that
\begin{equation}
|\Phi_+(\alpha)K_+(\alpha) + D_+(\alpha)| < |\alpha|^p \text{ as } |\alpha| \to \infty \text{ in } \tau > \tau_-, \nonumber
\end{equation}
and
\begin{equation}
|\Psi_-(\alpha)K_-(\alpha) + D_-(\alpha)| < |\alpha|^q \text{ as } |\alpha| \to \infty \text{ in } \tau < \tau_+, \tag{10}
\end{equation}

then the extended form of Liouville's theorem ensures that the function \( E(\alpha) \), appearing in equation (9) and (10), is a polynomial of degree less than or equal to the integral part of \( \min(p,q) \). Now we discuss here the extended form of Liouville's theorem.

**Theorem 5: Extended Liouville's Theorem**

If \( f(z) \) is an entire function such that \( f(z) = O(|z|^m) \) as \( |z| \to \infty \), then \( f(z) \) is a polynomial of degree not greater than \( |m| \), where \( |m| \) represents the largest integer not greater than \( m \). We note that the crucial step in the Wiener-Hopf technique described above involves the decomposition of a function regular in a strip into two parts, each part being regular in one of two half-planes which overlap with the strip of regularity of the function.

**Note:** The Wiener-Hopf technique described here to solve functional equation (1) to within an arbitrary entire function is applicable even if the known functions \( A(\alpha) \) and \( B(\alpha) \) possess zeros in their common strip of regularity, for the zeros of \( A(\alpha) \) can be absorbed into the function \( \Phi_+(\alpha) \) and the zeros of \( B(\alpha) \) can be absorbed into the function \( \Psi_-(\alpha) \) without their regularity properties.

### 2.7 Method of Stationary Phase

In the mathematical study of wave generation problems due to initial disturbances at the surface of a liquid, the solutions are obtained in terms of integrals which involve rapidly oscillating integrands. The asymptotic form of these integrals is more convenient in certain physical applications. The method of obtaining the asymptotic expansion of this type of integral is commonly known as the method of stationary phase (cf. Lamb...
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(1945), Stoker (1957), Jones (1964), Nayfeh (1993)). Now the stationary phase method is discussed here briefly. For this we consider an integral of the type

$$I(\alpha) = \int_{a}^{b} f(t) \exp(iah(t)) dt,$$

where $\alpha$ is a large positive real number and the functions $f(t), h(t)$ both are real.

If $h'(t)$ vanishes in the interval $[a, b]$ (i.e. the phase has stationary points), the contribution of the asymptotic expansion of the integral arises from the immediate neighbourhoods of the ends and stationary points but the major contribution arising from the neighbourhoods of the stationary points. For the rapid oscillations of $\exp[iah(t)]$, the contribution to the integral are neglected except in the neighbourhoods of the end and stationary points which are given by the equation $h'(t) = 0$. If we write $t = x + \xi$, where $a < x < b$ such that $h'(x) = 0$, then for small values of $\xi$, $h(t)$ can be expressed as

$$h(t) = h(x) + \frac{1}{2} \xi^2 h''(x),$$

approximately. Thus in the neighbourhood of the stationary point $x$, the dominant part of the integral (1) is given by

$$I(\alpha) \approx f(x) \exp(ih(x)) \int_{0}^{\infty} \exp \left\{ \frac{1}{2} iah''(x) \xi^2 \right\} d\xi.$$

If the derivative of a complex function $F(\xi)$, where $\xi$ is a complex variable, exists and is continuous inside and on a closed curve $c$, then

$$\int_{c} F(\xi) d\xi = 0.$$

This is known as Cauchy theorem. To evaluate the integral (3), we apply the Cauchy theorem, where $F(\xi) = \exp \left\{ \frac{1}{2} iah''(x) \xi^2 \right\}$ and the integration is carried around the closed curve $c$. For the evaluation of this integral, the fundamental step is to choose the contour $c$ in such a way that the original Fourier integral is transformed into a Laplace integral, that is the dominant part of the integrand is a real decaying exponential at infinity.
For the use of Cauchy theorem, we choose the closed contour $c$ in such a way that contour $c$ consists of the real axis $x$, a line making angle $45^0$ to the real axis, and an arc which forming an eighth of a circle with radius $R$, as shown in the following figure.

Deformation of the contour of integration

Hence,

$$\left(\int_{c_1} + \int_{c_2} + \int_{c_3}\right) F(\xi)d\xi = 0. \quad (5)$$

For $h''(x) < 0$, (5) implies

$$\int_0^\infty \exp \left\{ \frac{1}{2} i ah''(x) \xi^2 \right\} d\xi = \frac{\pi^{1/2} \exp \left( \frac{ix}{4} \right)}{\left| 2ah''(a) \right|^{1/2}} \text{ if } h''(a) < 0. \quad (6)$$

For $h''(x) > 0$, (5) implies

$$\int_0^\infty \exp \left\{ \frac{1}{2} i ah''(x) \xi^2 \right\} d\xi = \frac{\pi^{1/2} \exp \left( \frac{-ix}{4} \right)}{\left| 2ah''(a) \right|^{1/2}} \text{ for } h''(a) > 0. \quad (7)$$

By using equations (6) and (7), equation (4) reduces to

$$I(\alpha) \sim \frac{\pi^{1/2} f(x)}{\left| 2ah''(a) \right|^{1/2}} \exp \left[ i \left( h(x) \pm \frac{\pi}{4} \right) \right]. \quad (8)$$
To determine the validity of approximate expansion of the integral given in equation (4), we note that the next terms of $h(t)$ in equation (3) will be $\frac{1}{6} \xi^3 h'''(x)$ so that this approximation is valid only when $\xi^3 h'''(x)$ is small even when $\xi^2 h''(x)$ is a multiple of $2\pi$. It follows that the above method of approximation is valid under the condition that the ratio
\[
\frac{h'''(x)}{(|h''(x)|)^{3/2}},
\]
should be small.

In the presence of stationary point, it is shown that the principal contribution from the neighbourhood of a stationary point is $O(t^{-1/2})$ but in the absence of stationary point, the principal contribution is $O(t^{-1})$. Hence only the stationary points contribute to the leading term in the asymptotic expansion of $I(\alpha)$. The stationary phase method is used to obtain the asymptotic form of the interface depression or surface depression in a stratified fluid for some wave generation problems in Chapters 3 to 8 in this thesis.

2.8 Method of Steepest Descent

In the mathematical study of internal wave scattering in stratified fluid, the solution for the stream function describing the motion in the fluid is obtained in terms of an integral in which the arguments of exponents in the integrand are complex. The method used to evaluate the asymptotic form of this type of integrals is known as the method of Steepest descent (cf. Nayfeh (1993), Morse and Freshbach (1953), Withham (1977), Copson (1965), etc.). The method is discussed here briefly in connection with the approximate evaluation of an integral of the type
\[
I(z) = \int_C f(z) \exp(\alpha h(z)) \, dz,
\]
where $C$ is a contour in the complex $z$-plane, $|\alpha|$ is a large number and $f(z), h(z)$ both are complex analytic functions of $z$ where $z = x + iy$.

To evaluate this integral (1), the integration contour $C$ is deformed into a new contour $C'$ in such a way that either Re $\{\alpha h(z)\}$ or Im $\{\alpha h(z)\}$ is constant. Then the above
complex integral (1) is transformed into either a Fourier integral or a Laplace integral. Therefore the asymptotic development can be determined by using either the method of stationary phase or Laplace method. If the integral is transformed into a Laplace integral (i.e. $\text{Im} \{\alpha h(z)\} = \text{constant}$), the asymptotic development of a Laplace integral arises only from the immediate neighbourhood of the point where $\text{Re} \{\alpha h(z)\}$ is largest on $C'$. In contrast, the full asymptotic development of a Fourier integral depends, in general on the end points as well as all stationary points on $C'$. Suppose on the new contour, $\text{Im}[\alpha h(z)]$ is constant, then the integral (1) can be written as

$$I(z) = \exp\{i\text{Im}[\alpha h(z)]\} \int_{C'} \exp\{\text{Re}[\alpha h(z)]\} dz.$$  \hspace{1cm} (2)

Suppose $z = z_0$ is the maximum point of $h(z)$ which gives the largest value of $\text{Re}[\alpha h(z)]$, where

$$h'(z_0) = 0.$$ \hspace{1cm} (3)

It is particularly important that oscillations be suppressed at this point so that near this point

$$\text{Im}[\alpha h(z)] = \text{Im}[\alpha h(z_0)].$$

Also we note that if an analytic function assumes either a maximum or a minimum value within a closed contour, then it must be a constant function. However, points at which the analytic function has zero derivative are known as saddle points instead of maxima or minima.

Thus, from equation (2), the function $h(z)$ has no maximum or minimum at $z = z_0$, i.e. it cannot have a mountain top or a mountain valley. Therefore the real part of a complex function cannot have a maximum or minimum. This point must be a saddle point. Again, the constant phase contours are known as steepest descent and ascent contours.

To illustrate the above discussion, let us use this information to derive a general formula, giving the first term in the asymptotic expansion of $I(\alpha)$. Now we expand $h(z)$ about $z_0$, then

$$h(z) = h(z_0) + \frac{(z - z_0)^2}{2} h''(z_0) + \cdots$$ \hspace{1cm} (4)
The path of steepest descent must be such that the integrand is a decreasing exponential. Therefore let,

\[
\tau = \left[ \frac{\exp\{i(\pi + \phi)\} f''(t_0)}{2} \right]^{1/2} (z - z_0) ; \text{ where } \alpha = |\alpha| \exp(i\phi),
\]

then the integral (2) becomes,

\[
I(\alpha) = \left\{ \frac{\exp\{\alpha h(z_0)\}}{\exp\{i(\pi + \phi)\} f''(z_0)} \right\}^{1/2} \int_{c^t} \exp\{-|\alpha|\tau^2\} dt.
\] (5)

For sufficiently large $|\alpha|$, the integrand will become effectively zero outside the range in which equation (4) is valid. Therefore the complex integral is replaced by a real integral ranging from $-\infty$ to $\infty$. Finally we obtain from equation (5)

\[
I(\alpha) \sim \left\{ \frac{2\pi}{\alpha h''(z_0)} \right\}^{1/2} f(z_0) \exp\left[ \frac{\alpha h(z_0) + i\pi}{2} \right].
\] (6)

### 2.9 Derivation of the Solution of a Cauchy-Poisson Problem

The problems of generation of unsteady motion in a liquid due to initial disturbances at the free surface are called Cauchy-Poisson problems. When the disturbance is in the form of a initial depression or impulse concentrated at a point on the free surface, the problem was discussed in the treatise of Lamb (1945) and Stoker (1957) mentioned earlier. To solve this problem, they used the Fourier transform technique and the free surface elevation was evaluated asymptotically by applying the method of stationary phase. Possible extension of these problem to a liquid of uniform finite depth with the upper surface covered by an inertial surface together with its solution obtained by using Laplace transform in time and a suitable use of Green's integral theorem in the fluid region are considered here. The inertial surface elevation is obtained asymptotically by applying the method of stationary phase. Various known results are recovered by making the fluid depth tending to infinity.
Assuming linear theory, the two-dimensional unsteady irrotational motion in water of uniform finite depth \( h \) is considered. The upper surface of the fluid is covered by an inertial surface composed of a thin but uniform distribution of disconnected floating particles of area density \( \rho e \) (\( e > 0 \)), where \( \rho \) is the volume density of the liquid. Since the motion is starts from rest, it is irrotational and can be described by the velocity potential \( \phi(x, y; t) \) which satisfies the following initial value problem

\[
\begin{align*}
\nabla^2 \phi &= 0, \quad 0 \leq y \leq h, \quad t \geq 0, \\
\frac{\partial^2 \phi}{\partial t^2} \left( \phi - \epsilon \frac{\partial \phi}{\partial y} \right) - g \frac{\partial \phi}{\partial y} &= 0 \quad \text{on } y = 0, \quad t > 0, \\
\frac{\partial \phi}{\partial y} &= 0 \quad \text{on } y = h, \quad t \geq 0, \\
\phi - \epsilon \frac{\partial \phi}{\partial y} &= 0 \quad \text{at } t = 0 \quad \text{on } y = 0, \\
\frac{\partial}{\partial t} \left( \phi - \epsilon \frac{\partial \phi}{\partial y} \right) &= g f(x) \quad \text{at } t = 0 \quad \text{on } y = 0,
\end{align*}
\]

where \( f(x) \) is the initial depression of the inertial surface.

Let \( \Phi(x, y; p) \) denote the Laplace transform of \( \phi(x, y; t) \) in time defined as

\[
\Phi(x, y; p) = \int_0^\infty \phi(x, y; t) \exp(-pt) \, dt, \quad p > 0.
\]

By the use of Laplace transform, the initial value problem described by (1) to (5) reduces to the boundary value problem (BVP)

\[
\begin{align*}
\nabla^2 \Phi &= 0, \quad 0 \leq y \leq h, \\
p^2 \Phi - (g + \epsilon p^2) \frac{\partial \Phi}{\partial y} &= g f(x) \quad \text{on } y = 0, \\
\frac{\partial \Phi}{\partial y} &= 0 \quad \text{on } y = h.
\end{align*}
\]

This BVP is to be solved by a suitable use of Green's integral theorem.

Let \( G(x, y; X', Y'; p) \) denote the Green's function satisfying

\[
\nabla^2 G = 0, \quad 0 \leq y \leq h \quad \text{except at } (X', Y'),
\]
The solution for \( G(x, y; X', Y'; p) \) can be obtained as (cf. Mandal and Kundu \(1986))

\[
G = -\ln \frac{r}{r'} + 2 \int_0^\infty \left\{ \frac{\cosh k(h - y) \cosh k(h - Y')}{D(k)} \times \cos k(x - X') \cos k(x - X') \frac{\mu^2}{\mu^2 + p^2} \right\} dk,
\]

where

\[
\begin{align*}
    r' &= \{(x - X')^2 + (y - Y')^2\}^{1/2}, \\
    D(k) &= \cosh kh + \epsilon k \sinh kh, \\
    \mu^2 &= \frac{gk \sinh kh}{D(k)}.
\end{align*}
\]

By the use of Green's integral theorem to \( \Phi \) and \( G \) in the liquid region, we find

\[
\Phi(X', Y'; p) = \frac{1}{2\pi} \int_0^{\infty} \left( G - \frac{\partial G}{\partial y} \right)(x, 0; X', Y'; p) f(x) dx
\]

\[
= \frac{1}{\pi} \int_0^{\infty} \frac{\mu^2 \cosh k(h - Y')}{\mu^2 + p^2 \cosh kh} \int_{-\infty}^{\infty} \cos k(x - X') f(x) dx \, dk.
\]

Laplace inversion gives

\[
\phi(X', Y'; t) = \frac{1}{\pi} \int_0^{\infty} \mu \sin \mu \frac{\cosh k(h - Y')}{k \sinh kh} \int_{-\infty}^{\infty} \cos k(x - X') f(x) dx \, dk.
\]

The inertial surface depression \( \eta(X', t) \) at time \( t \) is given by
\[ \eta(X', t) = \frac{1}{g} \frac{\partial}{\partial t} \left\{ \phi(X', 0; t) - \frac{\epsilon}{g} \frac{\partial \phi}{\partial x} (X', 0; t) \right\} \]

\[ = \frac{1}{\pi} \int_0^\infty \cos \mu t \int_{-\infty}^\infty \cos k(x - X') f(x) dx \, dk. \quad (18) \]

If the initial inertial surface depression is concentrated at the origin, then \( f(x) = \delta(x) \), so that
\[ \eta(x, t) = \frac{1}{\pi} \int_0^\infty \cos \mu t \cos kx \, dk, \quad (19) \]
by choosing \( X' \) to \( x \).

Making \( h \to \infty \), equation (19) reduces to the result given in Mandal (1988a) as
\[ \eta(x, t) = \frac{1}{\pi} \int_0^\infty \cos \mu t \cos kx \, dk, \quad (20) \]
where \( \mu_1^2 = \frac{gk}{1 + \epsilon k} \).

Again, if we put \( \epsilon = 0 \) in equation (20) (i.e. liquid with a free surface), reduces to this result given in Stoker (1957) as
\[ \eta(x, t) = \frac{1}{\pi} \int_0^\infty \cos \omega t \cos kx \, dk, \quad (21) \]
where \( \omega^2 = gk \).

We now obtain the asymptotic form of the initial inertial surface depression by the method of stationary phase.

We rewrite equation (19) as
\[ \eta(x, t) = \frac{1}{2\pi} \text{Re} \int_0^\infty \left[ \exp\{it\psi(-k)\} + \exp\{it\psi(k)\} \right] dk, \quad (22) \]
where
\[ \psi(k) = \mu - \frac{kx}{t} \quad (23) \]

In the range of integration, the first integral of equation (22) has no stationary point but one stationary point exists for the second integral which is given by the equation
\[ \psi'(k) = 0, \]
where
\[ \psi'(k) = \frac{(gh)^{1/2}}{2} \left[ \frac{1}{\{D(k)\}^{3/2}} \left\{ \left( \frac{kh}{\sinh kh} \right)^{1/2} + \left( \frac{kh}{\sinh kh} \right)^{-1/2} \cosh kh \right\} \right] - \frac{x}{t}. \] (24)

The term in the square bracket of (24) decreases monotonically from 2 to 0 as \( k \) increases from 0 to \( \infty \). Also \( \psi'(0) = (gh)^{1/2} - \frac{\pi}{4} \) and \( \psi'(\infty) \) is negative.

Hence, for \( \frac{\pi}{4} > (gh)^{1/2} \), both the functions \( \psi'(0) \) and \( \psi'(\infty) \) are negative. Also \( \psi''(k) < 0 \), i.e. \( \psi'(k) \) is a monotonic decreasing function for \( k > 0 \) so that there exists no stationary points in the range of integration. However, for \( \frac{\pi}{4} < (gh)^{1/2} \), \( \psi'(0) \) is positive while \( \psi'(\infty) \) is negative. Since \( \psi'(k) \) is a monotonic decreasing function for \( k > 0 \), then \( \psi'(k) \) has a unique zero in the range of integration, so that there exists one stationary point at \( k = k_0 \), say. Finally, when \( \frac{\pi}{4} = (gh)^{1/2} \), \( \psi'(0) = 0 \) so that there exists a stationary point at \( k = 0 \). It gives a smaller contribution than the case \( \frac{\pi}{4} < (gh)^{1/2} \), so that its contribution can be neglected. Now applying the stationary phase method to the second integral in (22), we obtain

\[ \eta(x, t) \sim \frac{1}{2\pi} \left( \frac{2\pi}{|\psi''(k_0)|} \right)^{1/2} \cos \left( t\psi(k_0) - \frac{\pi}{4} \right). \] (25)

Making \( h \to \infty \), (25) reduces to the result given in Mandal (1988) as

\[ \eta(x, t) \sim \frac{1}{2\pi} \left( \frac{2\pi}{t|\psi''(k_0)|} \right)^{1/2} \cos \left( t\psi(k_0) - \frac{\pi}{4} \right), \] (26)

where

\[ \psi(k) = \left( \frac{gk}{1 + \epsilon k} \right)^{1/2} - \frac{kx}{t}. \] (27)

If we put \( \epsilon = 0 \) in equation (25), this result coincides with the result given in Stoker (1957), then

\[ \eta(x,t) \approx \frac{1}{2} \left( \frac{g}{\pi} \right)^{1/2} \frac{t}{x^{3/2}} \cos \left( \frac{gt^2}{4x} - \frac{\pi}{4} \right). \] (28)

If the motion is set up by the action of an initial surface impulse \( I(x) \) per unit area applied to the inertial surface, then only the conditions (4) and (5) are modifies to

\[ \phi - \epsilon \frac{\partial\phi}{\partial y} = - \frac{I(x)}{\rho} \] at \( t = 0 \) on \( y = 0 \), (29)
\[
\frac{\partial}{\partial t} \left( \phi - \epsilon \frac{\partial \phi}{\partial y} \right) = 0 \text{ at } t = 0 \text{ on } y = 0. \quad (30)
\]

By a similar analysis, we obtain instead of equation (16)
\[
\phi(X', Y'; t) = -\frac{1}{\pi \rho} \int_0^\infty \frac{\cos \mu t \cos k(h - Y')}{D(k)} \int_{-\infty}^\infty \cos k(x - X') I(x) dx \, dk. \quad (31)
\]

If the impulse is concentrated at the origin, then we find
\[
\phi(X', Y'; t) = -\frac{1}{\pi \rho} \int_0^\infty \frac{\cos k(h - Y')}{D(k)} \cos \mu t \cos kX' \, dk. \quad (32)
\]

In this case the form of the inertial surface depression is given by
\[
\eta(x, t) = \frac{1}{\pi \rho g^{1/2}} \int_0^\infty \left( \frac{k}{\cosh kh + ek} \right)^{1/2} \sin \mu t \cos kx \, dk. \quad (33)
\]

by choosing \( X' \) to \( x \).

Making \( h \to \infty \) in equation (32), it reduces to the result given in Mandal (1988a), then we obtain
\[
\eta(x, t) = \frac{1}{\pi \rho g^{1/2}} \int_0^\infty \left( \frac{k}{1 + ek} \right)^{1/2} \sin \mu t \cos kx \, dk, \quad (34)
\]

and for \( \epsilon = 0 \), equation (33) reduces to the result given in Stoker (1957) as
\[
\eta(x, t) = \frac{1}{\pi \rho g^{1/2}} k^{1/2} \sin \omega t \cos kx \, dk. \quad (35)
\]

By the use of the method of stationary phase, the asymptotic form of \( \eta(x, t) \) given in (33) becomes
\[
\eta(x, t) \sim \frac{1}{2\pi \rho g^{1/2}} \left( \frac{k_0}{\cosh k_0 h + \epsilon k_0} \right)^{1/2} \left( \frac{2\pi}{t |\psi''(k_0)|} \right)^{1/2} \sin \left( t\psi(k_0) - \frac{\pi}{4} \right). \quad (36)
\]

where \( \psi(k) \) is given in (23).

Making \( h \to 0 \), equation (36) reduces to the result given in Mandal (1988) as
\[
\eta(x, t) \sim \frac{1}{2\pi \rho g^{1/2}} \left( \frac{k_0}{1 + \epsilon k_0} \right)^{1/2} \left( \frac{2\pi}{t |\psi''(k_0)|} \right)^{1/2} \sin \left( t\psi(k_0) - \frac{\pi}{4} \right). \quad (37)
\]
where $\psi(k)$ is given in (27) and for $\epsilon = 0$, equation (37) reduces to the result given in Stoker (1957) as

$$\eta(x, t) \sim \frac{2}{\pi^{1/2} \rho g x t} \left( \frac{gt^2}{4x} \right)^{3/2} \sin \left( \frac{gt^2}{4x} - \frac{\pi}{4} \right).$$

(38)

This problem is generalised in stratified fluid due to various types of initial disturbances considered in Chapters 3 to 9 in this thesis.